Quantum Symmetry

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Symmetry

"The universe is built on a plan the profound symmetry of which is somehow present in the inner structure of our intellect."

– Paul Valery
"...symmetry is often a constituent of beauty..."

- Winston Churchill

Fig. 1: Mr. Mischief Maker

Fig. 2: Dr. Boom-Boom

Who is more beautiful?
GROUPS & SYMMETRY

Given an object $X$, a symmetry of $X$ is an invertible property-preserving transformation from $X$ to itself.

The collection of symmetries of an object $X$ forms a group $G$.

\[ G = \mathbb{Z}_2 \]

Fig. 4: Butterfly

\[ G = S_3 \]

Fig. 5: Configuration of 3 cups

\[ G = GL_2(\mathbb{R}) \]

Fig. 6: Real 2-space
Symmetries of affine varieties

An affine variety $X$ in affine $n$-space $\mathbb{A}^n$ over a ground field $\mathbb{k}$ is the vanishing set of a (finite) set of polynomials in $\mathbb{k}[x_1, \ldots, x_n]$.

Symmetries of affine varieties also form a group.

$$G = \mathbb{Z}_2 = \langle x \mapsto -x \rangle$$

![Image of a graph]

Fig. 7: $\forall (x^2 - y^3) \subset \mathbb{A}^2$

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<td>$\longleftrightarrow$</td>
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<td>Action of $\mathbb{Z}_2$ on affine varieties $X$</td>
<td>$\longleftrightarrow$</td>
<td>Action of $\mathbb{Z}_2$ on commutative rings $\text{ComAlg}$</td>
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<td>$\mathbb{Z}_2 \times X \to X$ is a morphism in $\text{Aff}$</td>
<td>$\longleftrightarrow$</td>
<td>$\mathbb{Z}_2 \times \text{ComAlg} \to \text{ComAlg}$ is a morphism in $\text{ComAlg}$</td>
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<tr>
<td>$G$ is a group of an affine variety, hence a linear algebraic group</td>
<td>$\longleftrightarrow$</td>
<td>$G$ is a commutative algebra with multiplication $\times = (2x_1, 2x_2) \mapsto (2x_1, 2x_2)$ and add in $\mathbb{k} = {0, 1}$ represented with structure ${0, 1} \times {0, 1}$</td>
</tr>
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<td>$x \mapsto -x$ is a morphism in $\text{Aff}$</td>
<td>$\longleftrightarrow$</td>
<td>$x \mapsto -x$ is a morphism in $\text{ComAlg}$</td>
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<td>Inclusion $\mathbb{k} \subset \text{ComAlg}$</td>
<td>$\longleftrightarrow$</td>
<td>Inclusion $\mathbb{k} \subset \text{ComAlg}$</td>
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<tr>
<td>$\text{ComAlg}$ is the category of commutative rings</td>
<td></td>
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</table>

Notes:
1. $\mathbb{Z}_2 = \{0, 1\}$
2. $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$
3. $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a group under addition modulo 2
Category of Affine Varieties $\text{Aff}_k$  

contravariant functor $X \mapsto \Theta(X)$...  

Action of a group $G$ on an affine variety $X$...  

$$G \times X \rightarrow X$$  

... is a morphism in $\text{Aff}_k$.

$. \cdot G$ is a group & an affine variety, hence a linear algebraic group  

$. \cdot G = (G, m, e, i)$

$m : G \times G \rightarrow G$, multiplication map  

(morphism in $\text{Aff}_k$)  

e $\in G$, identity element  

(object in $\text{Aff}_k$)  

$i : G \rightarrow G$, inversion map  

(morphism in $\text{Aff}_k$)  

satisfying group axioms:

(a) $m(\sigma, e) = m(e, \sigma) = \sigma$

(b) $m(\sigma, i(\sigma)) = m(i(\sigma), \sigma) = e$

(c) $m(\sigma, m(\tau, \gamma)) = m(m(\sigma, \tau), \gamma)$  

[associativity]  

for all $\sigma, \tau, \gamma \in G$. 

Classical Geometry  <->  Commutative Algebra

**Category of Affine Varieties** $\mathbf{Aff}_k$

- Contravariant functor $X \mapsto \mathcal{O}(X)$

Action of a group $G$ on an affine variety $X$...

$$G \times X \rightarrow X$$

... is a morphism in $\mathbf{Aff}_k$.

$\vdash G$ is a group & an affine variety, hence a linear algebraic group

$$\vdash G = (G, m, e, i)$$

$m : G \times G \rightarrow G$, multiplication map

$e \in G$, identity element

$i : G \rightarrow G$, inversion map

satisfying group axioms:

1. $m(\sigma, e) = m(e, \sigma) = \sigma$
2. $m(\sigma, i(\sigma)) = m(i(\sigma), \sigma) = e$
3. $m(\sigma, m(\tau, \gamma)) = m(m(\sigma, \tau), \gamma)$

for all $\sigma, \tau, \gamma \in G$.

**Category of Commutative Algebras** $\mathbf{ComAlg}_k$

- Coaction of the coordinate algebra $\mathcal{O}(G)$ on $\mathcal{O}(X)$.

$$\mathcal{O}(X) \rightarrow \mathcal{O}(X) \otimes \mathcal{O}(G)$$

... is a morphism in $\mathbf{ComAlg}_k$.

$\vdash \mathcal{O}(G)$ is a commutative algebra

with multiplication $m : \mathcal{O}(G) \otimes \mathcal{O}(G) \rightarrow \mathcal{O}(G)$ and unit $u : k \rightarrow \mathcal{O}(G)$

equipped with structure $\mathcal{O}(G) = (\mathcal{O}(G), \Delta, \varepsilon, S)$

- Comultiplication
  $$\Delta : \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(G),$$

- Counit
  $$\varepsilon : \mathcal{O}(G) \rightarrow k, f \mapsto f(1),$$

- Antipode
  $$S : \mathcal{O}(G) \rightarrow \mathcal{O}(G),$$

morphisms in $\mathbf{ComAlg}_k$ satisfying Hopf algebra axioms:

1. $(\text{id} \otimes \varepsilon) \Delta = (\varepsilon \otimes \text{id}) \Delta = \text{id}$ [counit axiom]
2. $m(S \otimes \text{id}) \Delta = m(\text{id} \otimes S) \Delta = u \varepsilon$ [antipode axiom]
3. $(\text{id} \otimes \Delta) \Delta = (\Delta \otimes \text{id}) \Delta$. [coassociativity]
Category of Commutative Algebras $\text{ComAlg}_k$

Coaction of the coordinate algebra $\mathcal{O}(G)$ on $\mathcal{O}(X)$:

$$\mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G)$$

... is a morphism in $\text{ComAlg}_k$.

$\therefore \mathcal{O}(G)$ is a commutative algebra with multiplication $m : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$ and unit $u : \mathbb{k} \to \mathcal{O}(G)$ equipped with structure $\mathcal{O}(G) = (\mathcal{O}(G), \Delta, \epsilon, S)$

$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$, \hspace{1cm} \text{comultiplication}

$\epsilon : \mathcal{O}(G) \to \mathbb{k}$, \hspace{0.5cm} f $\mapsto$ $f(e)$, \hspace{1cm} \text{counit}

$S : \mathcal{O}(G) \to \mathcal{O}(G)$, \hspace{1cm} \text{antipode}

morphism in $\text{ComAlg}_k$ satisfying Hopf algebra axioms:

(a) $(\text{id} \otimes \epsilon)\Delta = (\epsilon \otimes \text{id})\Delta = \text{id}$ \hspace{1cm} \text{[counit axiom]}
(b) $m(S \otimes \text{id})\Delta = m(\text{id} \otimes S)\Delta = u\epsilon$ \hspace{1cm} \text{[antipode axiom]}
(c) $(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta$. \hspace{1cm} \text{[coassociativity]}
QUANTUM SYMMETRIES
OF QUANTUM AFFINE VARIETIES

Noncommutative Geometry $\iff$ Noncommutative Algebra

Action of a Quantum Group on a Quantum Affine Variety

Coaction of a noncommutative Hopf algebra on a noncommutative algebra

Fig. 8: Quantum Variety
Hopf algebras

A Hopf algebra \( H = (H, m_H, u_H, \Delta, \varepsilon, S) \) over a field \( \mathbb{k} \) is an associative algebra \( (H, m_H, u_H) \), a coassociative coalgebra \( (H, \Delta, \varepsilon) \), with antipode map \( S \), satisfying compatibility conditions.

Take \( \tau : H \otimes H \to H \otimes H \), with \( h \otimes \ell \mapsto \ell \otimes h \).

\( H \) is commutative if \( (H, m_H, u_H) \) is commutative: i.e., \( m_H \circ \tau = m_H \).

\( H \) is cocommutative if \( (H, \Delta, \varepsilon) \) is cocommutative: i.e., \( \tau \circ \Delta = \Delta \).

Classical Examples:

- **group algebra \( \mathbb{k}G \):** we have for \( g \in G \)
  
  \[
  m \checkmark \quad u \checkmark \quad \Delta(g) = g \otimes g \quad \varepsilon(g) = 1_{\mathbb{k}} \quad S(g) = g^{-1}.
  \]

- **universal enveloping algebra of a Lie algebra \( U(g) \):** for \( x \in g \)
  
  \[
  m \checkmark \quad u \checkmark \quad \Delta(x) = 1_H \otimes x + x \otimes 1_H \quad \varepsilon(x) = 0_{\mathbb{k}} \quad S(x) = -x.
  \]

  \( \mathbb{k}G \) and \( U(g) \) are cocommutative.

- \( \mathbb{k}G \) (resp., \( U(g) \)) are commutative \( \iff \) \( G \) (resp., \( g \)) is abelian

- \( \mathcal{O}(G) \) is commutative.
Hopf actions on algebras

We say that a Hopf algebra $H = (H, m_H, u_H, \Delta, \varepsilon, S)$ over $\mathbb{k}$ acts on an algebra $A = (A, m_A, u_A)$ over $\mathbb{k}$ if

- $A$ is an $H$-module algebra:
- $A$ is an $H$-module, and $m_A$ and $u_A$ of $A$ are $H$-morphisms.

We need boxed equations to hold below for any $a, b \in A$ and $h \in H$ with $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler notation):

\[
\begin{align*}
H \otimes A \otimes A & \xrightarrow{\text{id}_H \otimes m_A} H \otimes A \\
A \otimes A & \xrightarrow{m_A} A
\end{align*}
\]

\[
\begin{align*}
A \otimes A & \xrightarrow{\text{id}_H \otimes u_A} H \otimes A \\
A & \xrightarrow{u_A} \mathbb{k}
\end{align*}
\]

\[
\begin{align*}
h \otimes a \otimes b & \xrightarrow{h \text{-action}} h \otimes ab \\
\sum (h_1 \cdot a) \otimes (h_2 \cdot b) & \xrightarrow{h \cdot (ab) = \sum (h_1 \cdot a) \otimes (h_2 \cdot b)}
\end{align*}
\]

\[
\begin{align*}
h \otimes 1_k & \xrightarrow{h \text{-action}} h \otimes 1_A \\
\varepsilon(h)1_k & \xrightarrow{h \cdot 1_A = \varepsilon(h)1_A}
\end{align*}
\]
Hopf coactions on algebras

We say that a Hopf algebra $H = (H, m_H, u_H, \Delta, \varepsilon, S)$ over $\mathbb{k}$ coacts on an algebra $A = (A, m_A, u_A)$ over $\mathbb{k}$ if

$A$ is an $H$-comodule algebra:

$A$ is an $H$-comodule via $\rho$, and $m_A$ and $u_A$ of $A$ are $H$-morphisms.

We need boxed equations to hold below for any $a, b \in A$ and $h \in H$: 

\[
\begin{align*}
A \otimes A & \xrightarrow{m_A} A \\
A \otimes H \otimes A & \xrightarrow{m_{A \otimes H}} A \otimes H \\
\mathbb{k} & \xrightarrow{u_A} A \\
\mathbb{k} \otimes H & \xrightarrow{u_A \otimes \text{id}_H} A \otimes H \\
\rho(a) \otimes \rho(b) & \xrightarrow{= \rho(ab)} \rho(ab) = \rho(a) \rho(b) \\
1_k \otimes 1_H & \xrightarrow{=} 1_A \otimes 1_H
\end{align*}
\]
Classical examples of Hopf (co)actions on algebras

**Action by cocommutative Hopf algebra on commutative algebra**

Take group alg. $\mathbb{k}(SL_2)$ gen. by matrices $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \in SL_2(\mathbb{k})$ with $e_{11}e_{22} - e_{12}e_{21} = 1$

$\mathbb{k}(SL_2)$ acts on $\mathbb{k}[u, v]$ by $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \cdot u = e_{11}u + e_{21}v, \quad \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \cdot v = e_{12}u + e_{22}v$

Take universal enveloping algebra $U(sl_2)$, as an algebra:

$\langle h, x, y \mid hx - xh = 2x, \; hy - yh = -2y, \; xy - yx = h \rangle$

$U(sl_2)$ acts on $\mathbb{k}[u, v, w]$ by $\begin{align*}
h \cdot u &= -2u, \\
h \cdot v &= 0, \\
h \cdot w &= 2w
\end{align*}$

$\begin{align*}
x \cdot u &= v, \\
x \cdot v &= 2w, \\
x \cdot w &= 0
\end{align*}$

$\begin{align*}
y \cdot u &= 0, \\
y \cdot v &= 2u, \\
y \cdot w &= v
\end{align*}$

**Coaction by commutative Hopf algebra on commutative algebra**

Take coordinate alg. of algebraic group $\mathbb{G}(SL_2) = \mathbb{k}[e_{ij}]_{i,j=1}^2/(e_{11}e_{22} - e_{12}e_{21} = 1)$, with $\Delta(e_{ij}) = \sum_{\ell=1}^2 e_{i\ell} \otimes e_{\ell j}, \quad \epsilon(e_{ij}) = \delta_{ij}, \quad S(e_{ij}) = (-1)^{i-j}e_{i+1,j+1} \text{ (indices mod 2)}$.

$\mathbb{G}(SL_2)$ coacts on $\mathbb{k}[u, v]$ by $u \mapsto u \otimes e_{11} + v \otimes e_{21}, \quad v \mapsto u \otimes e_{12} + v \otimes e_{22}$
Prototypical Examples of Quantum Symmetry: Actions / Coactions on the Quantum Plane

replace plane with coordinate ring:

\[ \mathbb{K}[x, y] = \frac{\mathbb{K}[x, y]}{(xy - yx)} \]

polynomial algebra

\[ \mathbb{K}_q[x, y] = \frac{\mathbb{K}[x, y]}{(xy - qyx)} \]

q-polynomial algebra

q-deform classical symmetries to get quantum symmetries....

Fig. 9: Affine plane

Fig. 10: Quantum plane
**Classical Symmetry:**

Coaction by **commutative Hopf algebra** on **commutative algebra**

Take coordinate algebra of algebraic group

\[
\mathcal{O}(SL_2) = \mathbb{k}[e_{ij}]_{i,j=1}^2 / (e_{11}e_{22} - e_{12}e_{21} = 1),
\]

with \( \Delta(e_{ij}) = \sum_{\ell=1}^2 e_{i\ell} \otimes e_{\ell j} \), \( \epsilon(e_{ij}) = \delta_{ij} \), \( S(e_{ij}) = (-1)^{j-i} e_{i+1,j+1} \) (indices mod 2).

\( \mathcal{O}(SL_2) \) coacts on \( \mathbb{k}[u,v] \) by \( u \mapsto u \otimes e_{11} + v \otimes e_{21}, \; \; v \mapsto u \otimes e_{12} + v \otimes e_{22} \)

---

**Quantum Symmetry:**

Coaction by **noncom. Hopf algebra** on **noncommutative algebra**

Take coordinate algebra of quantized algebraic group, for \( q \in \mathbb{k}^\times \),

\[
\mathcal{O}_q(SL_2) = \frac{\mathbb{k}\langle e_{11}, e_{12}, e_{21}, e_{22} \rangle}{\begin{pmatrix}
    e_{11}e_{12} = qe_{12}e_{11}, & e_{11}e_{21} = qe_{21}e_{11}, \\
    e_{12}e_{22} = qe_{22}e_{12}, & e_{21}e_{22} = qe_{22}e_{21}, \\
    e_{12}e_{21} = e_{21}e_{12}, & e_{11}e_{22} = e_{22}e_{11} + (q - q^{-1})e_{12}e_{21} \\
    e_{11}e_{22} = qe_{12}e_{21} = 1
\end{pmatrix}},
\]

with \( \Delta(e_{ij}) = \sum_{\ell=1}^2 e_{i\ell} \otimes e_{\ell j} \), \( \epsilon(e_{ij}) = \delta_{ij} \), \( S(e_{ij}) = (-q)^{j-i} e_{i+1,j+1} \) (indices mod 2).

\( \mathcal{O}_q(SL_2) \) coacts on \( \mathbb{k}_q[u,v] \) by \( u \mapsto u \otimes e_{11} + v \otimes e_{21}, \; \; v \mapsto u \otimes e_{12} + v \otimes e_{22} \)
ANOTHER EXAMPLE OF QUANTUM SYMMETRY

A = path algebra of a quiver (directed graph)

A vector space basis of $A$ = paths of quiver

Multiplication of $A$ = concatenation of paths, $0$ elsewhere

Example:

\[ \begin{array}{ccc}
| & a & |
\end{array} \]
\[ \begin{array}{ccc}
| & b & |
\end{array} \]

Eq. $a \cdot a = a$ in $A$, $ab \in A$, $a^2 = 0$ in $A$

\[ Z_2 = \langle g : g^2 = 1 \rangle \text{ acts on } A: \]

$g \cdot e_1 = e_b$, $g \cdot e_2 = e_a$, $g \cdot a = b$, $g \cdot b = a$

So $A$ admits classical symmetry

Example continued: $A$ also admits quantum symmetry

\[ \begin{array}{ccc}
| & a & |
\end{array} \]
\[ \begin{array}{ccc}
| & b & |
\end{array} \]
\[ \begin{array}{ccc}
| & c & |
\end{array} \]

$Z_2 = \langle g : g^2 = 1 \rangle$ acts on $A$:

$g \cdot e_1 = e_b$, $g \cdot e_2 = e_a$, $g \cdot a = b$, $g \cdot b = a$

Extend to action of the Sweedler Hopf alg. (semi, noncom, noncocom)

\[ H = \langle g, x : g^2 = 1, x^2 = 0, gx + xg = 0 \rangle \]

with $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$, $\epsilon(g) = 1$, $\epsilon(x) = 0$, $S(g) = g$, $S(x) = -xg$

$x \cdot e_1 = -g(e_1 + e_b)$, $x \cdot e_2 = g(e_1 + e_b)$, $x \cdot a = g(a - b) + \lambda e_1$, $x \cdot b = g(a - b) - \lambda e_1$

for $\gamma, \lambda \in k$
A = path algebra of a quiver (directed graph)

$k$-vector space basis of $A = \text{paths of quiver}$

Multiplication of $A = \text{concatenation of paths, 0 elsewhere}$

**Example:**

![Diagram of a quiver with vertices $e_1$ and $e_2$, and arrows $a$ and $b$.]

Eg. $e_1a = a$ in $A$, $ab \in A$, $a^2 = 0$ in $A$

$\mathbb{Z}_2 = \left\langle g : g^2 = 1 \right\rangle$ acts on $A$:

$g \cdot e_1 = e_2, \quad g \cdot e_2 = e_1, \quad g \cdot a = b, \quad g \cdot b = a$

So $A$ admits classical symmetry
Example continued: $A$ also admits quantum symmetry

\[ a \quad e_1 \bullet \quad e_2 \quad b \]

$\mathbb{Z}_2 = \langle g : g^2 = 1 \rangle$ acts on $A$:

\[ g \cdot e_1 = e_2, \quad g \cdot e_2 = e_1, \quad g \cdot a = b, \quad g \cdot b = a \]

Extend to action of the Sweedler Hopf alg. (4-diml, noncom, noncocom)

\[ H = \langle g, x : g^2 = 1, \ x^2 = 0, \ gx + xg = 0 \rangle \]

with $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$, $S(g) = g$

$\Delta(x) = 1 \otimes x + x \otimes g$, $\epsilon(x) = 0$, $S(x) = -xg$

\[ x \cdot e_1 = -\gamma(e_1 + e_2), \quad x \cdot e_2 = \gamma(e_1 + e_2) \]

\[ x \cdot a = \gamma(a - b) + \lambda e_1, \quad x \cdot b = \gamma(a - b) - \lambda e_2 \]

for $\gamma, \lambda \in \mathbb{k}$
**MAIN QUESTION**

When does there exist "genuine" quantum symmetry?

When are there actions (resp., coactions) of Hopf algebras that do not factor through actions (resp., coactions) of classical Hopf algebras?

Classical Hopf algebras = those that are com. or cocom. e.g. group algebras, universal enveloping algebras, coordinate algebras of algebraic groups
USEFUL HYPOTHESES TO IMPOSE TO ANSWER MAIN QUESTION
HYPOTHESES ON HOPF ALGEBRAS $H$

Could impose that $H$ is:

- **finite-dimensional** as a vector space, or
- **semisimple** as an algebra (which implies finite-dimensionality), or
- **cosemisimple** as a coalgebra
  
  [each $H$-comodule = direct sum of simple $H$-subcomods], or
- **involutory** [the square of the antipode $S$ of $H$ is the identity].

If $H$ is finite-dim'l and characteristic of ground field is $0$, then

**semisimple = cosemisimple = involutory.**

**pointed** [every simple $H$-comodule is 1-dimensional]

There is a very active program to classify finite-dimensional Hopf algebras
in the semisimple (resp. pointed) settings.
Group theoretic (resp. Lie theoretic) techniques are employed.
HYPOTHESES ON ALGEBRAS $A$

Could impose that $A$ is:

- **homologically nice**
  - finite global or injective dimension
  - a Koszulity condition (Koszul, $N$-Koszul, K2 condition)
  - a Calabi-Yau condition
- **ring-theoretically nice**
  - commutative
  - domain
  - Noetherian or coherent
  - if graded, polynomial growth of graded pieces (finite GK dim)
  - nice vector basis of monomials (PBW property)

There is a very active program to study homological analogues of commutative polynomial rings: "Artin-Schelter regular algebras", and more generally, "skew Calabi-Yau algebras".
HYPOTHESES ON ACTION OF HOPF ALGEBRA $H$ ON ALGEBRA $A$

If $A$ is graded (resp., filtered), then one could ask that the $H$-action on $A$ preserves this grading (resp., filtration).

Could also use the homological (co)determinant of $H$-(co)action on $A$.

Related to the quantum determinant in the literature.

Eg., to get an analogue of a result involving group actions with $G<\text{SL}(V)$, impose trivial homological determinant.

Avoiding technicalities here, $\text{hdet}(H,A)$ is an $H$-morphism from $H$ to the ground field; it is trivial if equal to counit map of $H$. $\text{hcodet}(H,A)$ arises as a "group-like element" in $H$; it is trivial if equal to the unit element of $H$. 

Main Results #1

No Quantum Symmetry

There are many No Quantum Symmetry results in the literature, but many are outside the scope of this field.

There is a No Quantum Symmetry result for the action of a group G on the set X that is not symmetric. This No Quantum Symmetry result has been observed.

No Quantum Symmetry results for the action of a group G on the set X

<table>
<thead>
<tr>
<th>Condition on G</th>
<th>on X</th>
<th>on A</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>G is abelian</td>
<td>yes</td>
<td>yes</td>
<td>symmetric</td>
</tr>
<tr>
<td>G is non-abelian</td>
<td>yes</td>
<td>no</td>
<td>asymmetric</td>
</tr>
</tbody>
</table>

Please see references.
Recall:

Actions of groups $G$ (or $kG$) and Lie algebras $g$ (or $U(g)$) are considered classical, and that $kG$ and $U(g)$ are cocommutative:

$\Delta = \tau \circ \Delta$, where $\tau(h \otimes \ell) = \ell \otimes h$, for $h, \ell \in H$.

**Theorem** (Cartier-Kostant-Milnor-Moore).

If $H$ is a cocommutative Hopf algebra over an alg. closed field of characteristic 0, then $H \cong U(g) \# kG$, for some $G \acts g$.

Further, if $H$ is finite-dimensional, then $H \cong kG$, for some group $G$.

---

Given an Hopf $H$-action on an algebra $A$, we say there is **No Quantum Symmetry** when this action must factor through the action of a cocommutative Hopf algebra.
No Quantum Symmetry results for

$H$-actions on **commutative domains**:

Below, Hopf actions must factor through the action of a cocom. Hopf algebra.

<table>
<thead>
<tr>
<th>Conditions on $k$</th>
<th>on $H$</th>
<th>on $A$</th>
<th>on action</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>char 0 alg. closed</td>
<td>semisimple ($\Rightarrow$ fin-dim &amp; coss)</td>
<td>commutative domain</td>
<td>(none)</td>
<td>[Etingof-W, 2014]</td>
</tr>
<tr>
<td>char $&gt; 0$ alg. closed</td>
<td>semisimple &amp; cosemisimple</td>
<td>commutative domain</td>
<td>(none)</td>
<td>[Etingof-W, 2014]</td>
</tr>
<tr>
<td>char $&gt; 0$ alg. closed</td>
<td>finite-dim'l &amp; cosemisimple</td>
<td>commutative domain</td>
<td>(none)</td>
<td>[Skryabin, 2016]</td>
</tr>
</tbody>
</table>
No Quantum Symmetry results for

\( H \)-actions on quantizations of com. domains and other algebras:

Below, Hopf actions must factor through the action of a cocom. Hopf algebra.

Here, \( \mathbb{k} \) is algebraically closed of characteristic 0.

<table>
<thead>
<tr>
<th>Conditions on ( H )</th>
<th>on ( A )</th>
<th>on action</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite-dim'</td>
<td>Weyl algebra ( A_n(\mathbb{k}) )</td>
<td>(none)</td>
<td>[Cuadra-Etingof-W, to appear]</td>
</tr>
<tr>
<td>semisimple</td>
<td>( U(\mathfrak{g}) ) for ( \mathfrak{g} ) fin dim', ( D(X) ) diff. op. on smooth aff var., generic Sklyanin algebras, twisted homog coord rings</td>
<td>(none)</td>
<td>[Etingof-W, submitted]</td>
</tr>
<tr>
<td>semisimple &amp; cosemisimple</td>
<td>division algebra ( D )</td>
<td>( \dim H ) &amp; (( \deg D ))! are coprime</td>
<td>[Cuadra-Etingof-W, to appear]</td>
</tr>
<tr>
<td>finite-dim'</td>
<td>( \frac{\mathbb{k}(x_1, \ldots, x_n)}{(x_i x_j - q_{ij} x_j x_i)} ) ( q_{ij} \in \mathbb{k}^x ) generic</td>
<td>pres. grading</td>
<td>[Chan-W-Zhang]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(none)</td>
<td>[Etingof-W, submitted]</td>
</tr>
</tbody>
</table>
There are lots of **No Quantum Symmetry results**
in the analytic setting
(outside of the scope of this talk).

There, \( A \) is the function algebra of a geometric object
(e.g. sphere, torus, certain manifolds).

So, \( A \) is a commutative domain.

Please see references.
Given an Hopf $H$-action on an algebra $A$, we say there is **Genuine Quantum Symmetry**
when this action does *not* factor through the action of a cocommutative Hopf algebra.

(Time permitting)
We discuss three occurrences of Genuine Quantum Symmetry...
Genuine Quantum Symmetry: on path algebras $kQ$

Hopf actions below do not factor through actions of smaller Hopf alg. quotients

<table>
<thead>
<tr>
<th>Conditions on $k$</th>
<th>on $H$</th>
<th>on $Q$</th>
<th>on action</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>contains a primitive $n$-th of root unity $\zeta$ for $n \geq 2$</td>
<td>pointed: $T_\zeta(n)$, Taft algebras $u_q(sl_2)$, small quan. group $D(T_\zeta(n))$, double of $T_\zeta(n)$</td>
<td>finite loopless &amp; no parallel arrows</td>
<td>preserves ascending path length filtration</td>
<td>[Kinser-W]</td>
</tr>
</tbody>
</table>

Example: We classify Sweedler Hopf $T(2)$-actions on the path algebra of $Q$ to the right.

The action of $\mathbb{Z}_2$ is given by $\bullet \longrightarrow \longrightarrow \longrightarrow \longrightarrow \bullet$
Genuine Quantum Symmetry: on commutative domains (fields)

Take $\mathbb{k}$ an algebraically closed field of characteristic 0.

Let $H$ be a Hopf algebra that acts on a field so that the action does not factor through a smaller Hopf algebra quotient; say such an $H$ is Galois-theoretical.

Below are noncocom., noncom., finite-dim'l, non-ss, pointed Galois-th'1 Hopf alg.

<table>
<thead>
<tr>
<th>$H$</th>
<th>“Cartan type”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taft algebras $T_\zeta(n)$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>Nichols Hopf algebras $E(n)$</td>
<td>$A_1^{\times n}$</td>
</tr>
<tr>
<td>the book algebra $h(\zeta, 1)$</td>
<td>$A_1 \times A_1$</td>
</tr>
<tr>
<td>the Hopf algebra $H_{81}$ of dimension 81</td>
<td>$A_2$</td>
</tr>
<tr>
<td>$u_q(sl_2)$</td>
<td>$A_1 \times A_1$</td>
</tr>
<tr>
<td>$u_q(gl_2)$</td>
<td>$A_1 \times A_1$</td>
</tr>
<tr>
<td>Twists $u_q(gl_n)^{J^+}$, $u_q(gl_n)^{J^-}$ for $n \geq 2$</td>
<td>$A_{n-1} \times A_{n-1}$</td>
</tr>
<tr>
<td>Twists $u_q(sl_n)^{J^+}$, $u_q(sl_n)^{J^-}$ for $n \geq 2$</td>
<td>$A_{n-1} \times A_{n-1}$</td>
</tr>
<tr>
<td>Twists $u_q^{&gt;0}(g)^J$ for $2^{\text{rank}(g) - 1}$ of such $J$</td>
<td>same type as $g$</td>
</tr>
</tbody>
</table>

$g$ is a finite-dimensional simple Lie algebra

Reference: [Etingof-W(2)]
Genuine Quantum Symmetry: on *commutative domains (fields)*

The *Galois-theoretical* property is preserved under taking:

- Hopf subalgebra
- \( \otimes \)

... so this allows one to cook up more quantum symmetries

The *Galois-theoretical* property is *not* preserved under taking:

- Hopf dual
- 2-cocycle deformation (twisting the multiplication)
- dual 2-cocycle deformation (twisting the comultiplication)

Reference: [Etingof-W(2)]
Galois-theoretical property & Galois extensions

Take $\mathbb{k}$ an algebraically closed field of characteristic 0. Say $H$ is finite-dimensional, Galois-theoretical with $H$-module field $L$.

If, further, $H$ is semisimple, then
$$ H \cong \mathbb{k}G \text{ and the extension } L^H = L^G \hookrightarrow L \text{ is Galois.} $$

On the other hand, if, further, $H$ is pointed, then
$$ L^H = L^{G(H)} \text{ and the extension } L^H \hookrightarrow L \text{ is Galois.} $$

Here, $G(H)$ is the group of group-like elements of $H$.
$$ G(H) = \{ h \in H \mid \Delta(h) = h \otimes h \} $$

Reference: [Etingof-W(2)]
Genuine Quantum Symmetry: on noncommutative domains

The Hopf actions (that do not factor through smaller Hopf actions) in the setting below are classified:

\( \mathbb{k} \) is an algebraically closed field of char. 0

\( H \) is a finite-dimensional Hopf algebra

\( A \) is an Artin-Schelter regular algebra of global dimension 2 (a homological analogue of \( \mathbb{k}[u, v] \))

\( H \)-action preserves the grading of \( A \), subject to trivial hom’l det.

....which is a generalization of the classical setting where \( G \leq SL_2(\mathbb{k}) \) acts on \( \mathbb{k}[u, v] \) linearly and faithfully

Reference: [Chan-Kirkman-W-Zhang]
More Results

Noncommutative Invariant Theory
given an $H$-action on $A$
study the invariant ring $A^H$ ...

Deformation Theory
given an $H$-action on $A$
study the smash product algebra $A#H$
and its deformations...
FURTHER QUESTIONS AND DIRECTIONS....

Computation.
Computations are a pain. Write a program to do this.

Classification results.
Pick a class of algebras. Pick a class of Hopf algebras. Perhaps impose some conditions on Hopf action. Is there quantum symmetry?

Fancy classification results.
Use the machinery of tensor categories/fusion categories to understand Hopf actions.

Make connections to other fields.
This has been done in functional analysis & geometry. Topology?

Physical Applications.
This will be useful to physicists. Investigate new applications. Then tell me about this.
Thanks for listening!
References: Background on Hopf algebras and Hopf (co)actions

[Andruskiewitsch]

[Drinfeld]

[Kassel]

[Majid]

[Montgomery]

[Radford]
References: No Quantum Symmetry (incomplete list)

[Chan-W-Zhang]

[Cuadra-Etingof-W]

[Cuadra-Etingof-W(2)]

[Etingof-W, 2014]

[Etingof-W, submitted]
Pavel Etingof and Chelsea Walton, Finite dimensional Hopf actions on algebraic quantizations, submitted.

[Skryabin]
Serge Skryabin, Finiteness of the number of coideal subalgebras, to appear in Proc AMS.
References: Genuine Quantum Symmetry (incomplete list)

[Chan-Kirkman-W-Zhang]

[Etingof-W(2)]

[Etingof-W(3)]

[Kinser-W]
References: Quantum Symmetry - analytic (incomplete list)

[Banica]

[Bichon]

[Bhowmick]
Jyotishman Bhowmick, Quantum Isometry Group of the n-tori, Proc. AMS, 2009

[Bhowmick-Goswami]

[Chirvasitu]

[Goswami]

[Wang]

[Woronowicz]
References: Noncommutative Invariant Theory and Deformation Theory arising from Hopf actions

see also references within

[Kirkman]

[Kirkman-Kuzmanovich-Zhang]

[Kirkman-Kuzmanovich-Zhang(2)]
Ellen Kirkman, James Kuzmanovich, and James Zhang, Gorenstein subrings of invariants under Hopf algebra actions, J. Algebra, 2009.

[Chan-Kirkman-W-Zhang(2, 3)]

[W-Witherspoon]

[W-Witherspoon(2)]
Quantum Symmetry

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AMS Sectional Meeting, U. Denver, October 2016
Thanks for listening!