Koszul algebras and the "master theorem"

Algebra Seminar
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Overview

- A combinatorial introduction: MacMahon’s “Master Theorem”

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- Generalized Koszul algebras: definition and a quick overview of some constructions of Manin

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- Application: Master Theorems via Koszul duality

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- Generalized Koszul algebras: definition and a quick overview of some constructions of Manin
- Application: Master Theorems via Koszul duality
- Recent work: superization and return to combinatorics

Part I: the “Master Theorem”

Generalized Matching Problem

At a reception, everybody checks their hats. There are n different types of hats, say

$$m_i = \#\text{ hats of type } i \qquad \text{(indistinguishable)}$$

If hats are returned randomly afterwards, what is the probability that nobody ends up with the type of hat they came with?

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The **original version** had all $m_i = 1$; it was solved by Montmort (1713).

Origins of the MT

Reformulation: Suppose $I = \coprod_{\ell=1}^{n} I_\ell$ with $|I_\ell| = m_\ell$. Determine the number of permutations $\sigma \in \mathfrak{S}_I$ so that

$$\forall \ell: \quad \sigma I_\ell \cap I_\ell = \varnothing$$

divided by $m_1! m_2! \cdots m_n!$

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Answer (after MacMahon, *Combinatory Analysis*, 1917)

It is the coefficient of $x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$ in the expansion of the following polynomial in commuting variables:

$$(x_2 + x_3 + \cdots + x_n)^{m_1} (x_1 + x_3 + \cdots + x_n)^{m_2} \dots (x_1 + \cdots + x_{n-1})^{m_n}$$

To solve the generalized matching problem and other combinatorial problems, MacMahon proved the following “Master Theorem”

MacMahon's Master Theorem (original version, 1917)

Given a matrix $A = (a_{ij})_{n\times n}$ *over some commutative ring* R *and commuting indeterminates* $x_1, \ldots, x_n$ *over* R*. For each* $(m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$*, the* R*-coefficient of* $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ *in*

$$\left(\sum_{j=1}^{n} a_{1j}x_j\right)^{m_1} \left(\sum_{j=1}^{n} a_{2j}x_j\right)^{m_2} \cdots \left(\sum_{j=1}^{n} a_{nj}x_j\right)^{m_n}$$

is identical to the corresponding coefficient in the power series

$$\det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right)^{-1}$$

The MT

Applying this to the **Matching Problem**, we obtain that the desired number is given by the coefficient of $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ in

$$\det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right)^{-1} \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix}$$

The MT

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Calculation of the determinant gives

$$\det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right) = 1 - e_2 - 2e_3 - \dots - (n-1)e_n$$

elem. symmetric poly's
in $x_1, \dots, x_n$

Proofs of the MT

MacMahon presents the theorem as a result in the theory of permutations:

98 APPLICATIONS OF THE THEOREM [SECT. III, CH. II

wherein the denominator is in symbolic form in such wise that on multiplication the factors a_1b_2, $a_1b_2c_3$, ... are to be placed in determinant brackets $|a_1b_2|$, $|a_1b_2c_3|$, ... and denote the co-axial minors of the determinant

$$|a_1b_2 \ldots n_n|,$$

which appertains to the matricular relation.

This is a master theorem in the Theory of Permutations.

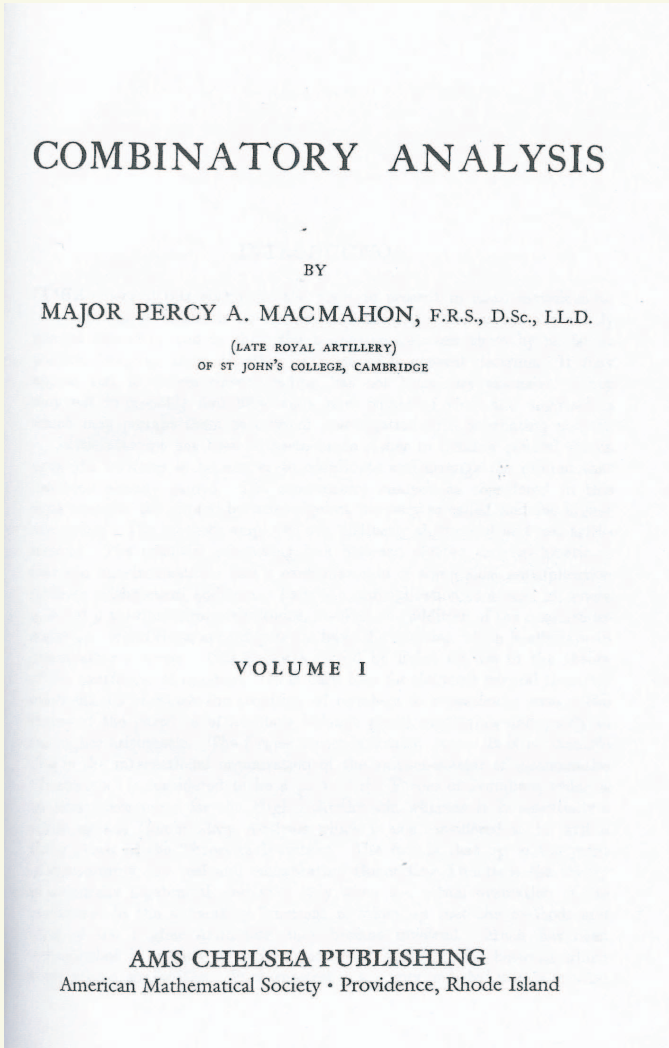
We will write

$$F_n$$

Proofs of the MT

The standard proof of the MT uses Lagrange inversion, and the result is often viewed in the analytic context:

I. J. Good, *A short proof of MacMahon's 'Master Theorem'*, Math. Proc. Cambridge Philos. Soc. **58** (1960), 160



COMBINATORY ANALYSIS

BY

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VOLUME I

AMS CHELSEA PUBLISHING

American Mathematical Society • Providence, Rhode Island

Percy Alexander MacMahon

1854 - 1929

Title page of MacMahon’s book containing the “Master Theorem”

(originally published at Cambridge, 1917)

Andrews' Problem:

Ref: George E. Andrews, *Problems and prospects for basic hypergeometric functions*, In: Theory and application of special functions, Academic Press, New York, 1975, pp. 191–224.

5. MacMahon's Master Theorem and the Dyson Conjecture.

PROBLEM 5. Are there q-analogs of MacMahon's Master Theorem and the Dyson Conjecture?

First let us recall:

MacMahon's Master Theorem (MacMahon (1894), (1915)). The coefficient of $X_1^{P_1} X_2^{P_2} \ldots X_n^{P_n}$ in

Noncommutative history

- First (somewhat) noncommutative version of MT in Cartier & Foata

  Problèmes combinatoires de commutation et réarrangements,

  SLN # 85 (1969)

- **Garoufalidis, Lê and Zeilberger**, *The quantum MacMahon master theorem*

  Proc. Nat. Acad. Sci. USA **103** (2006)

  arXiv: math.QA/0303319

- **Phùng Hô Hai and L.**: qMT is a consequence of "Koszul duality" for quantum affine space

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Noncommutative history

A considerable amount of research has been done by **mathematical physicists** on various quantum matrix identities; see, e.g.,

D. I. Gurevich, **P. N. Pyatov**, and **P. A. Saponov**:

The Cayley-Hamilton theorem for quantum matrix algebras of $\mathrm{GL}(m|n)$ *type*,

St. Petersburg Math. J. **17** (2006), 119-135; arXiv:math/0412192

The techniques employed are quite different from ours.

Part II: Generalized Koszul algebras

N-Koszul algebras

Notation: $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$ graded $\mathbb{k}$-algebra
connected: $\mathcal{A}_0 = \mathbb{k}$

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connected: $\mathcal{A}_0 = \Bbbk$

Minimal presentation: Write $\mathcal{A}_+ = \mathcal{A}_+^2 \oplus V$ for some graded subspace $V \subseteq \mathcal{A}_+ := \bigoplus_{n>0} \mathcal{A}_n$ to get

$$\boxed{\mathsf{T}(V)/(R) \xrightarrow{\sim} \mathcal{A}}$$

with a graded relation space $R \subseteq \mathsf{T}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$, chosen minimal: $(R) = R \oplus (V \otimes (R) + (R) \otimes V)$

N-Koszul algebras

Notation: $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$ graded $\Bbbk$-algebra
connected: $\mathcal{A}_0 = \Bbbk$

For fixed $N \geq 2$ define the "jump function"

$$\nu_N(i) := \begin{cases} \frac{i}{2}N & \text{if } i \text{ is even} \\ \frac{i-1}{2}N + 1 & \text{if } i \text{ is odd} \end{cases}$$

Lemma: *The relations R of $\mathcal{A}$ live in degrees $\geq N$ if and only if all $\mathrm{Tor}_i^{\mathcal{A}}(\Bbbk, \Bbbk)$ live in degrees $\geq \nu_N(i)$*

N-Koszul algebras

Notation: $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$ graded $\Bbbk$-algebra
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Definition: The algebra $\mathcal{A}$ is called N-Koszul if each
(R. Berger) $\mathrm{Tor}_i^{\mathcal{A}}(\Bbbk, \Bbbk)$ is concentrated in degree $\nu_N(i)$

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In this case, $\mathcal{A}$ is N-homogeneous:

- V is concentrated in degree 1 (b/c $\operatorname{Tor}_1^{\mathcal{A}}(\Bbbk, \Bbbk) \cong V$ and $\nu_N(1) = 1$)
- R is concentrated in degree N (b/c $\operatorname{Tor}_2^{\mathcal{A}}(\Bbbk, \Bbbk) \cong R$ and $\nu_N(2) = N$)

Some background ($N = 2$)

- 2-Koszul ("Koszul") algebras were introduced by S. Priddy in connection with his investigation of Yoneda algebras $\mathrm{Ext}_{\mathcal{A}}(\Bbbk, \Bbbk)$.

Reference: **S. Priddy**, *Koszul resolutions*, Trans AMS **152** 39–60 (1970)

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- Manin: 2-homogeneous ("quadratic") algebras provide a convenient framework for the investigation of quantum group actions on noncommutative spaces.

Reference: **Yu. I. Manin**, *Quantum groups and noncommutative geometry*, Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988

Some background (general $N \geq 2$)

- R. Berger: certain (Artin-Schelter) regular algebras are 3-Koszul, as are the "Yang-Mills algebras" defined by Connes and Dubois-Violette.

Reference: **R. Berger**, *Koszulity for nonquadratic algebras*, J. Algebra **239**, 705–734 (2001)

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Reference: **R. Berger**, *Koszulity for nonquadratic algebras*, J. Algebra **239**, 705–734 (2001)

- Berger, Dubois-Violette and Wambst extend Manin's theory to general N-homogeneous algebras

Reference: **R. Berger et al.**, *Homogeneous algebras*, J. Algebra **261**, 172–185 (2003)

Manin's constructions

$\ldots$ as extended to N-homogeneous algebras
by **Berger**, **Dubois-Violette** and **Wambst**

Manin's constructions

Notation:

$\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$ is N-homogeneous:

generators $V = \mathcal{A}_1$
relations $R \subseteq V^{\otimes N}$

$$\boxed{\mathcal{A} = A(V, R)}$$

Manin's constructions

$\mathcal{A} = A(V, R) \rightsquigarrow$ new N-homogeneous algebras:

- **dual algebra**: $\mathcal{A}^! = A(V^*, R^\perp)$
- **endomorphism bialgebra**:

$$\underline{\mathrm{end}}\,\mathcal{A} = \mathcal{A}^! \bullet \mathcal{A} = A\left(V^* \otimes V, \pi_N(R^\perp \otimes R)\right)$$

$\mathrm{End}(V)$: matrices

Manin's constructions

Example: Quantum affine n-space ($N = 2$)

For fixed scalars $0 \neq q_{ij} \in \Bbbk$ $(1 \leq i < j \leq n)$, define

$$A_{\mathbf{q}}^{n|0} := \Bbbk\langle x_1, \ldots, x_n\rangle / \left(x_j x_i - q_{ij} x_i x_j \mid 1 \leq i < j \leq n\right)$$

Get $\left(A_{\mathbf{q}}^{n|0}\right)^! =: A_{\mathbf{q}}^{0|n}$ with generators $x^1, \ldots, x^n$ and relations

$$x^\ell x^\ell = 0, \ x^k x^\ell + q_{k\ell} x^\ell x^k = 0 \quad (k < \ell)$$

Example: Quantum affine n-space ($N = 2$)

The endomorphism "semigroup" $\underline{\mathrm{end}}\, A_{\mathbf{q}}^{n|0}$ has generators $z_i^j := x^j \otimes x_i$ and relations

column relations: $z_j^\ell z_i^\ell = q_{ij}\, z_i^\ell z_j^\ell \quad$ (all $\ell, i < j$)

cross relations: $q_{ij} z_i^k z_j^\ell - q_{k\ell} z_j^\ell z_i^k = z_j^k z_i^\ell - q_{ij} q_{k\ell}\, z_i^\ell z_j^k$
$(i < j, k < \ell)$

Manin's constructions

For any N-homogeneous $\mathcal{A}$, there is an N-complex

$$\ldots \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}^{!\ *}_{i+1} \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}^{!\ *}_{i} \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{A} \longrightarrow 0$$

$$\boxed{d^N = 0}$$

The Koszul complex $K(\mathcal{A}) = \bigoplus_{n \geq 0} K(\mathcal{A})^n$ is the following contraction of this complex

$$\boxed{\ldots \xrightarrow{d^{N-1}} \mathcal{A} \otimes \mathcal{A}^{!\ *}_{N+1} \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}^{!\ *}_{N} \xrightarrow{d^{N-1}} \mathcal{A} \otimes \mathcal{A}^{!\ *}_{1} \xrightarrow{d} \mathcal{A} \longrightarrow 0}$$

Manin's constructions

Theorem: (a) $\mathcal{A}$ *is* N*-Koszul*
$\Leftrightarrow K(\mathcal{A})$ *is exact in degrees* > 0
$\Leftrightarrow$ *all* $K(\mathcal{A})^n$ $(n > 0)$ *are exact*

(b) $K(\mathcal{A})$ *and all* $K(\mathcal{A})^n$ *are complexes of* $\underline{\mathrm{end}}\,\mathcal{A}$*-comodules*

Part (a) is due to **R. Berger**, part (b) to **Phùng Hô Hai, Benoît Kriegk & L.**

Part III: MT from Koszul duality

The remainder of this talk is based on

- **Phùng Hô Hai** and **L.**,
  Koszul algebras and the quantum MacMahon master theorem, Bull. London Math. Soc. **39** (2007)
- **Phùng Hô Hai**, **Benoit Kriegk** and **L.**,
  N-homogeneous superalgebras,
  to appear in J. Noncomm. Geom.

Characters

Notation: $\mathcal{B}$ some bialgebra over $\Bbbk$ (later: $\mathcal{B} = \underline{\operatorname{end}}\,\mathcal{A}$)

$R_{\mathcal{B}}$ Grothendieck ring of all $\mathcal{B}$-comodules that are finite-dimensional$/\Bbbk$

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$R_{\mathcal{B}}$ Grothendieck ring of all $\mathcal{B}$-comodules that are finite-dimensional$/\Bbbk$

In more detail:

- $\mathcal{B}$-comodule $V \quad \rightsquigarrow \quad [V] \in R_{\mathcal{B}}$
- $0 \to U \to V \to W \to 0$ exact $\quad \rightsquigarrow \quad [V] = [U] + [W]$ in $R_{\mathcal{B}}$
- Multiplication in $R_{\mathcal{B}}$ is given by the tensor product of B-comodules

Characters

Def^n: Let V be a $\mathcal{B}$-comodule; so have $\delta_V : V \to V \otimes \mathcal{B}$.
The character of χ_V is the image of δ_V under the map

$$\mathrm{Hom}_{\Bbbk}(V, V \otimes \mathcal{B}) \xrightarrow{\sim} \mathrm{End}_{\Bbbk}(V) \otimes \mathcal{B} \xrightarrow{\mathrm{trace} \otimes \mathrm{Id}} \Bbbk \otimes \mathcal{B} = \mathcal{B}$$

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Lemma: *There is a commutative diagram of ring maps*

$$\begin{tikzcd} R_{\mathcal{B}} \arrow[r, "\chi"] \arrow[d, "\dim_{\Bbbk}"'] & \mathcal{B} \arrow[d, "\mathrm{counit}"] \\ \mathbb{Z} \arrow[r, "\mathrm{can.}"'] & \Bbbk \end{tikzcd}$$

Koszul duality

For any N-homogeneous algebra $\mathcal{A}$, define the following Poincaré series in $R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]$

$$P_{\mathcal{A}}(t) = \sum_i [\mathcal{A}_i] t^i \quad \text{and} \quad P_{\mathcal{A}^{!*},N}(-t) = \sum_i (-1)^i [A^{!\;*}_{\nu_N(i)}] t^{\nu_N(i)}$$

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Theorem: *If $\mathcal{A}$ is N-Koszul then, in $R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]$,*

$$\boxed{P_{\mathcal{A}}(t) \cdot P_{\mathcal{A}^{!*},N}(-t) = 1}$$

Koszul duality

Transport the identity in the Theorem from $R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]$ to $\underline{\mathrm{end}}\,\mathcal{A}[\![t]\!]$ via **characters**

$\rightsquigarrow$ a MT for any N-Koszul algebra $\mathcal{A}$

Koszul duality

Transport the identity in the Theorem from $R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]$ to $\underline{\mathrm{end}}\,\mathcal{A}[\![t]\!]$ via **characters**

$\rightsquigarrow$ a MT for any N-Koszul algebra $\mathcal{A}$

This can be **specialized** to "easier" $\mathcal{B}[\![t]\!]$ via algebra maps $\underline{\mathrm{end}}\,\mathcal{A} \to \mathcal{B}$

Summary

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}N\text{-Koszul algebra}\\ \mathcal{A}\end{array}} \arrow[r, Rightarrow] & \begin{array}{c}\text{Koszul complex}\\ K(\mathcal{A})\end{array} \arrow[d, Rightarrow] \\
\boxed{\begin{array}{c}\text{MT-equation in}\\ \underline{\mathrm{end}}\,\mathcal{A}[\![t]\!]\end{array}} & \begin{array}{c}\text{equation in}\\ R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]\end{array} \arrow[l, Rightarrow, "\text{characters}"']
\end{tikzcd}
$$

Examples

- $\mathcal{A} = \mathsf{S}(V)$ $\rightsquigarrow$ the original MT
  affine n-space (MacMahon)

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- $\mathcal{A} = A_{\mathbf{q}}^{n|0}$ $\rightsquigarrow$ qMT
  quantum n-space (Garoufalidis, Lê and Zeilberger)

Examples

- $\mathcal{A} = \mathsf{S}(V)$ affine n-space $\rightsquigarrow$ the original MT (MacMahon)

- $\mathcal{A} = A_{\mathbf{q}}^{n|0}$ quantum n-space $\rightsquigarrow$ qMT (Garoufalidis, Lê and Zeilberger)

- $\mathcal{A} = A\left(V, Y_N(V^{\otimes N})\right)$ N-symmetric algebra (Berger) $\rightsquigarrow$ N-MT (Etingof and Pak)

antisymmetrizer $\in \Bbbk[\mathfrak{S}_N]$

Part IV: Superization

Superization

This amounts to putting a $\mathbb{Z}/2$-grading ("parity": $\widehat{v}$) on the generating space V of $\mathcal{A} = A(V, R)$ and requiring the relations $R \subseteq V^{\otimes N}$ to be parity-graded.

Manin's constructions have to be carried out in the category of vector superspaces using the "rule of signs"

$$c_{U,V} : U \otimes V \xrightarrow{\sim} V \otimes U \ , \quad u \otimes v \mapsto (-1)^{\widehat{u}\widehat{v}} v \otimes u$$

Superization

$\rightsquigarrow$

- $\mathcal{A} = A(V,R)$ is a superalgebra
- $\underline{\mathrm{end}}\,\mathcal{A}$ is a super bialgebra
- $K(\mathcal{A})$ is a complex of super comodules/$\underline{\mathrm{end}}\,\mathcal{A}$
- have additional ring maps $\mathrm{sdim}\colon R_{\underline{\mathrm{end}}\,\mathcal{A}} \to \mathbb{Z}$ and $\chi^s\colon R_{\underline{\mathrm{end}}\,\mathcal{A}} \to \underline{\mathrm{end}}\,\mathcal{A}$:

$$\mathrm{sdim}\,V = \dim V_{\overline{0}} - \dim V_{\overline{1}}$$
$$\mathrm{strace}\left(F^i_j\right) = \sum (-1)^{\widehat{i}} F^i_i$$

Motivation

From **Konvalinka & Pak**, *Noncommutative extensions of the MacMahon master theorem* (arXiv: math.CO/0607737):

with each other. We do not need this observation for our telescoping argument.

Let us mention here that the inverse matrix $(I - A)^{-1}$ similarly appears in the study of quasi-determinants [GGRW] as well as in the non-commutative Lagrange inversion [PPR].

13.4. The relations for variables in our super-analogue are somewhat different from those studied in the literature (see e.g. [M3]). Note also that our super-determinant is different from the *Berezinian* [B] (see also [GGRW, M1]). We are somewhat puzzled by this and hope to obtain the "real" super-analogue in the future.

13.5. The relations studied in this paper always lead to quadratic algebras. While the deep reason lies in the Koszul duality, the fact that Koszulity can be extended to non-quadratic algebras is suggestive [Be]. The first such effort is made in [EP] where

The superalgebra $\mathsf{S}_N(V)$

Notation: $V \cong \Bbbk^{p|q}$: basis $\underbrace{x_1, \ldots, x_p}_{\text{even: } \widehat{i} = \overline{0}}, \underbrace{x_{p+1}, \ldots, x_{p+q}}_{\text{odd: } \widehat{i} = \overline{1}}$

$d = p + q$

assume $\operatorname{char} \Bbbk = 0$

The superalgebra $\mathsf{S}_N(V)$

Def^n^: The N-symmetric superalgebra $\mathsf{S}_N(V)$ is generated by $x_1, \ldots, x_d$ subject to the relations

$$\sum_{\sigma \in \mathfrak{S}_N} (\operatorname{sgn} \sigma)(-1)^{\sum_{(r,s) \in \operatorname{inv} \sigma} \widehat{i}_r \widehat{i}_s} x_{i_{\sigma^{-1}(1)}} \ldots x_{i_{\sigma^{-1}(N)}} = 0$$

with $\operatorname{inv} \sigma = \{(r,s) \mid r < s$ but $\sigma(r) > \sigma(s)\}$

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$$\sum_{\sigma \in \mathfrak{S}_N} (\operatorname{sgn} \sigma)(-1)^{\sum_{(r,s) \in \operatorname{inv} \sigma} \widehat{i_r} \widehat{i_s}} x_{i_{\sigma^{-1}(1)}} \ldots x_{i_{\sigma^{-1}(N)}} = 0$$

with $\operatorname{inv} \sigma = \{(r, s) \mid r < s \text{ but } \sigma(r) > \sigma(s)\}$

Example $\mathsf{S}_2(V) = \mathsf{S}(V) \cong \mathsf{S}(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}})$ is super-commutative: $[v, v'] = vv' - (-1)^{\widehat{v}\widehat{v'}} v'v = 0$

MT for $\mathcal{A} = \mathsf{S}_N(V)$

- $\mathcal{A} = \mathsf{S}_N(V)$ is indeed N-Koszul
- $\mathcal{A}_\ell$ has $\Bbbk$-basis the monomials $x_{\mathbf{i}} = x_{i_1} x_{i_2} \ldots x_{i_\ell}$ for sequences $\mathbf{i} = (i_1, \ldots, i_\ell) \in \{1, \ldots, d\}^\ell$ satisfying:

  $\mathbf{i}$ has no connected subsequence $(j_1, \ldots, j_N)$ with $j_1 < j_2 < \cdots < j_m \leq p < j_{m+1} \leq \cdots \leq j_N$ for some m

  Denote this collection of sequences $\mathbf{i}$ by

$$\boxed{\Lambda(p|q, N)_\ell}$$

MT for $\mathcal{A} = \mathsf{S}_N(V)$

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But: $\exists$ canonical

$$\underline{\mathrm{end}}\,\mathcal{A} \to \mathcal{O}(\mathsf{E}(V)) = \mathsf{S}(V^* \otimes V) = \Bbbk[x^i_j \mid 1 \le i, j \le d]$$

$x^i \otimes x_j$; parity $\widehat{i} + \widehat{j}$
supercommute

MT for $\mathcal{A} = \mathsf{S}_N(V)$

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$$\Downarrow$$

generic supermatrix of type $p|q$:

$$X = \left(x^i_j\right) = \left(\begin{array}{c|c} \text{even}_{p\times p} & \text{odd} \\ \hline \text{odd} & \text{even}_{q\times q} \end{array}\right)$$

Berezinian: Given a supermatrix

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (\text{even}, \text{odd})$$

over some supercommutative $\Bbbk$-superalgebra, one has:

- Φ is invertible $\Leftrightarrow$ A and D are invertible
- In this case, one defines

$$\boxed{\begin{aligned} \operatorname{ber}\Phi &= \det(A)\det(D - CA^{-1}B)^{-1} \\ &= \det(D)^{-1}\det(A - BD^{-1}C) \end{aligned}}$$

MT for $\mathcal{A} = \mathsf{S}_N(V)$

Notation: Put

$$y_i = \sum_j x_j \otimes x_i^j \in \mathcal{A} \otimes \mathcal{O}(\mathsf{E}(V))$$

In $\mathcal{A}_\ell \otimes \mathcal{O}(\mathsf{E}(V)) = \bigoplus_{\mathbf{i} \in \Lambda(p|q,N)_\ell} x_{\mathbf{i}} \otimes \mathcal{O}(\mathsf{E}(V))$ consider the elements

$$y_{\mathbf{i}} = x_{\mathbf{i}} \otimes X(\mathbf{i}) + \dots$$

with

$$\boxed{X(\mathbf{i}) \in \mathcal{O}(\mathsf{E}(V))_\ell}$$

“the coefficient”

MT for $\mathcal{A} = \mathsf{S}_N(V)$

Theorem: *Let* $X = \left(x^i_j\right)_{d \times d}$ *be the generic supermatrix of type* $p|q$. *Then, in the power series ring* $\Bbbk[x^i_j \mid \text{all } i, j\,]_{\bar{0}}[[t]]$,

$$\sum_{\ell} \sum_{\mathbf{i} \in \Lambda(p|q,N)_\ell} (-1)^{\widehat{\mathbf{i}}} X(\mathbf{i})\, t^\ell \cdot \sum_{m \equiv 0,1 \bmod N} (-1)^{m \bmod N} e_m t^m = 1$$

Here, e_m *is the* m^{th} *elementary supersymmetric function:*

$$\operatorname{ber}(1 + tX) = \sum_{n \geq 0} e_n t^n$$

MT for $\mathcal{A} = \mathsf{S}_N(V)$

- For $N = 2$ and V pure even ($q = 0$), this is the original MacMahon MT.
- For general N and V pure even, we obtain the N-MT of Etingof and Pak.

Back to combinatorics: a binomial identity

In general, applying the counit to the MT for $\mathcal{A}$ yields the super Hilbert series

$$H^s_{\mathcal{A}}(t) = \sum_{\ell \geq 0} \operatorname{sdim} \mathcal{A}_\ell \, t^\ell$$

Back to combinatorics: a binomial identity

In general, applying the counit to the MT for $\mathcal{A}$ yields the super Hilbert series

$$H^s_{\mathcal{A}}(t) = \sum_{\ell \geq 0} \operatorname{sdim} \mathcal{A}_\ell \, t^\ell$$

In our MT for $\mathcal{A} = \mathsf{S}_N(V)$, we have $X \mapsto 1_{d \times d}$, all coefficients $X(\mathbf{i}) \mapsto 1$ and

$$\operatorname{ber}(1 + tX) \mapsto \operatorname{ber}(1 + t1_{d\times d}) = (1+t)^{p-q}$$

This gives ...

Back to combinatorics: a binomial identity

$$\sum_{\ell \geq 0} \left(\sum_{\mathbf{i} \in \Lambda(p|q,N)_\ell} (-1)^{\widehat{\mathbf{i}}} \right) t^\ell$$

$$= \begin{cases} \left(\displaystyle\sum_{m \equiv 0,1 \bmod N} (-1)^{m \bmod N} \binom{p-q}{m} t^m \right)^{-1} & \text{if } p \geq q \\ \left(\displaystyle\sum_{m \equiv 0,1 \bmod N} (-1)^{\alpha_N(m)} \binom{m+q-p-1}{q-p-1} t^m \right)^{-1} & \text{if } p < q \end{cases}$$

$= m - (m \bmod N)$