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Integrality and Normalizing Extensions of Rings

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In this note we prove an integrality result for certain finite normalizing ring extensions. In particular this applies to the extension $R \subset R * G$, where R is any ring and $R * G$ denotes a crossed product of the finite group G over R. This result is then used to prove a Going Up Theorem for such crossed products. Indeed we show that if A is a G-prime ideal of R and if P is a prime ideal of $R * G$ with $P \cap R \subset A$, then there exists a prime ideal Q of $R * G$ with $P \subset Q$ and $Q \cap R = A$.

Let S be a ring and let R be a subring with the same 1. We say that the extension $R \subset S$ is *normalizing* if there exist elements x_i $(i \in I)$ in S such that:

(1) The elements x_i $(i \in I)$ generate S as a left R-module.

(2) They normalize R in the sense that $x_i R = R x_i$ $(i \in I)$.

The normalizing extension $R \subset S$ is called *finite* if S, viewed as a left R-module, is finitely generated or, equivalently, if the above set $\{x_i \mid i \in I\}$ can be chosen to be finite.

Normalizing extensions occur quite frequently. For example, if $R * G$ denotes a crossed product of the group G over the ring R (see [3] for definitions), then the extension $R \subset R * G$ is certainly normalizing. As a second example, observe that if $S = R\mathbb{C}_S(R)$, then $R \subset S$ is trivially a normalizing extension, since ${}_R S$ has generators which commute with R. These latter extensions, which one might call centralizing extensions, have proved useful in the study of PI-algebras [6, 7].

We begin our work by slightly generalizing a result of Paré and Schelter on matrix rings [5, Theorem 1]. The proof given here is just a slight modification of the original. For later applications it is necessary to deal with rings without 1, that is, rings which do not necessarily have a 1. Let A be such a ring, let $M_n(A)$ denote the ring of $n \times n$ matrices over A and let $D_n(A)$ denote the subring of $M_n(A)$ consisting of diagonal matrices. Then $D_n(A)$ is naturally isomorphic to $A \oplus A \oplus \cdots \oplus A$ (n times). A subring $B \subset D_n(A)$ is said to be a *transversal*

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if B is a subdirect sum of $A \oplus A \oplus \cdots \oplus A$ or, equivalently, if B projects onto each direct summand A. For example, A naturally embedded in $M_n(A)$ as scalar matrices is a transversal, but as we will see there are certainly other transversals of interest.

The following theorem asserts that every element of $M_n(A)$ is integral over each transversal. However, in order to properly state this, we must first consider the form of acceptable polynomials. Let B be a transversal in $M_n(A)$. If α_1, $\alpha_2, \ldots, \alpha_k \in M_n(A)$, then a *B-monomial* in $\alpha_1, \alpha_2, \ldots, \alpha_k$ is a product in some order of the α_i's, each occurring finitely often, and of elements of B with at least one element of B occurring. Thus, for example, if $b_1, b_2 \in B$, then $\alpha_1{}^2 b_1 \alpha_2 b_2 \alpha_1$ is a B-monomial in α_1 and α_2 but $\alpha_1{}^2\alpha_2\alpha_1$ is not. By the degree of any such monomial we will mean the total degree in the α_i's.

THEOREM 1. *Let A be a ring without 1 and let B be a transversal in $M_n(A)$. Then there exists an integer $t \geqslant 1$ depending only upon n such that for any $\alpha \in M_n(A)$,*

$$\alpha^t = \phi(\alpha),$$

where $\phi(\alpha)$ is a sum of B-monomials in α of degree less than t.

Proof. For each integer $1 \leqslant k \leqslant n$ let $M_k(A)$ denote the subring of $M_n(A)$ consisting of those matrices of the form $\left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $*$ a block of size k. Observe that $BM_k(A) \subset M_k(A)$ and $M_k(A)B \subset M_k(A)$. We show, by induction on k, that there exists an integer $s = s(k)$ such that if $\alpha \in M_k(A)$, then

$$\alpha^s = \phi(\alpha),$$

where $\phi(\alpha)$ is a sum of B-monomials in α of degree less than s. The theorem will then follow with $t = s(n)$. For notational convenience we will call the relation $\alpha^s - \phi(\alpha)$ satisfied by α a monic polynomial over B of degree s.

We start with $k = 1$, so that $\alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $a \in A$. Then since B is a transversal, there exists $b \in B$ with $b = \left(\begin{smallmatrix} a & 0 \\ 0 & * \end{smallmatrix}\right)$ and hence α satisfies $\alpha^2 - b\alpha = 0$. In particular, $s(1) = 2$.

Suppose now that the result holds for the integer $k < n$ and consider $M_{k+1}(A)$. Note that each matrix $\beta \in M_{k+1}(A)$ can be partitioned as

$$\beta = \begin{pmatrix} \beta' & \beta'' & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where β' denotes its initial k-block and where the remaining nonzero entries indicated come from the $(k+1)$st row and column. For this β, set

$$\tilde{\beta} = \begin{pmatrix} \beta' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_k(A)$$

and for any $\beta, \gamma \in M_{k+1}(A)$ write $\beta \equiv \gamma$ if and only if $\beta'' = \gamma''$.

Two observations are now in order. First, if $\gamma_1 \equiv \gamma_2$, then clearly $\tilde{\beta}\gamma_1 \equiv \tilde{\beta}\gamma_2$. Second, given β, γ there exists $b \in B$ with $\tilde{\beta}\gamma \equiv \beta\gamma - \beta b$. Indeed, since B is a transversal, we can choose $b \in B$ so that

$$\gamma - b = \begin{pmatrix} * & \gamma'' & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}$$

and then

$$\beta(\gamma - b) = \begin{pmatrix} * & \beta'\gamma'' & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \tilde{\beta}\gamma.$$

It now follows by induction on $j \geqslant 0$ that if $\beta_1, \beta_2, ..., \beta_j, \gamma \in M_{k+1}(A)$, then

$$\tilde{\beta}_j\tilde{\beta}_{j-1} \cdots \tilde{\beta}_1\gamma \equiv \beta_j\beta_{j-1} \cdots \beta_1\gamma + \psi,$$

where ψ denotes a sum of nonconstant B-monomials in the β_i's of degree smaller than $j + 1$. In fact the case $j = 0$ is trivial and if

$$\tilde{\beta}_{j-1}\tilde{\beta}_{j-2} \cdots \tilde{\beta}_1\gamma \equiv \beta_{j-1}\beta_{j-2} \cdots \beta_1\gamma + \bar{\psi}$$

for some suitable $\bar{\psi}$, then we have by the above

$$\begin{aligned} \tilde{\beta}_j\tilde{\beta}_{j-1} \cdots \tilde{\beta}_1\gamma &\equiv \tilde{\beta}_j(\beta_{j-1} \cdots \beta_1\gamma + \bar{\psi}) \\ &\equiv \beta_j(\beta_{j-1} \cdots \beta_1\gamma + \bar{\psi}) - \beta_j b \\ &= \beta_j\beta_{j-1} \cdots \beta_1\gamma + \psi, \end{aligned}$$

where b is a suitable element of B and $\psi = \beta_j\bar{\psi} - \beta_j b$ is an appropriate sum of B-monomials of degree less than $j + 1$.

Let $\alpha \in M_{k+1}(A)$. Then by induction, $\tilde{\alpha}$ satisfies a suitable monic polynomial $\tilde{\alpha}^s - \phi(\tilde{\alpha})$ over B of degree $s = s(k)$. Hence certainly $(\tilde{\alpha}^s - \phi(\tilde{\alpha}))\alpha \equiv 0$. We now apply the observation of the preceding paragraph to each of the monomials in this expression. Since $BM_{k+1}(A) \subset M_{k+1}(A)$ and $M_{k+1}(A)B \subset M_{k+1}(A)$, each nonconstant B-monomial in α is a product of factors all belonging to $M_{k+1}(A)$. With this, it follows easily from the above that

$$(\alpha^s - \phi(\alpha))\alpha + \psi(\alpha) \equiv (\tilde{\alpha}^s - \phi(\tilde{\alpha}))\alpha \equiv 0,$$

where $\psi(\alpha)$ is a suitable sum of nonconstant B-monomials in α of degree less than $s+1$. In other words, we have shown that there exists

$$\beta = \alpha^{s+1} - \phi(\alpha)\alpha + \psi(\alpha) \in M_{k+1}(A),$$

a monic polynomial over B in α of degree $s+1$, with $\beta = 0$.

Now $\tilde{\beta} \in M_k(A)$ also satisfies a monic polynomial $\tilde{\beta}^s - \theta(\tilde{\beta}) = 0$ over B and observe that the constant term in $\theta(\tilde{\beta})$ must surely belong to $B \cap M_k(A)$. Hence since $\beta = 0$, we see that

$$\gamma = \beta^s - \theta(\beta) = \begin{pmatrix} 0 & 0 & 0 \\ * & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $a \in A$. Thus if $b \in B$ with $b = \operatorname{diag}(*, a, *)$, then we have

$$\gamma^2 - b\gamma = 0.$$

Since the latter expression is a monic polynomial in β of degree $2s$ and hence a monic polynomial in α of degree $2s(s+1)$, the induction step clearly follows with $s(k+1) = 2s(k)(s(k)+1)$. As we observed earlier, this yields the result.

We remark again that the above proof is essentially due to Paré and Schelter [5]. In fact the only difference between the results here and in [5] is that here we allow B to be any transversal rather than just the scalar matrices, and also we take care to observe that all the monomials of smaller degree which occur are B-monomials. As we see below, both of these additions are extremely useful.

We now apply the above result to finite normalizing ring extensions $R \subseteq S = \sum_{i=1}^n Rx_i$. An ideal A of R is said to be *normal* if $x_iA = Ax_i$ for all $i = 1, 2, \ldots, n$. Thus for any normal ideal A of R, the right ideal AS of S is actually a two-sided ideal. In particular, AS is a subring without 1 of S containing A and, in analogy with our previous terminology, if $s \in AS$ we will speak of A-monomials in s and of the degree of such monomials. With this convention, we have the following consequence of Theorem 1.

COROLLARY 2. *Let $R \subseteq S = \sum_{i=1}^n Rx_i$ be a finite normalizing extension of rings and let A be a normal ideal of R. Then there exists an integer $t \geq 1$ depending only upon n such that for any $s \in AS$,*

$$s^t = \phi(s),$$

where $\phi(s)$ is a sum of A-monomials in s of degree less than t.

Proof. Set $F = R \oplus R \oplus \cdots \oplus R$ (n times) so that F is a free left R-module of rank n and let $\pi: F \to S$ be the left R-module homomorphism given by $(r_1, r_2, \ldots, r_n)^\pi = \sum_{i=1}^n r_i x_i$. If $K \subset F$ denotes the kernel of π, then $T = \{\varphi \in \mathrm{End}(_R F) \mid K^\varphi \subset K\}$ is a subring of $\mathrm{End}(_R F) = M_n(R)$. Furthermore, there is a natural ring homomorphism $\hat{}: T \to \mathrm{End}(_R S)$. Indeed, if $\varphi \in T$ and $s = \sum_{i=1}^n r_i x_i \in S$, then $\hat{\varphi} \in \mathrm{End}(_R S)$ is given by

$$s^{\hat{\varphi}} = (r_1, r_2, \ldots, r_n)^{\varphi\pi}.$$

Routine verifications show that $\hat{\varphi}$ is well defined and that the map $\hat{}$ thus obtained is a ring homomorphism.

Let B denote the set of all diagonal matrices of the form

$$\beta = \mathrm{diag}(a_1, a_2, \ldots, a_n) \in D_n(A) \subset M_n(R),$$

with $a_i x_i = x_i a$ for some $a \in A$. The normality of A implies easily that B is a transversal in $M_n(A)$. Furthermore, if $f = (r_1, r_2, \ldots, r_n) \in F$, then $f^\beta = (r_1 a_1, r_2 a_2, \ldots, r_n a_n)$ and

$$f^{\beta\pi} = \sum_i r_i a_i x_i = \sum_i r_i x_i a = f^\pi a.$$

It follows immediately from this that $\beta \in T$ and that the endomorphism $\hat{\beta}$ of $_R S$ is right multiplication by $a \in A$.

Now let $s \in AS$ be given. Since AS is a two-sided ideal of S, we have $x_i s \in AS$, so $x_i s = \sum_j a_{ij} x_j$ for suitable $a_{ij} \in A$. Of course the elements a_{ij} need not be uniquely determined. Let $\sigma = (a_{ij}) \in M_n(A)$. Then it follows easily as above that if $f = (r_1, r_2, \ldots, r_n) \in F$, then $f^{\sigma\pi} = f^\pi s$. Hence we see that $\sigma \in T$ and that the endomorphism $\hat{\sigma}$ of $_R S$ is right multiplication by $s \in S$.

By Theorem 1, since B is a transversal, there exists an integer $t \geqslant 1$ depending only upon n such that

$$\sigma^t = \psi(\sigma),$$

where $\psi(\sigma)$ is a sum of B-monomials in σ of degree less than t. Applying the ring homomorphism $\hat{}$ to this relation and letting the resulting R-homomorphism in $\mathrm{End}(_R S)$ act on $1 \in S$, we obtain a relation

$$s^t = \phi(s),$$

where $\phi(s)$ is a sum of monomials obtained from those in $\psi(\sigma)$ by replacing σ by s and each $\beta \in B$ by an appropriate $a \in A$. Thus $\phi(s)$ clearly has the desired form and the corollary is proved.

As a consequence, we can obtain a Lying Over Theorem for finite normalizing extensions. We first require a lemma which follows almost immediately from work of Amitsur [1]. However, instead of deriving this from [1], we will offer a brief self contained argument.

LEMMA 3. *Let I be a nonzero right ideal of R which is nil of bounded degree. Then I contains a nonzero nilpotent right ideal of R.*

Proof. If A is a subset of R and $t \geqslant 1$ is an integer, we let $A^{(t)} = \{a^t \mid a \in A\}$. By assumption, $I^{(k)} = 0$ for some integer $k \geqslant 1$ and hence $I^{(k)}I = 0$. Now choose $t \geqslant 1$ minimal such that there exists a nonzero right ideal $N \subset I$ with $N^{(t)}N = 0$. Certainly t and N exist and we show now that $N^2 = 0$. This is clearly true if $t = 1$, so assume $t > 1$.

First let $a \in N$ with $a^2N = 0$. Then for all $x \in N$, we have $(ax + a)^tN = 0$. But $a^2N = 0$ implies that $(ax + a)^t = (ax)^t + (ax)^{t-1}a + (ax)^{t-2}a^2$ and we know that $(ax)^tN = 0$. Hence $(ax)^{t-1}aN = 0$ and we see that $(aN)^{(t-1)}(aN) = 0$. By the minimality of t we conclude that $aN = 0$. In other words, we have shown that, for $a \in N$, $a^2N = 0$ implies that $aN = 0$. But $N \subset I$, so N is a nil right ideal. Thus if b is any element of N, then certainly some power of b annihilates N on the left. Therefore the above implies easily that $bN = 0$ and we conclude that $N^2 = 0$.

THEOREM 4. *Let $R \subset S = \sum_{i=1}^n Rx_i$ be a finite normalizing extension and let A be a normal ideal of R. If A is semiprime, then $AS \cap R = A$. Furthermore if A is prime, then there exists a prime ideal P of S with $P \cap R = A$.*

Proof. Set $I = AS \cap R$ so that I is an ideal of R containing A. Let $t \geqslant 1$ be the integer given by Corollary 2 and let $r \in I$. Then by that corollary, since $r \in AS$, we know that $r^t = \phi(r)$, where $\phi(r)$ is a sum of A-monomials in r. By definition, each such A-monomial is a finite product whose factors are either equal to r or belong to A and where at least one factor from A must occur. Thus each such A-monomial in r belongs to A and hence $r^t = \phi(r) \in A$. We have therefore shown that, in the ring $\bar{R} = R/A$, the ideal $\bar{I} = I/A$ is nil of bounded degree t. Since $\bar{R}$ is a semiprime ring, by assumption on A, we conclude immediately from Lemma 3 that $\bar{I} = 0$. Thus $A = I = AS \cap R$.

Finally suppose A is a prime ideal of R. Since $AS \cap R = A$, we can apply Zorn's lemma to find an ideal P of S maximal with respect to the property that $P \cap R = A$. If J_1 and J_2 are ideals of S properly containing P, then $J_i \cap R \supsetneqq A$, so $(J_1 \cap R)(J_2 \cap R) \not\subset A$, since A is prime. Thus $J_1J_2 \cap R \not\subset A$, so $J_1J_2 \not\subset P$ and P is a prime ideal of S.

We remark that, since normalizing extensions are closed under homomorphic images, an appropriate Going Up Theorem is an immediate consequence of the above Lying Over Theorem.

We now consider crossed products $S = R * G$ of the finite group G over the ring R (see [3] for appropriate definitions). In [3], the relationship between the prime ideals of $R * G$ and the G-prime ideals of R was studied in detail. In particular, it was shown that Lying Over [3, Lemma 1.1], Incomparability [3, Theorem 1.2], and Going Down [3, Theorem 1.3] hold for the extension $R \subset R * G$. Furthermore the primes of $R * G$ lying over a given G-prime ideal of R were reasonably well described. Here we complete the picture by proving the relevant Going Up Theorem.

THEOREM 5. *Let $R * G$ be a crossed product of the finite group G over the ring R. If A is a G-prime ideal of R and if P is a prime ideal of $R * G$ with $P \cap R \subset A$, then there exists a prime ideal Q of $R * G$ with $P \subset Q$ and $Q \cap R = A$.*

Proof. Let $S = R * G$ and let $\pi: S \to S/P$ denote the natural homomorphism. Then clearly S^π is a finite normalizing extension of R^π with generators $\bar{x}^\pi$ $(x \in G)$. Since A is G-invariant, we have $\bar{x}A = A\bar{x}$ for all $x \in G$ and hence A^π is certainly a normal ideal of R^π. Furthermore, since $P \cap R \subset A$ we have $R^\pi/A^\pi \simeq R/A$ and thus we conclude from [3, Lemma 3.1(i)] that A^π is a semiprime ideal. Theorem 4 applied to $R^\pi \subset S^\pi$ now yields $A^\pi S^\pi \cap R^\pi = A^\pi$ and hence, by taking complete inverse images, we have

$$(A * G + P) \cap R = A.$$

Thus we see that there exists an ideal I of $R * G$ such that $I \supset P$ and $I \cap R = A$ and, by Zorn's lemma, we can choose Q to be an ideal of $R * G$ maximal with respect to this property. Thus $Q \supset P$ and $Q \cap R = A$. Furthermore, if J_1 and J_2 are ideals of $R * G$ properly containing Q, then each $J_i \cap R$ is a G-invariant ideal of R properly containing A so $(J_1 \cap R)(J_2 \cap R) \not\subset A$, since A is G-prime. Thus $J_1 J_2 \cap R \not\subset A$ so $J_1 J_2 \not\subset Q$ and Q is a prime ideal of $R * G$. The result follows.

It is interesting to observe that the proof of Theorem 5 is really considerably more elementary than the work of [3]. Yet surprisingly the results of [3], including the description of the primes of $R * G$, offer little help in dealing with the Going Up Problem. We close this paper by briefly commenting on a few applications of the above theorem.

In [3, Theorem 4.4], the Going Down Theorem was used to show that the prime (or primitive) ranks of R and of $R * G$ are equal. In fact, what was actually proved was that corresponding prime (or primitive) ideals of R and of $R * G$ have the same height. In a similar manner, using the preceding Going Up Theorem, one can easily show that corresponding prime (or primitive) ideals of R and of $R * G$ have the same depth. In particular, if P is a maximal ideal of $R * G$, then $P \cap R = \bigcap_{x \in G} Q^{\bar{x}}$, where Q is maximal in R.

As a second application, we note that crossed products of finite groups are frequently useful in handling finite index problems which occur in the study of ordinary group rings. Indeed if N is a normal subgroup of G of finite index, then the group ring $R[G]$ of G over R is easily seen to equal $R[N] * (G/N)$, a suitable crossed product of the finite group G/N over $R[N]$. Hence the relationship between $R[G]$ and $R[N]$ is precisely the relationship between a crossed product and its ring of coefficients. We illustrate this method by extending a result of Farkas [2, Corollary 8(i)] on group algebras of polycyclic groups to group algebras of polycyclic-by-finite groups. Farkas' theorem states that if K is an uncountable field and if G is polycyclic group, then for any homomorhpic image S of the group algebra $K[G]$, the set Priv S of primitive ideals of S, endowed with the Jacobson topology, is a Baire space. Rings with the latter property are called Kaplansky rings and we refer the reader to [2] for additional relevant definitions.

THEOREM 6. *Let K be an uncountable field and let G be a polycyclic-by-finite group. Then the group algebra $K[G]$ is a Kaplansky ring.*

Proof. By [2, Lemmas 2 and 4] it suffices to prove the following assertion: Let P be a prime ideal of $S = K[G]$ and let $\{P_i \mid i \in I\}$ be a countable collection of prime ideals properly containing P. Then there exists a primitive ideal $Q \supset P$ which does not contain any of the ideals P_i.

Now the group G contains a normal polycyclic subgroup N of finite index. By the above remarks, $K[G]$ can be written as $K[G] = K[N] * (G/N)$ and hence, by [3, Theorem 1.2], Incomparability holds for the extension $R = K[N] \subseteq K[G] = S$. Therefore each $P_i \cap R$ is a G-stable ideal of R properly containing $P \cap R$. Note that Priv $R/(P \cap R)$ is a Baire space by Farkas' theorem. Furthermore, $P \cap R$ is a (G/N)-prime ideal of R and hence $R/(P \cap R)$ is a semiprime ring by [3, Lemma 3.1]. Since $R = K[N]$ is a Jacobson ring [4, Corollary 1.3], the latter implies that $R/(P \cap R)$ is in fact semiprimitive. We may therefore apply a slight generalization of [2, Lemma 3] to conclude that there exists a primitive ideal $T \supset P \cap R$ of R not containing any of the ideals $P_i \cap R$. The Going Up Theorem (Theorem 5), applied to P and the (G/N)-prime ideal $\Pi_{x \in G} T^x$ of R, now yields the existence of a prime ideal Q of S with $Q \supset P$ and $Q \cap R = \bigcap_{x \in G} T^x$. It follows from [3, Lemma 4.1(ii)] that Q is in fact primitive. Since Q clearly does not contain any of the P_i, the result follows.

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