Actions of finite dimensional Hopf algebras on commutative domains

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Interested in Hopf algebra actions on algebras in general
* difficult because computations can be horrible
* depends on the classification of Hopf algebras

In any case, there are three goals we could shoot for:

- Fix field $k$, $H =$ class of Hopf algebras over $k$, $A =$ class of algebras over $k$.

**No Quantum Symmetry**

$H^2 A$ factors through a cocommutative Hopf algebra $[\text{c.o.d} = \Delta]$.

In other words, if $H^2 A$ inner faithfully
$[\text{If nonzero Hopf ideal } I \text{ of } H \text{ with } IA = 0]$
when it is cocommutative.

**Some Quantum Symmetry**

Classify pairs $(H, A)$ so that $H^2 A$ inner faithfully
$[\text{at least one } H \text{ is non-cocommutative}]$

**All Quantum Symmetry**

Fix $H \neq 1$, $A \neq 1$, classify all inner faithful $H^2 A$
Today, \( k = k, \text{ chk} = 0 \)

\[ H = \text{finite dim. Hopf alg.}/k \quad \mathcal{A} = \text{commutative algebra}/k \]

Two tractable subclasses:

- \( H = \text{semisimple Hopf alg.}/k \)
  - (as a \( k \)-alg)

- \( H' = \text{pointed Hopf alg.}/k \)
  - (all simple \( H \)-comodules are 1-dim)
    \[ \Rightarrow H' \text{ is basic} \]

Classification programs are active:

- Two subclasses

\[ \text{Ex. } kG \text{ group algebra} \]
\[ (kG)^* = \text{dual} \]
\[ (kG)^J = \text{twists} \]
\[ \text{Extensions alg. s.s} \]
\[ H_0 \text{ noncon-normal} \]

'GROUP THEORETIC'

'\text{LIE THEORETIC}'

'\text{GROUP THEORETIC}'

Ex. (variants of)

- Small quantum groups

- Finite dim. Hopf extensions of \( U_q(g) \)

  \[ \text{at } g \text{ a root of } 1 \]

Ex. \( U_q(sl_2) \)

In fact, \( k \rightarrow \text{Hopf alg.}/k \) cocommutative \( \Rightarrow H = kG \times U_q(g) \)

\[ G \text{ finite}, G \text{ finite} \]

We show:

\( k = k, \text{ chk} = 0 \)

\( \mathcal{A} = \text{commutative algebra}/k \)

\[ H = \text{semisimple Hopf alg.}/k \quad \Rightarrow \text{finite} \quad \Rightarrow \text{NO QUANTUM SYMMETRY} \]

\[ H = \text{finite dim. pointed Hopf alg.}/k \quad \Rightarrow \text{SOME QUANTUM SYMMETRY} \]
Semisimple that alg. acts on commutative domains \( (k = \mathbb{K}, \chi, k = 0) \)

Thu [Ew/Schneider] \( H \rightarrow A \) inner faithfully \( \Rightarrow H \cong KG \) finite group.

Sketch of Proof: Want to show \( H^* \) is commutative (by [Ew/Schneider])

Recall that a right coroidal subalgebra. Beq \( H^* \) is

- a subalgebra of \( H^* \) so that
- \( \Delta(B) \subseteq B \otimes H^* \)
- \( H^0 \otimes H^* \)

Ex. \( A \)-commutative domain, \( H^2 A \) from the left

\( \rho: A \rightarrow A \otimes H^* \), the coaction of \( H^* \) on \( A \) from the right

\( \chi: A \rightarrow k \), character of \( A \)

\[ p_x(\chi \otimes \text{id}) \circ \rho: A \rightarrow A \otimes H^* \rightarrow H^* \]

\[ p_x(A) := A \chi \] is a right coroidal subalgebra of \( H^* \).

Thu [Ew/Schneider] A semisimple that alg. has finitely many coroidal subalgebras.

To proceed with proof: Reduce to case where \( A \) is finitely generated

- let \( X = \text{Spec}(A) \) (affine irreducible alg. variety \( /k \))

\[ \text{Ch}_{k/k} \leftrightarrow \text{characters } \chi: A \rightarrow k \]

- let \( \chi: A \rightarrow \text{Ch}_{k/k}(H^*) \) the set variety of dimension of coroidal subalgebra of \( H^* \)

\[ X \mapsto A \chi \]
do := \text{max} \{ \text{dim} k A x \} \quad \text{and} \quad X_0 = \{ x \in X \mid \text{dim} k A x = do \} \\

Get that: \( X_0 \) irreducible (\( X_0 + B \), open dense) \Rightarrow \text{\( X_0 \) is constant:} \\
- \text{\( X_0 \) regular} \\
- \text{CS do (H\( ^* \)) is finite (since H\( ^* \) semisimple)} \quad \forall x \in X_0 : \text{dim} k (x) = B \\
\text{for some coideal subalgebra } B \text{ of } H\( ^* \) of dimension do \\

Argue that: \( p: A \rightarrow A \otimes H\( ^* \) \quad \text{restricts to} \quad p': A \rightarrow A \otimes B \) (once &gt; squared by all A\( ^* \) in x) \\
inner-faithfulness \Rightarrow H\( ^* \) = B \text{ the image of } A \text{ (commutative)} \quad \text{ (= A for some x in x)} \\
\Rightarrow H\( ^* \) is commutative, as desired. \\

Finite and pointed Hopf alg acts on commutative domain. \\

+ Study boils down to acts on fields \\
  \text{Lem [Skryabin] A com. domain. On quotient field} \\
  H \text{ finite and } A \text{ inner-faithful } \Rightarrow H^2 \text{ On inner-faithful.} \\

\text{Dfn: A Hopf alg is Galois-theoretical (GT) if it acts } \\
\text{inner-faithfully and k-linearly on a field containing } k. \\

Prop \ 1 \ Any \ finite \ group \ algebra \ is \ GT. \\
2 \ H \text{ semisimple & GT } \Rightarrow H = k G \\
3 \ GT preserved under \text{- Hopf subalg} \\
\text{- Hopf dual} \\
\text{not preserved under } \\
\text{twists } \text{- coideal (twists unit)} \\
\text{- 2-cocycle (twists only) -} \\
\text{- dual (twists &).}
What about the invariant fields?

Thus H finite dim, pointed GT Hopf algebra w/ H-meanfield T.

Then

1. \( L^+ = L^\infty(H) \), \( G(H) = \text{group of grouplike elt} \text{s of } H \)
   \[ \Delta(g) = g g^\prime, \]

2. \( H^+ \in L^+ \) is Galois w/ Galois group \( G(H) \).

Have similar results for \( H \) not nec. pointed \( \Leftrightarrow \) A Azumaya domain.

\[ \text{action} \quad \text{in commutative domain } A \quad \text{on Azumaya algebra } A \]

\[
\begin{align*}
\text{H finite dim,} & \quad A^H = A^{H_0} \quad \mathbb{Z} \cap A^H = \mathbb{Z} \cap A^{H_0} \\
\text{pointed} & \quad A^H = A^G \quad \mathbb{Z}^H = \mathbb{Z}^G
\end{align*}
\]

Example \( H = U_q(sl_2) \) is GT

\[ U_q(sl_2) \text{ generated by } E, F, K \]

\[ \text{growth: } \lambda \]

\[ \text{relations } EF - FE = \frac{q^2 - 1}{q - q^{-1}}, \quad K E = q^2 E K, \quad K F = q^2 F K, \quad K^2 = 1, \quad K^m = 1, \quad E = F = 0 \]

\[ \Delta(K) = K \otimes K, \quad \Delta(E) = K E + E \otimes 1, \quad \Delta(F) = 1 \otimes K F + F \otimes K^{-1} \]

\[ \mathbb{G}(H) = \mathbb{Z}^m \]

\[ H^\times \text{ in } k[u] \text{ by } \quad K \cdot u = q^2 u \quad E \cdot u = 1 \quad F \cdot u = -q u^2 \]

\[ k(u) \text{ is cycl} \text{g} \text{al ext of degree } m. \]
We have examples (classification results) of finite dual pointed Hopf algebras of finite Cartan type.

Alg. Topo. [Andruskiewitsch-Schneider-Angiono]

$H$ finite dual pointed with $G(H)$ abelian

$\Rightarrow H$ generated by grouplike elements $g$ skew-primitive elements $\{x_i, x_j\}$

$\Delta(x_i) = g_i \otimes x_i + x_i \otimes g_i \in G(H)$

Say $H$ is of finite Cartan type if $g_i x_j = g_{ij} x_j g_i$, $g_{ii} = 1$

where $\prod_{ij} g_{ij} g_{ji} = g_{ii} g_{jj}$

$\{a_{ij}\} = $ Cartan matrix of $\mathfrak{g}$ as Lie algebra

Example $H = uq(sl_2)$

$g_1 = g_2 = K$  
$x_1 = E$  
$x_2 = KF$

$\begin{pmatrix}
  q & 0 \\
  0 & q^{-1}
\end{pmatrix}
$  

$\begin{pmatrix}
  2 & 0 \\
  0 & 2
\end{pmatrix}$

$\Rightarrow$ type $A_1 \times A_1$

Thus examples of finite dual pointed GT Hopf algebras of finite Cartan type:

- Taft algebras $uq^{\varpi}(sl_2)$ (type $A_1$)
- $uq(\mathfrak{sl}_2)$ (type $A_1 \times A_1$)
- Some Bernstein twist of $uq(\mathfrak{gl}_n)$, $uq(\mathfrak{su}_n)$, $uq^{\varpi}(g)$ as Lie algebra (type $A_{n-1} \times A_{n-1}$)
- Nichols Hopf algebras $E_n$ (type $A_1 \times n$)

Non-examples include $gr(uq(sl_2))$ (generalized Taft algebra (Taft))

Classification results for type $A_1^{\infty}$, e.g., "rank two" in preparation.

ex. $uq(g)$ is GT $\otimes \mathfrak{g} = \mathfrak{sl}_2$