



*Ring theoretic methods in the
representation theory of
Hopf algebras*

Ben-Gurion University 05/26/2010

Martin Lorenz
Temple University, Philadelphia



A MORITA CONTEXT RELATED TO FINITE AUTOMORPHISM GROUPS OF RINGS

MIRIAM COHEN

Let R be a semiprime ring, G a finite group of automorphisms of R and R^G the fixed ring. We investigate the associated Morita context (R^G, R, R, R^*G) , where R^*G is the skew group ring. We then apply these results to two situations: (1) G is X -outer (2) R is $|G|$ -torsion free.

0. Introduction and preliminaries. Let R be a ring, G a finite group of automorphisms of R and $R^G = \{x \in R \mid x^g = x \text{ for all } g \in G\}$. There has been considerable interest in the past years in studying connections between R^G and R . The two major ways to approach the subject were the direct approach, and via the skew group ring R^*G which we denote by S . In this paper we investigate a third way which was used in the commutative case by Chase, Harrison and Rosenberg [5], and was suggested to us by S.A. Amitsur.



(I) Ring theory:

- Background on **Frobenius algebras**



(I) Ring theory:

- Background on **Frobenius algebras**
- Separability locus



(I) Ring theory:

- Background on **Frobenius algebras**
- Separability locus
- Characters



(I) Ring theory:

- Background on **Frobenius algebras**
- Separability locus
- Characters
- Integrals



(II) Hopf algebras:

- Frobenius Hopf algebras



(II) Hopf algebras:

- Frobenius Hopf algebras
- Grothendieck rings



(II) Hopf algebras:

- Frobenius Hopf algebras
- Grothendieck rings
- A theorem of Shenglin Zhu



(III) More ring theory:

- Projectives over Frobenius algebras



(III) More ring theory:

- Projectives over Frobenius algebras
- The Cartan-Brauer triangle



(III) More ring theory:

- Projectives over Frobenius algebras
- The Cartan-Brauer triangle

(IV) More Hopf algebras:

- Bialgebra cohomology



(III) More ring theory:

- Projectives over Frobenius algebras
- The Cartan-Brauer triangle

(IV) More Hopf algebras:

- Bialgebra cohomology
- Some problems (if time)



Reference

- “*Some applications of Frobenius algebras to Hopf algebras*”
preprint covering parts (I) and (II) & ...

Article & **pdf file of this talk** available on my web page:

<http://math.temple.edu/~lorenz/>



Frobenius algebras



Notation

For the remainder of this talk,

R denotes a commutative ring

A is an associative R -algebra that is
fin. gen. projective (“finite”) over R



Definitions

Put $A^* = \text{Hom}_R(A, R)$; this is an (A, A) -bimodule via the (A, A) -bimodule structure on A :

$$(afb)(x) = f(bxa) \quad (a, b, x \in A, f \in A^*).$$

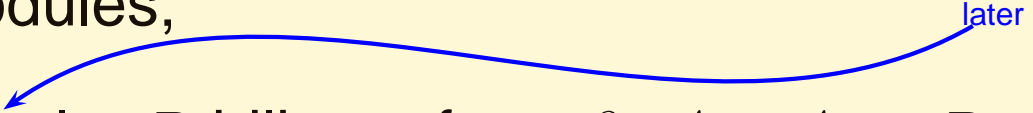


Definitions

Put $A^* = \text{Hom}_R(A, R)$; this is an (A, A) -bimodule via the (A, A) -bimodule structure on A :

$$(afb)(x) = f(bxa) \quad (a, b, x \in A, f \in A^*).$$

The algebra A is called **Frobenius** if it has the following equivalent properties:

- $A \cong A^*$ as right A -modules;
 - there exists a nonsingular R -bilinear form $\beta: A \times A \rightarrow R$ that is associative: $\beta(ab, c) = \beta(a, bc)$;
 - $A \cong A^*$ as left A -modules.
- 



Definitions

Put $A^* = \text{Hom}_R(A, R)$; this is an (A, A) -bimodule via the (A, A) -bimodule structure on A :

$$(afb)(x) = f(bxa) \quad (a, b, x \in A, f \in A^*).$$

Similarly, A is called **symmetric** if it has the following equivalent properties

- $A \cong A^*$ as (A, A) -bimodules;
- there exists a nonsingular associative R -bilinear form $\beta: A \times A \rightarrow R$ that is *symmetric*: $\beta(a, b) = \beta(b, a)$.



Nonsingularity

The R -bilinear form $\beta: A \times A \rightarrow R$ is said to be **nonsingular** if it satisfies the following equivalent conditions:

- the map $A \rightarrow A^*$, $a \mapsto \beta(\cdot, a)$, is an isomorphism;
- there exist **dual bases** $\{x_i\}_1^n \subseteq A$ and $\{y_i\}_1^n \subseteq A$ satisfying

$$a = \sum_i \beta(a, y_i) x_i \quad \text{for all } a \in A.$$

These conditions are actually left-right symmetric, even if β is not symmetric.



Examples and remarks

- (a) Symmetry and the Frobenius property are stable under base change $R \rightarrow R'$.



Examples and remarks

(a) Symmetry and the Frobenius property are stable under base change $R \rightarrow R'$.

(b) **Theorem of Endo and Watanabe (1967):**

faithful & separable $/R \Rightarrow$ symmetric $/R$



Examples and remarks

(a) Symmetry and the Frobenius property are stable under base change $R \rightarrow R'$.

(b) **Theorem of Endo and Watanabe (1967):**

$$\text{faithful \& separable } /R \Rightarrow \text{symmetric } /R$$

(c) Each (A, A) -bimodule isomorphism $A \xrightarrow{\sim} A^*$ restricts to

center

$$f(ab) = f(ba)$$

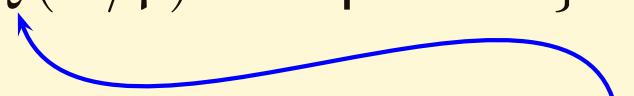
$$Z(A) \xrightarrow{\sim} A_{\text{trace}}^*$$



Separability locus

Goal: For a given Frobenius R -algebra A , determine

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid A \otimes_R Q(R/\mathfrak{p}) \text{ is separable}\}$$

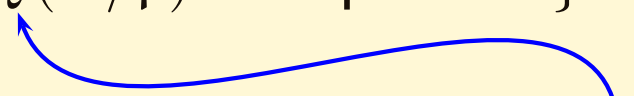
$$\iff A \otimes_R R_{\mathfrak{p}} \text{ is separable}$$




Separability locus

Goal: For a given Frobenius R -algebra A , determine

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid A \otimes_R Q(R/\mathfrak{p}) \text{ is separable}\}$$

$$\iff A \otimes_R R_{\mathfrak{p}} \text{ is separable}$$


We use techniques and results from

Donald G. Higman, *On orders in separable algebras*,
Canad. J. Math. **7** (1955), 509–515



Separability locus

For a fixed nonsingular associative R -bilinear form $\beta: A \times A \rightarrow R$, define the **Casimir operator**

$$c_\beta: A \rightarrow \mathcal{Z}(A), \quad a \mapsto \sum_i y_i a x_i$$

where $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ are dual bases for β .



Separability locus

For a fixed nonsingular associative R -bilinear form $\beta: A \times A \rightarrow R$, define the **Casimir operator**

$$c_\beta: A \rightarrow \mathcal{Z}(A), \quad a \mapsto \sum_i y_i a x_i$$

where $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ are dual bases for β .

One checks:

- c is independent of the choice of dual bases;
- $c(A)$ is an ideal of $\mathcal{Z}(A)$ which is independent of the choice of β (“Casimir ideal”).



Separability locus

For a fixed nonsingular associative R -bilinear form $\beta: A \times A \rightarrow R$, define the **Casimir operator**

$$c_\beta: A \rightarrow \mathcal{Z}(A), \quad a \mapsto \sum_i y_i a x_i$$

where $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ are dual bases for β .

Of particular importance will be the **Casimir element**

$$z_\beta = c(1) = \sum_i y_i x_i = \sum_i x_i y_i \in \mathcal{Z}(A)$$

for A symmetric



Separability locus

For a fixed nonsingular associative R -bilinear form $\beta: A \times A \rightarrow R$, define the **Casimir operator**

$$c_\beta: A \rightarrow \mathcal{Z}(A), \quad a \mapsto \sum_i y_i a x_i$$

where $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ are dual bases for β .

Thm 1 $A \otimes_R Q(R/\mathfrak{p})$ is separable $\iff \mathfrak{p} \not\subseteq c(A) \cap R.$



Characters

M a left A -module, f.g. projective $/R$

Trace

$$\text{Tr}: \text{End}_R(M) \cong M \otimes_R M^* \xrightarrow{\text{eval.}} R$$

Rank

$$\text{rank}_R M = \text{Tr}(1_M) \in R \quad (\text{Hattori-Stallings})$$

Character

$$\chi_M: A \rightarrow R, \quad a \mapsto \text{Tr}(a_M)$$



Characters

M a left A -module, f.g. projective $/R$, $\text{End}_A(M) \cong R$

Trace

$$\text{Tr}: \text{End}_R(M) \cong M \otimes_R M^* \xrightarrow{\text{eval.}} R$$

Rank

$$\text{rank}_R M = \text{Tr}(1_M) \in R \quad (\text{Hattori-Stallings})$$

Character

$$\chi_M: A \rightarrow R, \quad a \mapsto \text{Tr}(a_M)$$

Central character

$$\begin{aligned} \omega_M: \mathcal{Z}(A) &\rightarrow R, & a_M &= \omega_M(a)1_M \\ & & \downarrow & \\ \chi_M(a) &= \omega_M(a) \text{rank}_R M \end{aligned}$$



Characters

(A, β) a symmetric R -algebra, with dual bases $\{x_i\}, \{y_i\} \subseteq A$
 M a left A -module, f.g. projective $/R$,

$$\begin{array}{ccc}
 A_{\text{trace}}^* & \xrightarrow[\beta]{\sim} & \mathcal{Z}(A) \\
 \rightsquigarrow & \Psi & \Psi \\
 \chi_M & \leftrightarrow & z_\beta(M) = \sum_i \chi_M(x_i) y_i
 \end{array}$$



Characters

(A, β) a symmetric R -algebra, with dual bases $\{x_i\}, \{y_i\} \subseteq A$
 M a left A -module, f.g. projective $/R$,

$$A_{\text{trace}}^* \xrightarrow[\beta]{\sim} \mathcal{Z}(A)$$


 Ψ
 Ψ

$$\chi_M \leftrightarrow z_\beta(M) = \sum_i \chi_M(x_i) y_i$$

Example: $M = A$ yields the **regular character**

Casimir element

$$\chi_{\text{reg}} := \chi_A \leftrightarrow z_\beta = \sum_i y_i x_i = \sum_i x_i y_i$$



Characters

(A, β) a symmetric R -algebra, with dual bases $\{x_i\}, \{y_i\} \subseteq A$
 M a left A -module, f.g. projective $/R$,

$$\begin{array}{ccc} A_{\text{trace}}^* & \xrightarrow[\beta]{\sim} & \mathcal{Z}(A) \\ \rightsquigarrow & \Psi & \Psi \\ \chi_M & \leftrightarrow & z_\beta(M) = \sum_i \chi_M(x_i)y_i \end{array}$$

If $\text{End}_A(M) \cong R$ then define the **index**

$$[A : M]_\beta := \omega_M(z(M)) \in R$$



Theorem 2 Assume that A is *separable* and that M is cyclic with $\text{End}_A(M) \cong R$. Then:

(a) $[A : M]$ is invertible in R .

(b) $e(M) := [A : M]^{-1}z(M) \in \mathcal{Z}(A)$ is an idempotent such that $e(M)_M = 1_M$ and $xe(M) = \omega_M(x)e(M)$ ($x \in \mathcal{Z}(A)$).

(c) $\chi_{\text{reg}} e(M) = (\text{rank}_R M)\chi_M$



Let (A, β) be a Frobenius algebra, **augmented** by

$$\varepsilon: A \rightarrow R .$$



Integrals

Let (A, β) be a Frobenius algebra, **augmented** by

$$\varepsilon: A \rightarrow R .$$

Define $\Lambda_\beta \in A$ by $\beta(\Lambda_\beta, \cdot) = \varepsilon$. With dual bases $\{x_i\}, \{y_i\}$ for β ,

$$\Lambda_\beta = \sum_i \varepsilon(y_i) x_i .$$



Integrals

Let (A, β) be a Frobenius algebra, **augmented** by

$$\varepsilon: A \rightarrow R .$$

Define $\Lambda_\beta \in A$ by $\beta(\Lambda_\beta, \cdot) = \varepsilon$. With dual bases $\{x_i\}, \{y_i\}$ for β ,

$$\Lambda_\beta = \sum_i \varepsilon(y_i) x_i .$$

The **right integrals** in A are given by

$$\int_A^r := \{t \in A \mid ta = \varepsilon(a)t \ \forall a \in A\} = R\Lambda_\beta \cong R$$



Integrals

Let (A, β) be a Frobenius algebra, **augmented** by

$$\varepsilon: A \rightarrow R .$$

Similarly, one defines the R -module \int_A^l of **left integrals** and shows

$$\int_A^l = R\Lambda'_\beta \cong R \quad \text{with } \Lambda'_\beta = \sum_i \varepsilon(x_i)y_i$$



Integrals

Let (A, β) be a Frobenius algebra, **augmented** by

$$\varepsilon: A \rightarrow R .$$

If A is symmetric then $\Lambda_\beta = \Lambda'_\beta$ and

$$\int_A^r = \int_A^l =: \int_A$$



Hopf algebras



Notations

Throughout, we consider Hopf algebras that are **finite** $/R$:

$$H = (H, u, m, \varepsilon, \Delta, \mathcal{S})$$

The bimodule operations for ${}_H H^* {}_H$ and ${}_{H^*} H {}_{H^*}$ will be denoted by \rightharpoonup and \leftharpoonup :

$$\begin{aligned} \langle a \rightharpoonup f \leftharpoonup b, c \rangle &= \langle f, bca \rangle & (a, b, c \in H, f \in H^*) , \\ \langle e, f \rightharpoonup a \leftharpoonup g \rangle &= \langle gef, a \rangle & (e, f, g \in H^*, a \in H) . \end{aligned}$$

Here, $\langle \cdot, \cdot \rangle: H^* \times H \rightarrow R$ denotes the evaluation pairing.



Theorem 3 (Larson-Sweedler '69, Pareigis '71, Oberst-Schneider '73)

- (a) *The antipode S is bijective. Consequently, $\int_H^l = S(\int_H^r)$.*
- (b) *H is Frobenius if and only if $\int_H^r \cong R$. This is automatic if $\text{Pic } R = 1$. Furthermore, if H is Frobenius then so is H^* .*
- (c) *Assume H is Frobenius. Then H is symmetric iff*
 - (i) *H is unimodular (i.e., $\int_H^l = \int_H^r$), and*
 - (ii) *S^2 is an inner automorphism of H .*



Frobenius Hopf algebras

Assume H Frobenius and **fix** a generator $\Lambda \in \int_H^r$.



Frobenius Hopf algebras

Assume H Frobenius and **fix** a generator $\Lambda \in \int_H^r$.



bilinear form: There is a unique $\lambda \in \int_{H^*}^l$ with $\langle \lambda, \Lambda \rangle = 1$.

$$\beta(a, b) = \langle \lambda, ab \rangle$$

dual bases: $\{x_i\} = \{\Lambda_2\}$, $\{y_i\} = \{\mathcal{S}(\Lambda_1)\}$ ($\Delta(\Lambda) = \sum \Lambda_1 \otimes_R \Lambda_2$)



Frobenius Hopf algebras

Assume H Frobenius and **fix** a generator $\Lambda \in \int_H^r$.



The **Casimir operator** is given by the right adjoint action of Λ on H :

$$c: H \rightarrow \mathcal{Z}(H), \quad a \mapsto \sum \mathcal{S}(\Lambda_1) a \Lambda_2$$

Casimir element:

$$z = c(1) = \langle \varepsilon, \Lambda \rangle \in R$$



Application: the separability locus of H

Theorem 1 gives a classical result due to Larson and Sweedler:

Corollary 1 *The separability locus of a Frobenius Hopf algebra H over R is*

$$\text{Spec } R \setminus V(\langle \varepsilon, \int_H^r \rangle)$$



Grothendieck rings

In this part:

\mathbb{k} is an alg. closed field, $\text{char } \mathbb{k} = 0$

H is a semisimple Hopf algebra $/\mathbb{k}$

$\text{Irr } H$ is a full set of irreducible H -modules

The **Grothendieck ring** is

the (tensor) category of f.g. left H -modules

$$G_0(H) = K_0(H\text{-mod}) = \bigoplus_{V \in \text{Irr } H} \mathbb{Z} [V]$$



Grothendieck rings

The Grothendieck ring $G_0(H)$ is a symmetric \mathbb{Z} -algebra.

bilinear form: $\beta([V], [W]) = \dim_{\mathbb{k}} \text{Hom}_H(V, W^*)$

\mathbb{k} -linear dual

dual bases: $\{[V] \mid V \in \text{Irr } H\}, \{[V^*] \mid V \in \text{Irr } H\}$



The Grothendieck ring $G_0(H)$ is a symmetric \mathbb{Z} -algebra.

Casimir element:

$$z = \sum_{V \in \text{Irr } H} [V^*][V] = [H_{\text{ad}}] ,$$

the class of $\text{ad} = \text{ad}_l: H \rightarrow \text{End}_{\mathbb{k}}(H)$, $\text{ad}(h)(k) = \sum h_1 k \mathcal{S}(h_2)$.



Application: separability locus of $G_0(H)$

By Thm 1, the issue is to determine the ideal $c(G_0(H)) \cap \mathbb{Z}$ for the **Casimir operator**

$$c: G_0(H) \rightarrow \mathcal{Z}(G_0(H)), \quad [M] \mapsto \sum_{V \in \text{Irr } H} [V^*][M][V].$$



Application: separability locus of $G_0(H)$

By Thm 1, the issue is to determine the ideal $c(G_0(H)) \cap \mathbb{Z}$ for the **Casimir operator**

$$c: G_0(H) \rightarrow \mathcal{Z}(G_0(H)), \quad [M] \mapsto \sum_{V \in \text{Irr } H} [V^*][M][V].$$

Here is what I know . . .



Application: separability locus of $G_0(H)$

Theorem 4

- (a) *If p divides $\dim_{\mathbb{k}} H$ then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is not semisimple.*
- (b) *$G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple for $p = 0$ and all $p > \dim_{\mathbb{k}} H$.*
- (c) *If $G_0(H)$ is commutative then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple if and only if p does not divide $\dim_{\mathbb{k}} H$.*



Application: separability locus of $G_0(H)$

Theorem 4

- (a) *If p divides $\dim_{\mathbb{k}} H$ then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is not semisimple.*
- (b) *$G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple for $p = 0$ and all $p > \dim_{\mathbb{k}} H$.*
- (c) *If $G_0(H)$ is commutative then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple if and only if p does not divide $\dim_{\mathbb{k}} H$.*

Part (a) follows from the augmentation $\dim = \dim_{\mathbb{k}}: G_0(H) \rightarrow \mathbb{Z}$

$$(\dim_{\mathbb{k}} H) = (\dim z) = \dim c(G_0(H)) \supseteq c(G_0(H)) \cap \mathbb{Z} .$$



Application: separability locus of $G_0(H)$

Theorem 4

- (a) *If p divides $\dim_{\mathbb{k}} H$ then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is not semisimple.*
- (b) *$G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple for $p = 0$ and all $p > \dim_{\mathbb{k}} H$.*
- (c) *If $G_0(H)$ is commutative then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple if and only if p does not divide $\dim_{\mathbb{k}} H$.*

For (b) and (c), I use the norm of the adjoint class $z = c(1) = [H_{\text{ad}}]$.
(class equation, Sommerhäuser)



The character map

The *character map*

$$\chi: G_0(H) \rightarrow H^* , \quad [V] \mapsto \chi_V$$

is a ring embedding.



The character map

The *character map*

$$\chi: G_0(H) \rightarrow H^*, \quad [V] \mapsto \chi_V$$

is a ring embedding. In fact, χ yields an isomorphism

$$G_0(H) \otimes_{\mathbb{Z}} \mathbb{k} \cong H_{\text{trace}}^* = \bigoplus_{V \in \text{Irr } H} \mathbb{k} \chi_V \subseteq H^*$$

This is a semisimple \mathbb{k} -algebra (Thm 4) with \mathbb{Z} -form $G_0(H)$.



The character map

The *character map*

$$\chi: G_0(H) \rightarrow H^*, \quad [V] \mapsto \chi_V$$

is a ring embedding. It respects

- augmentations:

$$\begin{array}{ccc} G_0(H) & \xrightarrow{\chi} & H^* \\ \text{dim} \downarrow & & \downarrow u^* \\ \mathbb{Z} & \xrightarrow{\quad} & \mathbb{k} \end{array}$$



The character map

The *character map*

$$\chi: G_0(H) \rightarrow H^*, \quad [V] \mapsto \chi_V$$

is a ring embedding. It respects

- augmentations:

$$\begin{array}{ccc} G_0(H) & \xrightarrow{\chi} & H^* \\ \text{dim} \downarrow & & \downarrow u^* \\ \mathbb{Z} & \xrightarrow{\quad} & \mathbb{k} \end{array}$$

- integrals:

$$\int_{G_0(H)} = \mathbb{Z}[H_{\text{reg}}] \hookrightarrow \mathbb{k}\chi_{\text{reg}} = \int_{H^*}$$



The character map

The *character map*

$$\chi: G_0(H) \rightarrow H^*, \quad [V] \mapsto \chi_V$$

is a ring embedding. It respects

- augmentations:

$$\begin{array}{ccc} G_0(H) & \xhookrightarrow{\chi} & H^* \\ \text{dim} \downarrow & & \downarrow u^* \\ \mathbb{Z} & \xhookrightarrow{\quad} & \mathbb{k} \end{array}$$

- integrals:

$$\int_{G_0(H)} = \mathbb{Z}[H_{\text{reg}}] \hookrightarrow \mathbb{k}\chi_{\text{reg}} = \int_{H^*}$$

- involutions:

$$\chi_{V^*} = \mathcal{S}^*(\chi_V)$$



Application: a theorem of Shenglin Zhu

Theorem 5 (S. Zhu '93)

If $V \in \text{Irr } H$ satisfies $\chi_V \in \mathcal{Z}(H^)$ then $\dim_{\mathbb{k}} V$ divides $\dim_{\mathbb{k}} H$.*



Application: a theorem of Shenglin Zhu

Theorem 5 (S. Zhu '93)

If $V \in \text{Irr } H$ satisfies $\chi_V \in \mathcal{Z}(H^)$ then $\dim_{\mathbb{k}} V$ divides $\dim_{\mathbb{k}} H$.*

Sketch of proof (an application of Thm 2):



Application: a theorem of Shenglin Zhu

Theorem 5 (S. Zhu '93)

If $V \in \text{Irr } H$ satisfies $\chi_V \in \mathcal{Z}(H^)$ then $\dim_{\mathbb{k}} V$ divides $\dim_{\mathbb{k}} H$.*

Sketch of proof (an application of Thm 2): Put $\Lambda = \chi_{H_{\text{reg}}^*} \in \int_H$ and consider

$$b: H^* H^*_{H^*} \xrightarrow{\sim} H^* H_{H^*}, \quad f \mapsto (f \rightharpoonup \Lambda = \Lambda \leftarrow f)$$



Application: a theorem of Shenglin Zhu

Theorem 5 (S. Zhu '93)

If $V \in \text{Irr } H$ satisfies $\chi_V \in \mathcal{Z}(H^*)$ then $\dim_{\mathbb{k}} V$ divides $\dim_{\mathbb{k}} H$.

Sketch of proof (an application of Thm 2): Put $\Lambda = \chi_{H_{\text{reg}}^*} \in \int_H$ and consider

$$b: H^* H^*_{H^*} \xrightarrow{\sim} H^* H_{H^*}, \quad f \mapsto (f \rightharpoonup \Lambda = \Lambda \leftarrow f)$$

Thm 2(c) gives $b(\chi_{V^*}) = \frac{\dim_{\mathbb{k}} H}{\dim_{\mathbb{k}} V} e(V)$; so we need to show:

$$b(\chi_{V^*}) \text{ is integral } / \mathbb{Z}$$



Application: a theorem of Shenglin Zhu

Theorem 5 (S. Zhu '93)

If $V \in \text{Irr } H$ satisfies $\chi_V \in \mathcal{Z}(H^)$ then $\dim_{\mathbb{k}} V$ divides $\dim_{\mathbb{k}} H$.*

Sketch of proof (an application of Thm 2): Our goal is

$b(\chi_{V^*})$ is integral $/\mathbb{Z}$

where $b: {}_{H^*}H^*_{H^*} \xrightarrow{\sim} {}_{H^*}H_{H^*}$, $f \mapsto (f \rightharpoonup \Lambda = \Lambda \leftarrow f)$.



Application: a theorem of Shenglin Zhu

Theorem 5 (S. Zhu '93)

If $V \in \text{Irr } H$ satisfies $\chi_V \in \mathcal{Z}(H^)$ then $\dim_{\mathbb{k}} V$ divides $\dim_{\mathbb{k}} H$.*

Sketch of proof (an application of Thm 2): Our goal is

$b(\chi_{V^*})$ is integral $/\mathbb{Z}$

where $b: {}_{H^*}H^*_{H^*} \xrightarrow{\sim} {}_{H^*}H_{H^*}$, $f \mapsto (f \rightharpoonup \Lambda = \Lambda \leftarrow f)$. Now,

$$\chi_{V^*} \underset{\text{hypo.}}{\in} \mathcal{Z}(H^*)^{\text{alg. int.}} = \bigoplus_{M \in \text{Irr } H^*} \mathbb{k}^{\text{alg. int.}} e(M) \xrightarrow[\text{Thm 2(c)}]{b} \chi(G_0(H^*)) \mathbb{k}^{\text{alg. int.}} \subseteq H$$



Application: a theorem of Shenglin Zhu

Theorem 5 (S. Zhu '93)


If $V \in \text{Irr } H$ satisfies $\chi_V \in \mathcal{Z}(H^)$ then $\dim_{\mathbb{k}} V$ divides $\dim_{\mathbb{k}} H$.*

Sketch of proof (an application of Thm 2): Our goal is

$b(\chi_{V^*})$ is integral $/\mathbb{Z}$

where $b: {}_{H^*}H^*_{H^*} \xrightarrow{\sim} {}_{H^*}H_{H^*}$, $f \mapsto (f \rightharpoonup \Lambda = \Lambda \leftarrow f)$. Now,

$$\chi_{V^*} \underset{\text{hypo.}}{\in} \mathcal{Z}(H^*)^{\text{alg. int.}} = \bigoplus_{M \in \text{Irr } H^*} \mathbb{k}^{\text{alg. int.}} e(M) \xrightarrow[\text{Thm 2(c)}]{b} \chi(G_0(H^*)) \mathbb{k}^{\text{alg. int.}} \subseteq H$$

 Finally, all elements of $\chi(G_0(H^*)) \mathbb{k}^{\text{alg. int.}}$ are integral $/\mathbb{Z}$. ■

Ring theory again . . .



Notations

In this part,

(F, R, \mathbb{k}) consists of a complete d.v.r. R with residue field $\mathbb{k} = R/\mathfrak{m}$ and quotient field $F = Q(R)$

A is a Frobenius R -algebra, finite $/R$

$\overline{}$ denotes “reduction mod \mathfrak{m} ”: $\overline{(\cdot)} = \mathbb{k} \otimes_R (\cdot)$

$\cdot_{(F)}$ denotes base change $R \rightarrow F$: $\cdot_{(F)} = F \otimes_R \cdot$



Projectives over Frobenius algebras

The following transports standard standard methods and results for finite group algebras to the setting of general Frobenius algebras.

Reference: J.-P. Serre, *Linear Representations of Finite Groups*, Chapters 14 and 15

The framework I use is that of “Frobenius extensions”.



Projectives over Frobenius algebras

Proposition 5

(a) *Let V be a f.g. A -module. Then:*

$$V \text{ is projective} \iff V|_R \text{ and } \bar{V} \text{ are projective}$$

(b) *Let P and P' be f.g. projective A -modules. Then:*

$$P \cong P' \iff \bar{P} \cong \bar{P}'$$

(c) *For every f.g. projective \bar{A} -module Q , there exists a f.g. projective A -module P such that $Q \cong \bar{P}$.*



Projectives over Frobenius algebras

Now consider the Grothendieck group

the category of f.g. projective left A -modules

$$K_0(A) = K_0(A\text{-proj})$$

This is a partially ordered abelian group with positive cone

$$K_0^+(A) = \{[P] \mid P \text{ in } A\text{-proj}\}$$



Projectives over Frobenius algebras

Now consider the Grothendieck group

the category of f.g. projective left A -modules

$$K_0(A) = K_0(A\text{-proj})$$

This is a partially ordered abelian group with positive cone

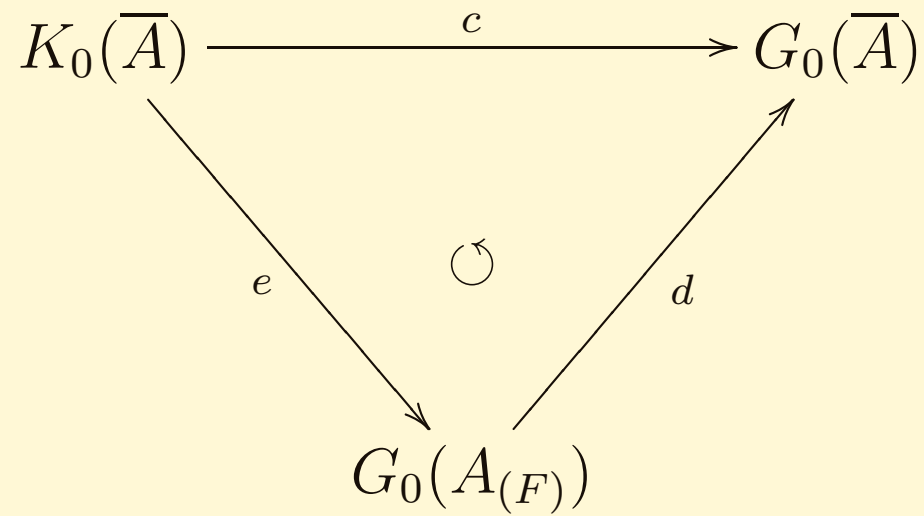
$$K_0^+(A) = \{[P] \mid P \text{ in } A\text{-proj}\}$$

Corollary 6

- (a) $[P] \mapsto [\bar{P} = P/\mathfrak{m}P]$ gives an isomorphism of p. ordered abelian groups, $K_0(A) \xrightarrow{\sim} K_0(\bar{A})$.
- (b) For P and Q in $A\text{-proj}$, $[P] = [Q]$ in $K_0(A)$ iff $P \cong Q$.



The Cartan-Brauer triangle



The Cartan-Brauer triangle

$$\begin{array}{ccc} K_0(\overline{A}) & \xrightarrow{c} & G_0(\overline{A}) \\ & \searrow e & \nearrow d \\ & & G_0(A_{(F)}) \end{array}$$


↻

- c is the **Cartan map**, from $\overline{A}\text{-proj} \hookrightarrow \overline{A}\text{-mod}$



The Cartan-Brauer triangle

$$\begin{array}{ccc}
 K_0(\overline{A}) & \xrightarrow{c} & G_0(\overline{A}) \\
 & \searrow e & \nearrow d \\
 & & G_0(A_{(F)})
 \end{array}$$




- c is the **Cartan map**, from $\overline{A}\text{-proj} \hookrightarrow \overline{A}\text{-mod}$

- $e = \left(K_0(\overline{A}) \xrightarrow[\text{Cor. 6}]{\sim} K_0(A) \xrightarrow{F \otimes_R (\cdot)} K_0(A_{(F)}) \xrightarrow{\text{Cartan}} G_0(A_{(F)}) \right)$



The Cartan-Brauer triangle

$$\begin{array}{ccc}
 K_0(\overline{A}) & \xrightarrow{c} & G_0(\overline{A}) \\
 & \searrow e & \nearrow d \\
 & & G_0(A_{(F)})
 \end{array}$$



- c is the **Cartan map**, from \overline{A} -proj \hookrightarrow \overline{A} -mod
- $e = \left(K_0(\overline{A}) \xrightarrow[\text{Cor. 6}]{\cong} K_0(A) \xrightarrow[F \otimes_R (\cdot)]{} K_0(A_{(F)}) \xrightarrow[\text{Cartan}]{} G_0(A_{(F)}) \right)$
- d is the **decomposition map**: given V in $A_{(F)}$ -mod choose L in A -mod such that $L \subseteq V$, $V = FL$ and put $d([V]) := [\overline{L}]$; this is independent of the choice of L . (R. G. Swan)



The Cartan-Brauer triangle

Proposition 7 *Assume that \bar{A} is semisimple. Then $A_{(F)}$ is semisimple as well. Moreover:*

- (a) *The maps d and e are order preserving isomorphisms that are inverse to each other:*

$$\begin{array}{ccc} & \xrightarrow{e} & \\ G_0(\bar{A}) & & G_0(A_{(F)}) \\ & \xleftarrow{d} & \end{array}$$

- (b) *In particular, d and e yield a dimension preserving one-to-one correspondence $\text{Irr } \bar{A} \leftrightarrow \text{Irr } A_{(F)}$.*

- (c) *$A_{(F)}$ is F -split if and only if \bar{A} is \mathbb{k} -split.*



Back to Hopf algebras . . .



Bialgebra cohomology (Gerstenhaber-Schack)

Notation:

\mathbb{k} is a field

B a \mathbb{k} -bialgebra



Bialgebra cohomology (Gerstenhaber-Schack)

Notation:

\mathbb{k} is a field

B a \mathbb{k} -bialgebra

Define a bicomplex $\mathbf{B} = (\mathbf{B}^{p,q})_{p,q \geq 0}$ with $\mathbf{B}^{p,q} = \text{Hom}_{\mathbb{k}}(B^{\otimes p}, B^{\otimes q})$ and with $\delta_{\text{Hochschild}}^{p,q} : \mathbf{B}^{p,q} \rightarrow \mathbf{B}^{p+1,q}$ and $\delta_{\text{Cartier}}^{p,q} : \mathbf{B}^{p,q} \rightarrow \mathbf{B}^{p,q+1}$.

$$H_{\text{GS}}^{\bullet}(B) := H^{\bullet}(\text{Tot } \mathbf{B})$$



Bialgebra cohomology (Gerstenhaber-Schack)

For the purposes of studying bialgebra deformations, the following variant is most useful.

Let $\widehat{\mathbf{B}}$ denote the sub-bicomplex of \mathbf{B} where the edge row and column are replaced by zeroes.

$$\widehat{H}_{\text{GS}}^{\bullet}(B) := H^{\bullet+1}(\text{Tot } \widehat{\mathbf{B}})$$



Bialgebra cohomology (Gerstenhaber-Schack)

Proposition 8 *Each of the following conditions implies that $\widehat{H}_{\text{GS}}^\bullet(B) = 0$.*

- (a) *B is separable as \mathbb{k} -algebra and commutative;*
- (b) *the dual algebra B^* is separable and commutative;*
- (c) *B is a bi-semisimple Hopf algebra. (D. Ştefan)*

e.g., $B = \mathbb{k}G$ a finite group algebra



Proposition 9 (Etingof-Gelaki)

Let (F, R, \mathbb{k}) be as before and let B be a finite-dimensional \mathbb{k} -bialgebra.

- (a) If $\widehat{H}_{\text{GS}}^3(B) = 0$ then there exists an R -free R -bialgebra \widetilde{B} such that $\widetilde{B} \otimes_R \mathbb{k} = B$. Moreover, if B is a Hopf algebra then so is each such \widetilde{B} .*
- (b) If $\widehat{H}_{\text{GS}}^2(B) = 0$ then, up to isomorphism, there is at most one \widetilde{B} as in (a).*



Problem: lifting to characteristic 0

Is it conceivable that all semisimple Hopf algebras over a field \mathbb{k} of $\text{char } p > 0$ lift to $\text{char } 0$?



Problem: adjoint representation and Chevalley property

Notation:

\mathbb{k} is an alg. closed field, $\text{char } \mathbb{k} = p > 0$

G is a finite group

