Given a graded algebra \( A = \bigoplus_{i \in \mathbb{N}} A_i \) and a filtered algebra \( D = \bigcup_{i \in \mathbb{N}} F_i \), \( 0 \leq F_i \leq \cdots \leq D \).

Say \( D \) is a PBW deformation of \( A \) if \( \gamma(D) = A \) as algebras.

\[ \bigoplus_{i \in \mathbb{N}} F_i/F_i-1 \]

**Examples**

- **Weyl algebra** \( A_n(\mathbb{C}) = \mathbb{C}\langle x_1, \ldots, x_n \rangle / (x_i x_j - x_j x_i - 1) \)

  is a PBW deformation of \( \mathbb{C}[x_1, \ldots, x_n] \)

- \( V = \) finite dimensional \( \mathbb{C}^k \)

  \[ L_k = \mathbb{C}^k \]

  \[ K: \text{Span}_{\mathbb{C}} \mathbb{C}^k - \mathbb{C}^k \to V \]

  \[ W_k \]

  \[ \bigoplus_{k \geq 0} W_k \]

  \[ S(V) = \mathbb{C}^k \bigoplus_{k \geq 0} W_k \]

  \[ D(V) = \bigoplus_{k \geq 0} W_k \]

  \[ \text{K is skew symmetric} \]

  \[ \text{k satisfies Jacobi identity} \]

  \[ 0_k = U(V) \]

  \[ \gamma(U(V)) = \mathbb{C}[x_1, \ldots, x_n] \]

**Properties preserved under PBW deformation**

- integral domain
- prime
- (right) noetherian
- \( \text{gcd}(A) \leq \text{gcd}(D) \)
- \( \text{Ker}(\delta) \leq \text{Ker}(\delta_A) \)
- \( \text{g.l.d.m.}(D) \leq \text{g.l.d.m.}(A) \)

**By properly investigating** [Faeger-Tojborg, 2013, Wu-Chu, 2014]
Representations of PDW deformations of small pro-CG algebras has been of great (recent) interest to symplectic reflection algebras and certain Cherednik algebras, various types of Hecke algebras.

Some results on PDW deformations of \( B \# H \), \( B = T(V)/R \) quadratic algebra.

Have nice enough conclusions for

\[
\mathcal{L}_k = T(V) \# H \quad \text{with} \quad (r - k(r)) \in R
\]
\[ k : R \to H \quad \text{is linear} \]

\[ \text{to be a PDW of } B \# H \]

\[ \text{some of these results hold with } k \text{ as above} \]

\[ k : R \to H \oplus (\text{vov}) \]

\[ \text{all kernel are } \text{affine} \] algebra

Theorem [K-Witten, etc.]: \( B = T(V)/R \) Koszul algebra \( (R \subseteq V \oplus V) \)

\[ \text{Let } H \text{ be a finite algebra with bijective antipode } \]

\[ \text{that acts on } B \text{ (preserving grading)} \]

Then, \( \mathcal{L}_k \) is a PDW deformation of \( B \# H \) if and only if

\[ k \text{ is } H \text{-invariant } \quad k(\theta \cdot r) = k(r) \cdot k(\theta) \quad \text{and} \quad \theta \cdot m = 0 \quad \text{for } (\theta \cdot m) \in (\text{ov}) \text{vov} \]

need more considerations
Outline of proof. (Following Braverman-Gaitsgory)

\[ \Rightarrow \quad \text{(injective direction)} \]

\[ \begin{align*}
\left( \ell \otimes_{A} V \right)^{q} & = \{ \text{graded def. of } B \oplus H \text{ over } C[V] \} \\
\text{where} & \\
0 \otimes_{A} V & = A \otimes_{A} C[V], \\
0 & \end{align*} \]

\( \otimes_{A} V \)

\[ \left( \ell \otimes_{A} V \right)^{q} \]

\[ \left( \ell \otimes_{A} V \right)^{q} = \{ \text{graded def. of } B \oplus H \text{ over } C[V] \} \\
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\text{where} \\
0 \otimes_{A} V & = A \otimes_{A} C[V], \\
0 & \]
Future work: Study PoW deformations of B\oplus H where

\[
\begin{array}{ccc}
B & \oplus & H \\
\text{Reg} & & \\
\end{array}
\]

1) \quad N-Koszul \quad H

\[
\begin{bmatrix}
N-Koszul \\
G \\
\text{Coxing-Eshlan}
\end{bmatrix}
\]

2) "Good\-\"coproduct" of cochains

\[
\begin{array}{c}
S_g(V) \\
\text{Lm} \\
\text{Mm}
\end{array}
\quad U_g(s,m)
\]

\[
\begin{array}{c}
S_g(V) \\
\text{Lm} \\
\text{Mm}
\end{array}
\quad U_g(s,\ell_2)
\]

3) The recent work of Dvir, Tzalgarchik, Lasz, Tzalgarchik, Shkarchke,

\[
S(V) \\
V = \mathbb{C}^n \oplus \mathbb{C}^n \\
V = \mathbb{C}^n
\]

\[
U(g) \\
\frac{\text{Lm}}{\text{Mm}} \\
\text{Lm}
\]