Collisions at infinity in hyperbolic manifolds

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Abstract

For a complete, finite volume real hyperbolic $n$-manifold $M$, we investigate the map between homology of the cusps of $M$ and the homology of $M$. Our main result provides a proof of a result required in a recent paper of Frigerio, Lafont, and Sisto.

1. Introduction

Let $M$ be a cusped finite volume hyperbolic $n$-orbifold. Recall that the thick part of $M$ is the quotient $M_0 = X_M/\pi_1(M)$, where $X_M$ is the complement in $H^n$ of a maximal $\pi_1(M)$-invariant collection of horoballs (see for instance [11]). It is known that $M$ and $M_0$ are homotopy equivalent and $M_0$ is a compact orbifold with boundary components $E_1, \ldots, E_r$. Each $E_j$ is called a cusp cross-section of $M$. Since horoballs in $H^n$ inherit a natural Euclidean metric, each cusp cross-section is naturally a flat $(n-1)$-orbifold. Changing the choice of horoballs preserves the flat structure up to similarity.

The aim of the present note is to provide a proof of a result required in Frigerio, Lafont, and Sisto [3] for their construction in every $n \geq 4$ of infinitely many $n$-dimensional graph manifolds that do not support a locally CAT(0) metric. Specifically, the following is our principal result.

Theorem 1.1. For every $n \geq 3$ and $n > k \geq 2$, there exist infinitely many commensurability classes of orientable non-compact finite volume hyperbolic $n$-manifolds $M$ containing a properly embedded totally geodesic hyperbolic $k$-submanifold $N$ with the following properties. Let $\mathcal{E} = \{E_1, \ldots, E_r\}$ be the cusp cross-sections of $M$ and $\mathcal{F} = \{F_1, \ldots, F_s\}$ the cusp cross-sections of $N$. Then:

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2. The proof of Theorem 1.2

The proof of Theorem 1.2 is an easy consequence of the virtual retract property of [7] (see also [2]) which has found significant applications in low-dimensional topology and geometric group theory of late (see [7], [2] and the references therein).

Definition. Let $G$ be a group and $H < G$ be a subgroup. Then $G$ virtually retracts onto $H$ if there exists a finite index subgroup $G' < G$ with $H < G'$ and a homomorphism
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\(\rho: G' \to H\) such that \(\rho|_H = \text{id}_H\). In addition we say that \(G'\) retracts onto \(H\), and \(\rho\) is called the retraction homomorphism.

With this definition we note the following lemma.

**Lemma 2.1.** Let \(G\) be a group and \(H < G\) a subgroup such that \(G\) retracts onto \(H\). Then two subsets \(S_1, S_2\) of \(H\) are conjugate in \(G\) if and only if they are conjugate in \(H\).

**Proof.** One direction is trivial. Suppose that there exists \(g \in G\) such that \(S_1 = gS_2g^{-1}\).

Then \(S_1 = \rho(S_1) = \rho(gS_2g^{-1}) = \rho(g)S_2\rho(g)^{-1}\), so \(S_1\) and \(S_2\) are conjugate in \(H\). \(\square\)

**Proof of Theorem 1.2** Let \(M = \mathbb{H}^n/\Gamma\) be a cusped finite volume hyperbolic \(n\)-manifold, \(N = \mathbb{H}^k/\Lambda\) be a noncompact finite volume totally geodesic hyperbolic \(k\)-manifold immersed in \(M\) such that \(\Gamma\) virtually retracts onto \(\Lambda\). Let \(F_1, \ldots, F_r\) be the cusp cross-sections of \(N\) and \(\Delta_1, \ldots, \Delta_r < \Lambda\) representatives for the associated \(\Lambda\)-conjugacy classes of peripheral subgroups, i.e., \(\Delta_j = \pi_1(F_j)\).

Two ends \(F_{j_1}\) and \(F_{j_2}\) of \(N\) collide at infinity in \(M\) if and only if any two representatives \(\Delta_{j_1}\) and \(\Delta_{j_2}\) for the associated \(\Lambda\)-conjugacy classes of peripheral subgroups are conjugate in \(\Gamma\) but not in \(\Lambda\). Let \(\Gamma_N\) denote the finite index subgroup of \(\Gamma\) that retracts onto \(\Lambda\), and \(\rho: \Gamma_N \to \Lambda\) the retracting homomorphism. By Lemma 2.1, \(\Delta_{j_1}\) and \(\Delta_{j_2}\) are not conjugate in \(\Gamma_N\) for any \(j_1 \neq j_2\). Thus \(N\) has no collisions at infinity inside \(M' = \mathbb{H}^n/\Gamma_N\).

Moreover, since \(\Lambda\) is a retract of \(\Gamma_N\), it follows that \(\Lambda\) is separable in \(\Gamma_N\) (see Lemma 9.2 of [4]). Now a well-known result of Scott [12] shows that we can pass to a further covering \(M''\) of \(M'\) such that the immersion of \(N\) into \(M'\) lifts to an embedding in \(M''\). This proves the theorem. \(\square\)

**Remark.**

(i) Examples where the virtual retract property holds are abundant. From [2], if \(M = \mathbb{H}^n/\Gamma\) is any non-compact finite volume hyperbolic \(n\)-manifold, which is arithmetic or arises from the construction of Gromov–Piatetskii-Shapiro, then \(\Gamma\) has the required virtual retract property. Briefly, the arithmetic case follows from Theorem 1.4 of [2] and the discussion at the very end of §9 of [2], and for the examples from the Gromov–Piatetskii-Shapiro construction it follows from Theorem 9.1 of [2] and the same discussion at the very end of §9.

(ii) We have in fact shown something stronger, namely that two essential loops in a cusp cross-section \(F_j\) of \(N\) are homotopic inside \(M'\) if and only if they are freely homotopic in \(N\). Therefore, the kernel of the induced map from \(H_*(F; \mathbb{Q})\) to \(H_*(M'; \mathbb{Q})\) is precisely equal to the kernel of the homomorphism from \(H_*(N; \mathbb{Q})\) to \(H_*(N; \mathbb{Q})\).

(iii) Lemma 2.1 also implies that \(N\) cannot have positive-dimensional essential self-intersections inside \(M'\). In particular, if \(n < 2k\), then \(N\) automatically embeds in \(M'\).

3. Covers with torus ends

The following will complete the proof of Theorem 1.1.
Let more than \( k \) hyperbolic known that the above reduction quotients \( \Gamma / Q \) covering of \( M \) subgroup of \( Q \) is a peripheral subgroup of \( \Gamma \). For each \( \Delta_j \) the Bieberbach Theorem [11, §7.4] gives a short exact sequence

\[
1 \to \mathbb{Z}^{n-1} \to \Delta_j \to \Theta_j \to 1
\]

where \( \Theta_j \) is finite. Then \( E_j \) is a flat \((n-1)\)-torus if and only if \( \Theta_j \) is the trivial group. Note in the case when \( M \) is a surface, the statement is trivial and thus we will assume \( n > 2 \).

Let \( \gamma_{j,1}, \ldots, \gamma_{j,r_j} \) be lifts of the distinct nontrivial elements of \( \Theta_j \) to \( \Delta_j \). Since \( n \geq 3 \), it is a well-known consequence of Weil Local Rigidity (see [10, Thm. 7.67]) that we can conjugate \( \Gamma \) in \( \text{PO}_0(n,1) \) inside \( \text{GL}_N(\mathbb{C}) \) so that it has entries in some number field \( k \).

Since \( \Gamma \) is finitely generated, we can further assume that it has entries in some finitely generated subring \( R \subset k \). Then \( R/p \) is finite for every prime ideal \( p \subset R \).

This determines a homomorphism from \( \Gamma \) to \( \text{GL}_N(R/p) \). For every \( \gamma_{j,k} \), the image of \( \gamma_{j,k} \) in the finite group \( \text{GL}_N(R/p) \) is nontrivial for almost every prime ideal \( p \) of \( R \). Indeed, any off-diagonal element is congruent to zero modulo \( p \) for only finitely many \( p \) and there are only finitely many \( p \) so that a diagonal element is congruent to 1 modulo \( p \). Since there are finitely many \( \gamma_{j,k} \), this determines a finite list of prime ideals \( P = \{p_1, \ldots, p_s\} \) such that \( \gamma_{j,k} \) has nontrivial image in \( \text{GL}_N(R/p) \) for any \( p \notin P \) and every \( j,k \). If \( \Gamma(p) \) is the kernel of this homomorphism, then \( \Gamma(p) \) contains no conjugate of any of the \( \gamma_{j,k} \).

The peripheral subgroups of \( \Gamma(p) \) are all of the form \( \Gamma(p) \cap \gamma_{j,k}^{-1} \gamma_1 \gamma^{-1} \) for some \( \gamma \in \Gamma \).

Since no conjugate of any \( \gamma_{j,k} \) is contained in \( \Gamma(p) \), we see that \( \Gamma(p) \cap \gamma_{j,k}^{-1} \gamma_1 \gamma^{-1} \) is contained in the kernel of the above homomorphism \( \gamma_{j,k}^{-1} \gamma_1 \gamma^{-1} \to \Theta_j \). It follows that every cusp cross-section of \( \mathbb{H}^n/\Gamma(p) \) is a flat torus. This proves the second part of the theorem.

To complete the proof of Theorem 3-1, it suffices to show that if \( M \) is a noncompact hyperbolic \( n \)-manifold with \( k \) ends, then \( M \) has a finite sheeted covering \( M' \) with strictly more than \( k \) ends. We recall the following elementary fact from covering space theory. Let \( \rho: \Gamma \to Q \) be a homomorphism of \( \Gamma \) onto a finite group \( Q \) and \( \Gamma_p \) be the kernel of \( \rho \). If \( \Delta_j \) is a peripheral subgroup of \( \Gamma \), then the number of ends of \( \mathbb{H}^n/\Gamma_p \) covering the associated end of \( \mathbb{H}^n/\Gamma \) equals the index \( [Q : \rho(\Delta_j)] \) of \( \rho(\Delta_j) \) in \( Q \). Therefore, it suffices to find a finite quotient \( Q \) of \( \Gamma \) and a peripheral subgroup \( \Delta_j \) of \( \Gamma \) that \( \rho(\Delta_j) \) is a proper subgroup of \( Q \).

In our setting, the proof is elementary. From above, we can pass to a finite sheeted covering of \( M \), for which all the cusp cross-sections are tori, i.e., all peripheral subgroups are abelian. It follows that for \( \rho|_{\Delta_j} \) to be onto, \( \rho(\Gamma) \) must be abelian. However, it is well-known that the above reduction quotients \( \Gamma/\Gamma(p) \) are central extensions of non-abelian finite simple groups for all but finitely many prime ideals \( p \) [8, Chapter 6]. The theorem follows.

Remark. Constructing examples with a small number of ends is much more difficult. For example, there are no known one-cusped hyperbolic \( n \)-orbifolds for \( n > 11 \). Furthermore, it is shown in [13] that for every \( d \), there is a constant \( c_d \) such that \( d \)-cusped arithmetic hyperbolic \( n \)-orbifolds do not exist for \( n > c_d \). For example, in the case \( d = 1 \), there are no 1-cusped arithmetic hyperbolic \( n \)-orbifolds for any \( n \geq 30 \). Very recently,
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Kolpakov and Martelli [5] announced the first construction of a one-cusped hyperbolic 4-manifold.

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