Group actions and stratifications of prime spectra

“New Trends in Noncommutative Algebra” – UW 08/10/2010

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Thank you

MR744454 (85j:16027) 16A46 (16A54 18F25)
Goodearl, K. R. (1-UT)
Simple Noetherian rings not isomorphic to matrix rings over domains.
Comm. Algebra **12** (1984), *no. 11-12,* 1421–1434.

Let L be a field of characteristic zero containing a primitive nth root ξ of unity, where $n > 1$ is an integer. Let $S = A_1(L)$ be the Weil algebra over L, that is, the algebra over L generated by symbols x and θ subject to the sole relation $\theta x - x\theta = 1$. Let α be the automorphism of S which sends x into ξx and θ into $\xi^{-1}\theta$. Finally let R be the skew group ring $S^*\langle\alpha\rangle$. It is known that R is a simple Noetherian ring. The author proves that R is not isomorphic to a $k \times k$ matrix ring for any $k > 1$. He mentions that the first example of a simple Noetherian ring not isomorphic to a matrix ring over a domain was constructed by the reviewer and O. M. Neroslavskii [Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk 1975, no. 5, 38–42; MR0389968 (52 #10797)] and produced the ring R above for $n = 2$. Thus the author's construction extends that example. It is more important however that the author uses a new machinery for his proof. Namely he uses a theorem of Quillen on the functor K_0 for rings with a nonnegative filtration.

Reviewed by *A. E. Zalesskiĭ*

Overview

- **Background**: quantized coordinate algebras and enveloping algebras

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- **Tool**: the Amitsur-Martindale ring of quotients
- Some noncommutative spectra: $\operatorname{Spec} R$, $\operatorname{Rat} R$, ...
- Stratification of $\operatorname{Spec} R$

References

- *“Group actions and rational ideals”*,
  Algebra and Number Theory **2** (2008), 467-499

- *“Algebraic group actions on noncommutative spectra”*,
  Transformation Groups **14** (2009) 649-675

Papers & **pdf file of this talk** available on my web page:

http://math.temple.edu/~lorenz/

Background

Quantized coordinate rings

Goal: For $R = \mathcal{O}_q(\Bbbk^n)$, $\mathcal{O}_q(\mathrm{M}_n)$, $\mathcal{O}_q(G)$ ... a quantized coordinate ring, describe

$$\operatorname{Spec} R = \{\text{prime ideals of } R\}$$

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Typically, some algebraic torus T acts rationally by $\Bbbk$-algebra automorphisms on R; so have

$$\operatorname{Spec} R \longrightarrow \operatorname{Spec}^T R = \{T\text{-stable primes of } R\}$$

$$P \longmapsto P:T \underset{\text{def}}{=} \bigcap_{g \in T} g.P$$

Quantized coordinate rings

$\rightsquigarrow$ **T-stratification of $\operatorname{Spec} R$**

(Goodearl & Letzter; see also the monograph by Brown & Goodearl)

$$\operatorname{Spec} R = \bigsqcup_{I \in \operatorname{Spec}^T R} \operatorname{Spec}_I R$$

$\operatorname{Spec}_I R$: $\{P \in \operatorname{Spec} R \mid P : T = I\}$

Quantized coordinate rings

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Finite group algebras

For $R = \Bbbk G$, the group algebra of a finite group G, one has

$$\operatorname{Spec} R \overset{\text{1-1}}{\longleftrightarrow} \operatorname{IrrRep} R$$

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Clifford's Thm *Given $P \in \operatorname{Spec} R$ and $N \trianglelefteq G$, there is a unique, up to G-conjugacy, $Q \in \operatorname{Spec} \Bbbk N$ with*

$$P \cap \Bbbk N = Q : G$$

Enveloping algebras

Goal: For $R = U(\mathfrak{g})$, the enveloping algebra of a finite-dim'l Lie algebra $\mathfrak{g}$ over an algebraically closed field $\Bbbk$, describe

$$\operatorname{Prim} R = \{\text{primitive ideals of } R\}$$

kernels of (generally infinite-dimensional)

irreducible rep^{s} $R \longrightarrow \operatorname{End}_{\Bbbk}(V)$

Enveloping algebras

Dixmier's Problem 11 (from *Algèbres enveloppantes*, 1974)
aims for an analog of Clifford's Thm:

334 PROBLÈMES

10. On suppose que tr ad $x = 0$ pour tout $x \in \mathfrak{g}$. Est ce que $Z(\mathfrak{g}) \neq k$?

11. Soient $\mathfrak{k}$ un idéal de $\mathfrak{g}$, I un idéal primitif de $U(\mathfrak{g})$. Les propriétés suivantes sont-elles vraies : (*a*) il existe un idéal primitif de $U(\mathfrak{k})$ générique pour $U(\mathfrak{k}) \cap I$; (*b*) deux tels idéaux sont conjugués par le groupe adjoint algébrique de $\mathfrak{g}$; (*c*) soit L un tel idéal; il existe une représentation simple σ de $\mathfrak{k}$ de noyau L, et une représentation simple ρ de $\mathfrak{st}(\sigma, \mathfrak{g})$, telles que $\rho \mid \mathfrak{k}$ soit un multiple de σ et que ind $(\rho, \mathfrak{g})$ soit simple de noyau I. *Cf.* 4.5.9, 5.4.3, 5.4.4, 5.6.5.

Enveloping algebras

- solved for $\operatorname{char} \Bbbk = 0$ by **Mœglin & Rentschler**, even for noetherian or Goldie algebras R

Orbites d'un groupe algébrique dans l'espace des idéaux rationnels d'une algèbre enveloppante, Bull. Soc. Math. France **109** (1981), 403–426.

Sur la classification des idéaux primitifs des algèbres enveloppantes, Bull. Soc. Math. France **112** (1984), 3–40.

Sous-corps commutatifs ad-stables des anneaux de fractions des quotients des algèbres enveloppantes; espaces homogènes et induction de Mackey, J. Funct. Anal. **69** (1986), 307–396.

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Idéaux G-rationnels, rang de Goldie, preprint, 1986.

- for $\operatorname{char} \Bbbk$ arbitrary and under weaker Goldie hypotheses by **N. Vonessen**

Actions of algebraic groups on the spectrum of rational ideals, J. Algebra **182** (1996), 383–400.

Actions of algebraic groups on the spectrum of rational ideals. II, J. Algebra **208** (1998), 216–261.

Notation and hypotheses

Throughout the remainder of this talk,

$\Bbbk$	denotes an **algebraically closed** base field
R	is an associative $\Bbbk$-algebra (with 1)
G	is an affine algebraic $\Bbbk$-group acting rationally on R; so R is a $\Bbbk[G]$-comodule algebra.

Equivalently, we have a rational representation

$$\rho = \rho_R \colon G \to \mathrm{Aut}_{\Bbbk\text{-alg}}(R)$$

Notation and hypotheses

Occasionally, I will assume that R sat[s] the weak Nullstellensatz:

$$\mathrm{End}_R(V) = \mathbb{k} \quad \text{for all } V \in \mathrm{IrrRep}\, R$$

Example: R any affine $\mathbb{k}$-algebra, $\mathbb{k}$ uncountable Amitsur

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…or even the Nullstellensatz: weak Nullstellensatz & Jacobson property

semiprime $\equiv \bigcap$ primitives

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Examples:

- R affine noetherian / uncountable $\Bbbk$ — Amitsur
- R an affine PI-algebra — Kaplansky, Procesi
- $R = U(\mathfrak{g})$ — Quillen, Duflo
- $R = \Bbbk\Gamma$ with Γ polycyclic-by-finite — Hall, L., Goldie & Michler
- $\mathcal{O}_q(\Bbbk^n)$, $\mathcal{O}_q(M_n(\Bbbk))$, $\mathcal{O}_q(G)$, ...

Tool: The Amitsur-Martindale ring of quotients

The definition

$$\mathrm{Q_r}(R) = \varinjlim_{I \in \mathscr{E}} \mathrm{Hom}(I_R, R_R)$$

where $\mathscr{E} = \{I \trianglelefteq R \mid \mathrm{l.\,ann}_R\, I = 0\}$, a filter of ideals of R.

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- Elements are equivalence classes of right R-module maps

$$f : I_R \to R_R \quad (I \in \mathscr{E}) \ ,$$

with $f \sim f' : I'_R \to R_R$ if $f = f'$ on some $J \subseteq I \cap I'$, $J \in \mathscr{E}$.

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  with $f \sim f' : I'_R \to R_R$ if $f = f'$ on some $J \subseteq I \cap I'$, $J \in \mathscr{E}$.
- $+$ and $\cdot$ come from addition and composition of maps.
- $R \hookrightarrow \mathrm{Q_r}(R)$ via $r \mapsto (x \mapsto rx)$.

Extended center and central closure

Def^s & Facts:

- The extended center of R is defined by

$$\mathcal{C}(R) = \mathcal{Z}\,\mathrm{Q_r}(R)$$

center

If R is prime then $\mathcal{C}(R)$ is a $\Bbbk$-field.

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- R is said to be centrally closed if

$$R = \widetilde{R} := R\mathcal{C}(R) \subseteq \mathrm{Q_r}(R)$$

If R is semiprime then $\widetilde{R}$ is centrally closed.

Original references

for prime rings R:

W. S. Martindale, III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.

for general R:

S. A. Amitsur, *On rings of quotients*, Symposia Math., Vol. VIII, Academic Press, London, 1972, pp. 149–164.

Examples (R semiprime)

- R a finite product of **simple** rings $\Longrightarrow \mathrm{Q_r}(R) = R$.

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- R **rt Goldie** $\Longrightarrow \mathrm{Q_r}(R) = \{q \in \mathrm{Q_{cl}}(R) \mid qI \subseteq R$ for some $I \in \mathscr{E}\}$.
  In particular,

classical quotient ring of R

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$$\boxed{\mathcal{C}(R) = \mathcal{Z}\mathrm{Q_{cl}}(R)}$$

- If $U(\mathfrak{g}) \twoheadrightarrow R$ then $\mathrm{Q_r}(R) = \{$ ad $\mathfrak{g}$-finite elements of $\mathrm{Q_{cl}}(R)\}$.

Noncommutative spectra

Rational ideals

Want: an intrinsic characterization of “primitivity”, ideally

in detail . . .

Rational ideals

"coeur"

"Herz"

"heart"

"core"

Definition

- Recall: $\mathcal{C}(R/P)$ is a $\Bbbk$-field for any $P \in \operatorname{Spec} R$.
  We call P rational if $\mathcal{C}(R/P) = \Bbbk$.

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- Recall: $\mathcal{C}(R/P)$ is a $\Bbbk$-field for any $P \in \operatorname{Spec} R$. We call P rational if $\mathcal{C}(R/P) = \Bbbk$.
- Put $\operatorname{Rat} R = \{P \in \operatorname{Spec} R \mid P \text{ is rational}\}$; so

$$\operatorname{Rat} R \subseteq \operatorname{Spec} R$$

Connection with irreducible representations

Lemma (Martindale) *Given* $V \in \operatorname{IrrRep} R$, *let* $P = \operatorname{ann}_R V \in \operatorname{Prim} R$. *There is an embedding of* $\Bbbk$*-fields*

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Consequently, if R sat^s the weak Nullstellensatz then

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In fact, in most of the aforementioned examples, it has been shown that **equality** holds under mild restrictions on $\Bbbk$ or q.

Group action: G-prime and G-rational ideals

G-action on $R \rightsquigarrow$ G-actions on $\{$ ideals of R $\}$, $\operatorname{Spec} R$, $\operatorname{Rat} R$, ...

$G\backslash ?$ denotes the orbit sets in question.

Group action: G-prime and G-rational ideals

Definition: A proper G-stable ideal $I \lhd R$ is called G-prime if $A, B \underset{G\text{-stab}}{\trianglelefteq} R$, $AB \subseteq I \Longrightarrow A \subseteq I$ or $B \subseteq I$. Put

$$G\text{-Spec}\, R = \{G\text{-prime ideals of } R\}$$

Group action: G-prime and G-rational ideals

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$$G\text{-}\operatorname{Spec} R = \{G\text{-prime ideals of } R\}$$

Prop$^{\text{n}}$ *The assignment* $\gamma \colon P \mapsto P : G = \bigcap_{g \in G} g.P$ *yields surjections*

$$\begin{array}{ccc} \operatorname{Spec} R & \overset{\gamma}{\twoheadrightarrow} & G\text{-}\operatorname{Spec} R \\ {\scriptstyle \textit{can.}} \big\downarrow & \nearrow & \\ G \backslash \operatorname{Spec} R & & \end{array}$$

Group action: G-rational ideals

Given $I \in G\text{-}\operatorname{Spec} R$, the group G acts on $\mathcal{C}(R/I)$ and the invariants $\mathcal{C}(R/I)^G$ are a $\Bbbk$-field.

Definition: We call I G-rational if $\mathcal{C}(R/I)^G = \Bbbk$ and put

$$G\text{-}\operatorname{Rat} R = \{G\text{-rational ideals of } R\}$$

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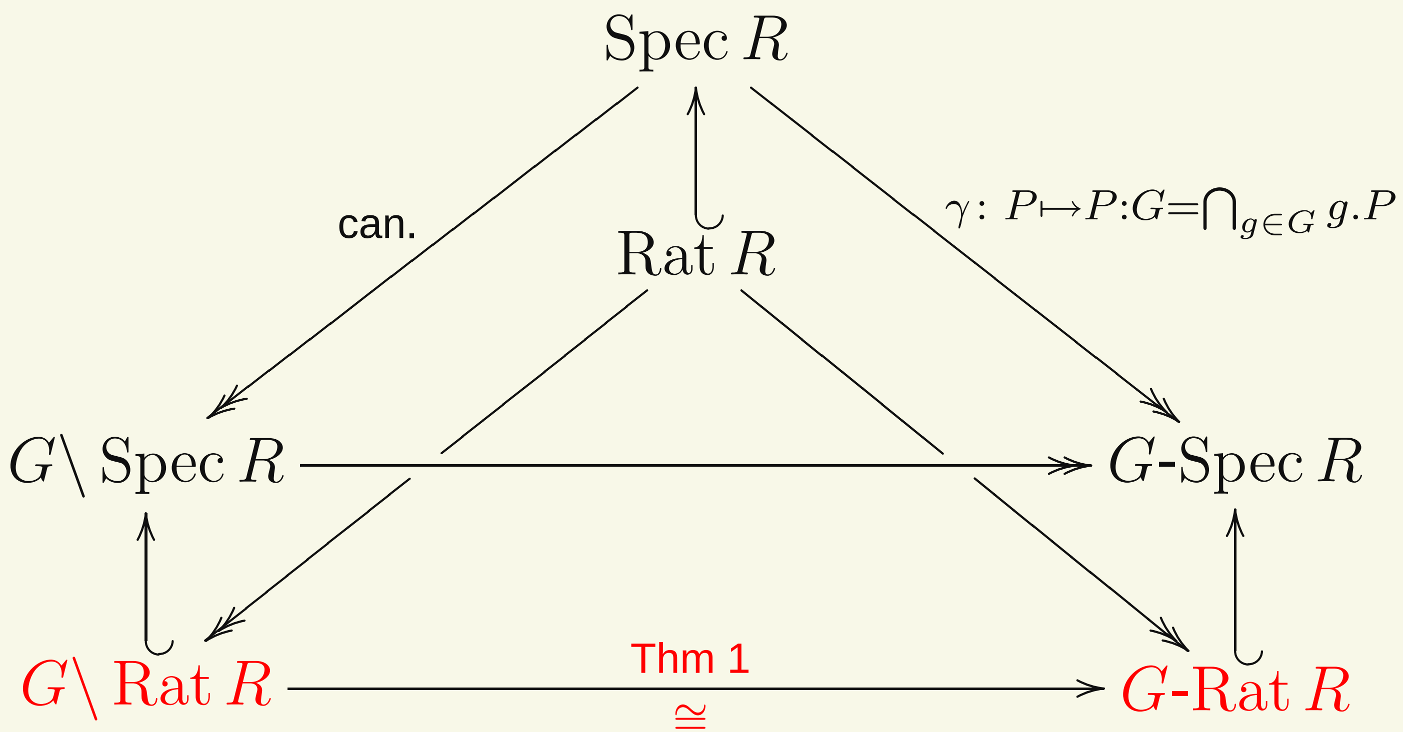
The following result solves Dixmier's Problem # 11 (a),(b) for arbitrary algebras.

Theorem 1

$$\begin{array}{ccc} G\backslash \operatorname{Rat} R & \xrightarrow{\textit{bij.}} & G\text{-}\operatorname{Rat} R \\ \Cup & & \Cup \\ G.P & \longmapsto & P:G \end{array}$$

Noncommutative spectra



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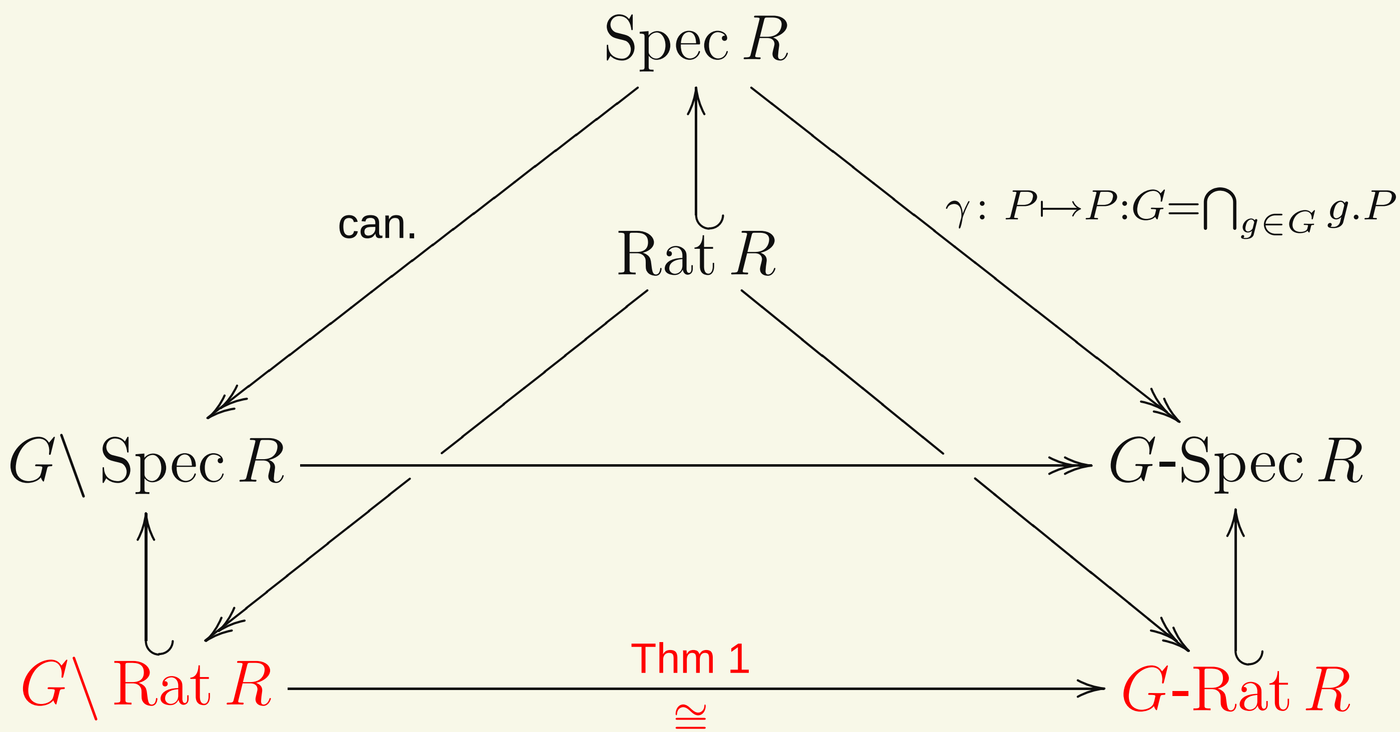
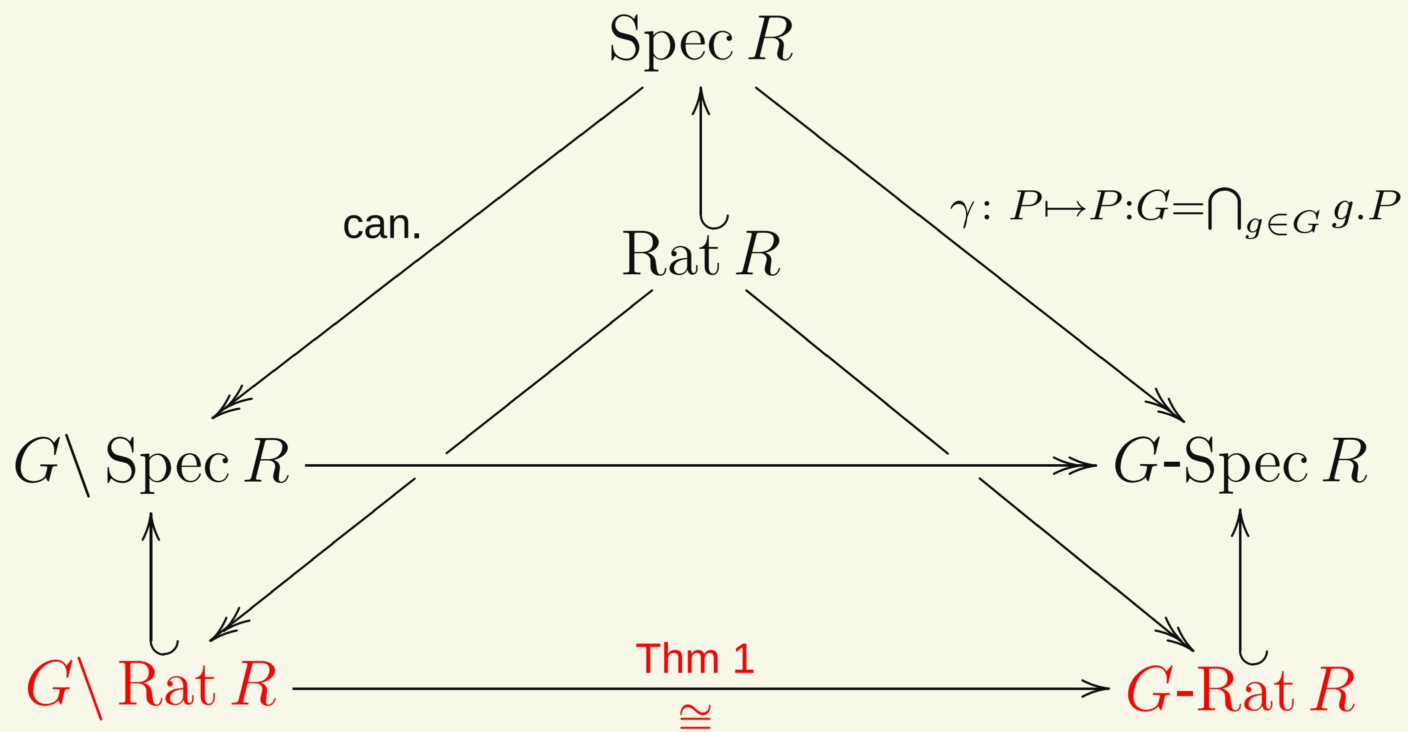


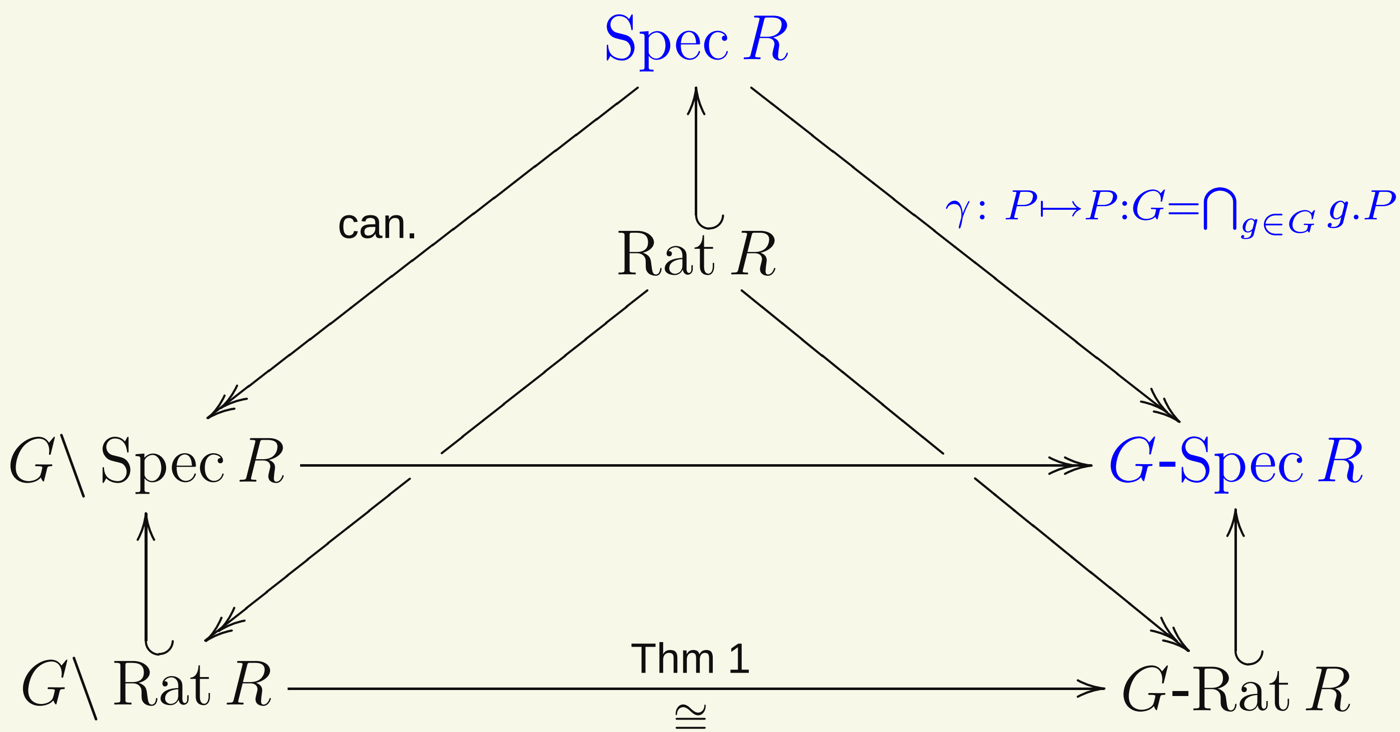
Spec R carries the Jacobson-Zariski topology: closed subsets are those of the form $\mathbf{V}(I) = \{P \in \operatorname{Spec} R \mid P \supseteq I\}$ where $I \subseteq R$.

Noncommutative spectra



- $\twoheadrightarrow$ is a surjection whose target has the final topology,
- $\hookrightarrow$ is an inclusion whose source has the induced topology, and
- $\cong$ is a homeomorphism

Noncommutative spectra



Next, we turn to $\operatorname{Spec} R$ and the map γ . . .

Stratification of the prime spectrum

Reminder: the Goodearl-Letzter stratification

Recall: the map $\gamma\colon \operatorname{Spec} R \twoheadrightarrow G\text{-}\operatorname{Spec} R$, $P \mapsto P:G = \bigcap_{g\in G} g.P$, yields the G-stratification

$$\operatorname{Spec} R = \bigsqcup_{I \in G\text{-}\operatorname{Spec} R} \operatorname{Spec}_I R$$

with G-strata

$$\operatorname{Spec}_I R = \gamma^{-1}(I) = \{P \in \operatorname{Spec} R \mid P:G = I\}$$

Goal: ?

Reminder: the Goodearl-Letzter stratification

For simplicity, I assume G to be **connected**; so $\Bbbk[G]$ is a domain. In particular,

$$G\text{-}\operatorname{Spec} R = \operatorname{Spec}^G R = \{G\text{-stable primes of } R\}$$

The rings T_I

For a given $I \in G\text{-}\mathrm{Spec}\, R$, put

$$T_I = \mathcal{C}(R/I) \otimes_{\Bbbk} \Bbbk(G)$$

$\mathrm{Fract}\, \Bbbk[G]$

This is a commutative domain, a tensor product of two fields.

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G-actions:

- on $\mathcal{C}(R/I)$ via the given action on R, $\rho\colon G \longrightarrow \mathrm{Aut}_{\Bbbk\text{-alg}}(R)$
- on $\Bbbk(G)$ by the right and left regular actions $\rho_r\colon (x.f)(y) = f(yx)$ and $\rho_\ell\colon (x.f)(y) = f(x^{-1}y)$
- on T_I by $\rho \otimes \rho_r$ and $\mathrm{Id} \otimes \rho_\ell$ $\longleftarrow$ commute!

The rings T_I

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This is a commutative domain, a tensor product of two fields.

Put

$$\operatorname{Spec}^G T_I = \{(\rho \otimes \rho_r)(G)\text{-stable primes of } T_I\}$$

Stratification Theorem

Theorem 2 *Given $I \in G$-$\operatorname{Spec} R$, there is a bijection*

$$c\colon \operatorname{Spec}_I R \longrightarrow \operatorname{Spec}^G T_I$$

having the following properties:

(a) G-equivariance*:* $c(g.P) = (\operatorname{Id} \otimes \rho_\ell)(g)(c(P))$*;*

(b) inclusions*:* $P \subseteq P' \iff c(P) \subseteq c(P')$*;*

(c) hearts*:* $\mathcal{C}\,(T_I/c(P)) \cong \mathcal{C}\,(R/P \otimes \Bbbk(G))$ *as* $\Bbbk(G)$*-fields;*

(d) rationality*:* P *is rational* $\iff T_I/c(P) = \Bbbk(G)$*.*

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(d) rationality: P *is rational* $\iff T_I/c(P) = \Bbbk(G)$.

Cor: Rational ideals are maximal in their strata

Application: local closedness

Recall: locally closed = open $\cap$ closed

Theorem 3 *Let* $P \in \operatorname{Rat} R$*. Then* $\{P\}$ *is loc. closed in* $\operatorname{Spec} R$ *iff* $\{P : G\}$ *is loc. closed in* G-$\operatorname{Spec} R$*.*

Application: local closedness

Recall: locally closed = open ∩ closed

Theorem 3 *Let $P \in \operatorname{Rat} R$. Then $\{P\}$ is loc. closed in $\operatorname{Spec} R$ iff $\{P:G\}$ is loc. closed in G-$\operatorname{Spec} R$.*

Pf of easy direction: Suppose $I = P:G$ is loc. closed in G-$\operatorname{Spec} R$. Then the preimage $\gamma^{-1}(I) = \operatorname{Spec}_I R$ under the continuous map $\gamma\colon \operatorname{Spec} R \to G\text{-}\operatorname{Spec} R$ is locally closed in $\operatorname{Spec} R$, and hence so is $\operatorname{Spec}_I R \cap \overline{\{P\}}$. Finally, by the Corollary, $\operatorname{Spec}_I R \cap \overline{\{P\}} = \{P\}$.

Application: local closedness

Recall: locally closed = open $\cap$ closed

Theorem 3 *Let $P \in \operatorname{Rat} R$. Then $\{P\}$ is loc. closed in $\operatorname{Spec} R$ iff $\{P:G\}$ is loc. closed in G-$\operatorname{Spec} R$.*

Cor *If $P \in \operatorname{Rat} R$ is loc. closed in $\operatorname{Spec} R$ then the orbit $G.P$ is open in its closure in $\operatorname{Rat} R$.*

Pf: By Thm 3, $\{P:G\}$ is loc. closed in G-$\operatorname{Spec} R$. Hence the fiber of $f\colon \operatorname{Rat} R \hookrightarrow \operatorname{Spec} R \overset{\gamma}{\twoheadrightarrow} G\text{-}\operatorname{Spec} R$ over $P:G$ is loc. closed in $\operatorname{Rat} R$. Finally $f^{-1}(P:G) = G.P$ by Thm 1.

Finiteness of G-Spec R

The following result is an application of Thms 1 - 3 . . .

Finiteness of G-Spec R

Prop^{n} *Assume that R sat^{s} the Nullstellensatz. Then the following are equivalent:*

(a) $G\text{-}\operatorname{Spec} R$ *is finite;*

(b) $G \backslash \operatorname{Rat} R$ *is finite;*

(c) R *sat^{s}*
(1) *ACC for G-stable semiprime ideals,*
(2) *the Dixmier-Mœglin equivalence, and*
(3) $G\text{-}\operatorname{Rat} R = G\text{-}\operatorname{Spec} R$.

If these conditions are satisfied then rational ideals of R are exactly the primes that are maximal in their G-strata.

locally closed = primitive = rational

Finiteness of G-Spec R

Example: If G is an algebraic torus then a sufficient condition for the equality $G\text{-}\mathrm{Spec}\, R = G\text{-}\mathrm{Rat}\, R$ is

$$\dim_{\Bbbk} R_\lambda \leq 1 \quad \text{for all } \lambda \in X(G)$$

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For a commutative domain R, this is also necessary. Thus:

Cor (classical) *Let R be an affine commutative domain / $\Bbbk$ and let G be an algebraic $\Bbbk$-torus acting rationally on R. Then:*

$$G\backslash \mathrm{Rat}\,R \text{ is finite} \iff \dim_{\Bbbk} R_\lambda \leq 1 \text{ for all } \lambda \in X(G).$$

Finiteness of G-Spec R

Example: If G is an algebraic torus then a sufficient condition for the equality $G\text{-Spec}\, R = G\text{-Rat}\, R$ is

$$\dim_{\Bbbk} R_\lambda \leq 1 \quad \text{for all } \lambda \in X(G)$$

In general, this condition is not nececessary, however, and I do not yet understand when $G\text{-Spec}\, R = G\text{-Rat}\, R$.

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- To do:
  - apply to interesting new examples
  - finiteness of G-Spec
  - $\ldots$