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Frobenius divisibility for Hopf algebras

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To Don Passman on the occasion of his 75^{th} *birthday*

Abstract. We present a unified ring theoretic approach, based on properties of the Casimir element of a symmetric algebra, to a variety of known divisibility results for the degrees of irreducible representations of semisimple Hopf algebras in characteristic 0. All these results are motivated by a classical theorem of Frobenius on the degrees of irreducible complex representations of finite groups.

Introduction

0.1. A result of Frobenius from 1896 [**4**, §12] states that the degree of every irreducible representation of a finite group G over an algebraically closed field $\Bbbk$ characteristic 0 divides the order of G. According to a conjecture of Kaplansky, the 6^{th} in his celebrated list of conjectures in [**7**], an analogous fact is expected to hold more generally for any finite-dimensional involutory Hopf $\Bbbk$-algebra H:

FD: the degrees of all irreducible representations of H divide $\dim_{\Bbbk} H$.

Recall that H is said to be involutory if the antipode of H satisfies $\mathcal{S}^2 = \mathrm{Id}_H$ – it is a standard fact that, over any base field $\Bbbk$ of characteristic 0, this condition amounts to semisimplicity of H; see [**11**, Theorem 16.1.2]. Frobenius' original theorem is the special case of FD where H is the group algebra of G over $\Bbbk$; therefore, the statement FD is referred to as ***Frobenius divisibility***. While Kaplansky's conjecture remains open as of this writing, it is in fact known to hold in several instances. It is our aim in this article to present a unified approach to a number of these generalizations using some general observations on symmetric algebras. Some of this material can be found in the earlier article [**9**] in the slightly more general setting of Frobenius algebras, and the authors have also greatly benefitted from a reading of the preprint [**1**] by Cuadra and Meir.

0.2. Here is a brief overview of the contents of this article. Section 1 is entirely devoted to symmetric algebras, with particular focus on the special case of a finite-dimensional semisimple algebra A. The main results, Proposition 4 and Theorem 5, both concern the so-called ***Casimir element*** of A; the former gives an expression

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of the central primitive idempotents of A in terms of the Casimir element while the latter gives a representation theoretic description of the Casimir element. Under certain hypotheses, these results allow us to deduce that a version of property FD for A is in fact equivalent to the apparently simpler condition that the Casimir element is integral over $\mathbb{Z}$; this is spelled out in Corollary 6. As a first application, we quickly derive Frobenius' original divisibility theorem for finite group algebras in §1.9.

Section 2 then concentrates on a semisimple Hopf algebra H over a field $\Bbbk$ of characteristic 0. We start with a formulation of Corollary 6, due to Cuadra and Meir [**1**], that is specific to H; the result (Theorem 8) states that FD for H is equivalent to integrality (over $\mathbb{Z}$) of the Casimir element c_λ of H that is associated to the unique integral of $\lambda \in H^*$ satisfying $\langle \lambda, 1\rangle = 1$. Further results derived in Section 2 from the material on symmetric algebras are: a theorem of Zhu [**14**] on irreducible representations of H whose character is central in H^*, the so-called class equation [**6**], [**15**], and a theorem of Schneider [**13**] on factorizable Hopf algebras. All these results ultimately are consequences of certain ring theoretic properties of the subalgebra $\mathscr{R}_\Bbbk(H)$ of H^* that is spanned by the characters of representations of H: the algebra $\mathscr{R}_\Bbbk(H)$ is semisimple and defined over $\mathbb{Z}$, that is, $\mathscr{R}_\Bbbk(H) \cong \Bbbk \otimes_\mathbb{Z} \mathscr{R}(H)$ for some subring $\mathscr{R}(H)$. Moreover, $\mathscr{R}(H)$ is finitely generated over $\mathbb{Z}$ and the Casimir element of $\mathscr{R}_\Bbbk(H)$ does in fact belong to $\mathscr{R}(H)^{\otimes 2}$, thereby ensuring that the requisite integrality property holds. We emphasize that none of the results in Section 2 are new; we take credit only for the presentation and the unified approach. However, we hope that the methods of this article and the point of view promulgated here will prove useful in making further progress toward a resolution of Kaplansky's conjecture.

0.3. With gratitude and admiration, the authors dedicate this article to Don Passman. The senior author has benefitted throughout his career from Don's mathematical insights, his generosity in sharing ideas and his long lasting support and friendship. This paper bears witness to the fact that Don has profoundly influenced generations of algebraists.

Notations and conventions. Throughout, we work over a base field $\Bbbk$ and $\otimes = \otimes_\Bbbk$. For any $\Bbbk$-vector space V, we let $\langle\,.\,,\,.\,\rangle \colon V^* \times V \to \Bbbk$ denote the evaluation pairing. The center of a $\Bbbk$-algebra A is denoted by $\mathscr{Z}A$ and the unit group by $A^\times$. Furthermore, $\operatorname{Irr} A$ will denote a full representative set of the isomorphism classes of irreducible representations of A and $\mathsf{rep}\,A$ is the category of finite-dimensional representations of A or, equivalently, finite-dimensional left A-modules. Our notation and terminology concerning Hopf algebras is standard and follows Montgomery [**10**] and Radford [**11**].

1. Symmetric Algebras

1.1. Symmetric Algebras and Frobenius Forms. Every $\Bbbk$-algebra A carries the "regular" (A, A)-bimodule structure: the left action of $a \in A$ on A is given by the left multiplication operator, a_A, and the right action by right multiplication, ${}_Aa$. This structure gives rise to a bimodule structure on the linear dual $A^* = \operatorname{Hom}_\Bbbk(A, \Bbbk)$ for which the following notation is customary in the Hopf literature:

$$a \rightharpoonup f \leftharpoonup b \stackrel{\text{def}}{=} f \circ b_A \circ {}_A a \qquad \text{or} \qquad \langle a \rightharpoonup f \leftharpoonup b, c\rangle = \langle f, bca\rangle$$

for $a, b, c \in A$ and $f \in A^*$. The algebra A is said to be ***symmetric*** if $A \cong A^*$ as (A, A)-bimodules. Note that even a mere $\Bbbk$-linear isomorphism $A^* \cong A$ forces A to be finite-dimensional; so symmetric algebras will necessarily have to be finite-dimensional.

The image of $1 \in A$ under any (A, A)-bimodule isomorphism $A \xrightarrow{\sim} A^*$ is a linear form $\lambda \in A^*$ satisfying $a \rightharpoonup \lambda = \lambda \leftharpoonup a$ for all $a \in A$ or, equivalently, $\langle \lambda, ab \rangle = \langle \lambda, ba \rangle$ for all $a, b \in A$. Linear forms satisfying this condition will be called ***trace forms***. Moreover, $a \rightharpoonup \lambda = 0$ forces $a = 0$, which is equivalent to the fact that λ does not vanish on any nonzero ideal of A; this condition will be referred to as ***nondegeneracy***. Conversely, if A is a finite-dimensional $\Bbbk$-algebra that is equipped with a nondegenerate trace form $\lambda \in A^*$, then we obtain an (A, A)-bimodule isomorphism

$$\begin{array}{ccc} A & \xrightarrow{\quad\sim\quad} & A^* \\ \Psi & & \Psi \\ a & \longmapsto & a \rightharpoonup \lambda = \lambda \leftharpoonup a \end{array} \tag{1}$$

Thus, A is symmetric. We will refer to λ as a ***Frobenius form*** for A. Note that λ is determined up to multiplication by a central unit of A: the possible Frobenius forms of A are given by

$$\lambda' = u \rightharpoonup \lambda \qquad \text{with} \qquad u \in \mathscr{Z} A \cap A^\times$$

We will think of a symmetric algebra as a pair (A, λ) consisting of the algebra A together with a fixed Frobenius trace form λ. A ***homomorphism*** $f: (A, \lambda) \to (B, \mu)$ of symmetric algebras is a $\Bbbk$-algebra map $f: A \to B$ such that $\lambda = \mu \circ f$.

1.2. First Examples of Symmetric Algebras. To put the above definitions into perspective, we offer the following examples of symmetric algebras. Another important example will occur later in this article (§2.5).

1.2.1. *Finite Group Algebras.* The group algebra $\Bbbk G$ of any finite group G is symmetric. Indeed, any $a \in \Bbbk G$ has the form $a = \sum_{g \in G} \alpha_g g$ with unique scalars $\alpha_g \in \Bbbk$ and a Frobenius form for $\Bbbk G$ is given by

$$\lambda(\sum_{g \in G} \alpha_g g) \stackrel{\text{def}}{=} \alpha_1$$

It is straightforward to check that λ is indeed a non-degenerate trace form.

1.2.2. *Finite-dimensional Semisimple Algebras.* Any finite-dimensional semisimple algebra A is symmetric. In order to obtain a Frobenius form for A, it suffices to construct a nonzero trace form for all simple Wedderburn components of A; the sum of these trace forms will then be a Frobenius form for A. Thus, we may assume that A is a finite-dimensional simple $\Bbbk$-algebra. Letting $\overline{K}$ denote an algebraic closure of the $\Bbbk$-field $K = \mathscr{Z} A$, we have $A \otimes_K \overline{K} \cong \operatorname{Mat}_d(\overline{K})$ for some d. The ordinary matrix trace $\operatorname{Mat}_d(\overline{K}) \to \overline{K}$ yields a nonzero $\Bbbk$-linear map $A \hookrightarrow A \otimes_K \overline{K} \to \overline{K}$ that vanishes on the space $[A, A]$ of all Lie commutators in A. Thus, $A/[A, A] \neq 0$ and we may pick any $0 \neq \lambda_0 \in (A/[A, A])^*$ to obtain a nonzero trace form $\lambda = \lambda_0 \circ \operatorname{can}: A \twoheadrightarrow A/[A, A] \to \Bbbk$. Non-degeneracy of λ is clear, because A has no nonzero ideals.

1.3. The Casimir Element. Let (A,λ) be a symmetric algebra. The element $c_\lambda \in A\otimes A$ that corresponds to $\mathrm{Id}_A \in \mathrm{End}_{\Bbbk}(A)$ under the isomorphism $\mathrm{End}_{\Bbbk}(A) \cong A\otimes A$ coming from (1) is called the ***Casimir element*** of (A,λ):

$$\begin{tikzcd} \mathrm{End}_{\Bbbk}(A) \arrow[r, "\sim", "\text{can.}"'] & A\otimes A^* \arrow[r, "\sim", "(1)"'] & A\otimes A \\ \mathrm{Id}_A \arrow[u, phantom, "\cup" rotate=90] \arrow[rr, mapsto] & & c_\lambda \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd} \tag{2}$$

Writing $c_\lambda = \sum_i x_i \otimes y_i$ with $\{x_i\}$ $\Bbbk$-linearly independent, (2) means explicitly that $a = \sum_i x_i\langle\lambda, ay_i\rangle$ for all $a\in A$ or, equivalently, the x_i form a $\Bbbk$-basis of A such that $\langle\lambda, x_iy_j\rangle = \delta_{i,j}$ for all i,j. It then follows that the y_i also form a $\Bbbk$-basis of A satisfying $\langle\lambda, y_ix_j\rangle = \delta_{i,j}$. Hence,

$$c_\lambda = \sum_i x_i\otimes y_i = \sum y_i\otimes x_i \tag{3}$$

Thus, the Casimir element c_λ is fixed by the switch map $\tau \in \mathrm{Aut}_{\mathsf{Alg}_{\Bbbk}}(A\otimes A)$ given by $\tau(a\otimes b) = b\otimes a$.

LEMMA 1. *Let (A,λ) be a symmetric algebra. Then, for all $z\in A\otimes A$, we have $zc_\lambda = c_\lambda\tau(z)$. Consequently, $c_\lambda^2 \in \mathscr{Z}(A\otimes A) = \mathscr{Z}A\otimes\mathscr{Z}A$.*

PROOF. Recall that $a = \sum_i x_i\langle\lambda, ay_i\rangle = \sum_i y_i\langle\lambda, ax_i\rangle$ for all $a\in A$. Using this, we compute

$$\sum_i ax_i\otimes y_i = \sum_{i,j} x_j\langle\lambda, ax_iy_j\rangle\otimes y_i = \sum_{i,j} x_j\otimes y_i\langle\lambda, y_jax_i\rangle = \sum_j x_j\otimes y_ja$$

Thus, $(a\otimes 1)c_\lambda = c_\lambda(1\otimes a)$. Applying the switch automorphism τ to this equation and using the fact that c_λ is stable under τ, we also obtain $(1\otimes b)c_\lambda = c_\lambda(b\otimes 1)$. Hence, $(a\otimes b)c_\lambda = c_\lambda(b\otimes a)$, which implies $c_\lambda^2\in\mathscr{Z}(A\otimes A)$. □

1.4. The Casimir Trace. The following operator was originally introduced by D.G. Higman [**5**]:

$$\begin{tikzcd} \gamma_\lambda\colon & A \arrow[r, two heads, "\text{can.}"] & A/[A,A] \arrow[r] & \mathscr{Z}A \\ & a \arrow[u, phantom, "\cup" rotate=90] \arrow[rr, mapsto] & & \sum_i x_iay_i = \sum_i y_iax_i \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd} \tag{4}$$

The following lemma justifies the claims, implicit in (4), that γ_λ is center-valued and vanishes on $[A,A]$. We will refer to γ_λ as the ***Casimir trace*** of (A,λ).

LEMMA 2. *Let (A,λ) be a symmetric algebra. Then $a\gamma_\lambda(bc) = \gamma_\lambda(cb)a$ for all $a,b,c\in A$.*

PROOF. The identity in Lemma 1 states that $\sum_i ax_i\otimes by_i = \sum_i x_ib\otimes y_ia$ in $A\otimes A$. Multiplying this identity on the right with $c\otimes 1$ and then applying the multiplication map $A\otimes A\to A$ gives $\sum_i ax_icby_i = \sum_i x_ibcy_ia$ or, equivalently, $a\gamma_\lambda(cb) = \gamma_\lambda(bc)a$ as claimed. □

1.5. A Trace Formula. The Casimir element c_λ can be used to give a convenient trace formula for endomorphisms of A:

LEMMA 3. *Let (A, λ) be a symmetric algebra with Casimir element $c_\lambda = \sum_i x_i \otimes y_i$. Then, for any $f \in \operatorname{End}_{\Bbbk}(A)$, we have $\operatorname{trace}(f) = \sum_i \langle \lambda, f(x_i) y_i \rangle = \sum_i \langle \lambda, x_i f(y_i) \rangle$.*

PROOF. By (2), $f(a) = \sum_i f(x_i) \langle \lambda, a y_i \rangle$ for all $a \in A$. Thus,

$$\begin{array}{cccccc} \text{trace}: & \operatorname{End}_{\Bbbk}(A) & \xrightarrow[\text{can.}]{\sim} & A \otimes A^* & \xrightarrow[\text{evaluation}]{\hspace{4em}} & \Bbbk \\ & \cup\!\!\!| & & \cup\!\!\!| & & \cup\!\!\!| \\ & f & \longmapsto & \sum_i f(x_i) \otimes (y_i \rightharpoonup \lambda) & \longmapsto & \sum_i \langle \lambda, f(x_i) y_i \rangle \end{array}$$

This proves the first equality; the second follows from (3). □

With $f = b_A \circ {}_A a$ for $a, b \in A$, Lemma 3 gives the formula

$$\operatorname{trace}(b_A \circ {}_A a) = \langle \lambda, b \gamma_\lambda(a) \rangle = \langle \lambda, \gamma_\lambda(b) a \rangle = \operatorname{trace}(a_A \circ {}_A b)$$

In particular, we obtain the following expressions for the ***regular character*** of A:

$$\langle \chi_{\text{reg}}, a \rangle \stackrel{\text{def}}{=} \operatorname{trace}(a_A) = \operatorname{trace}({}_A a) = \langle \lambda, \gamma_\lambda(a) \rangle = \langle \lambda, \gamma_\lambda(1) a \rangle \tag{5}$$

1.6. Primitive Central Idempotents. Now let A be a finite-dimensional semisimple $\Bbbk$-algebra and let $\operatorname{Irr} A$ denote a full representative set of the isomorphism classes of irreducible representations of A. For each $S \in \operatorname{Irr} A$, we let $D(S) = \operatorname{End}_A(S)$ denote the Schur division algebra of S and $a_S \in \operatorname{End}_{D(S)}(S)$ the operator given by the action of a on S. Consider the Wedderburn isomorphism

$$\begin{array}{ccc} A & \xrightarrow{\sim} & \prod_{S \in \operatorname{Irr} A} \operatorname{End}_{D(S)}(S) \\ \cup\!\!\!| & & \cup\!\!\!| \\ a & \longmapsto & (a_S) \end{array} \tag{6}$$

The ***primitive central idempotent*** $\boldsymbol{e}(S) \in \mathscr{Z} A$ is the element corresponding to $(0, \dots, 0, \operatorname{Id}_S, 0, \dots, 0) \in \prod_{S \in \operatorname{Irr} A} \operatorname{End}_{D(S)}(S)$ under the this isomorphism; so

$$\boldsymbol{e}(S)_T = \delta_{S,T} \operatorname{Id}_S \qquad (S, T \in \operatorname{Irr} A)$$

The following proposition gives a formula for $\boldsymbol{e}(S)$ using data coming from the structure of A as a symmetric algebra (§1.2.2) and the ***character*** χ_S of S, defined by

$$\langle \chi_S, a \rangle = \operatorname{trace}(a_S) \qquad (a \in A)$$

PROPOSITION 4. *Let A be a finite-dimensional semisimple $\Bbbk$-algebra with Frobenius trace form λ. Then, for each $S \in \operatorname{Irr} A$, we have the following formula in $A = \Bbbk \otimes A$:*

$$\gamma_\lambda(1)\, \boldsymbol{e}(S) = d(S)\, (\chi_S \otimes \operatorname{Id}_A)(c_\lambda) = d(S)\, (\operatorname{Id}_A \otimes \chi_S)(c_\lambda)$$

where $d(S) = \dim_{D(S)} S$. In particular, $\gamma_\lambda(1)_S = 0$ if and only if $\chi_S = 0$ or $d(S)\, 1_{\Bbbk} = 0$.

PROOF. Since $(\chi_S \otimes \mathrm{Id}_A)(c_\lambda) = (\mathrm{Id}_A \otimes \chi_S)(c_\lambda)$ by (3), we only need to show that $\gamma_\lambda(1)\, \boldsymbol{e}(S) = d(S)\, (\chi_S \otimes \mathrm{Id}_A)(c_\lambda)$. This amounts to the condition $\langle \lambda, \gamma_\lambda(1)\, \boldsymbol{e}(S)\, a\rangle = d(S)\, \langle \lambda, (\chi_S \otimes \mathrm{Id}_A)(c_\lambda) a\rangle$ for all $a \in A$ by nondegeneracy of λ. But $a = \sum_i x_i \langle \lambda, y_i a\rangle$ by (3) and so

$$\langle \lambda, (\chi_S \otimes \mathrm{Id}_A)(c_\lambda) a\rangle \underset{(3)}{=} \langle \lambda, \textstyle\sum_i \langle \chi_S, x_i\rangle y_i a\rangle = \sum_i \langle \chi_S, x_i\rangle \langle \lambda, y_i a\rangle = \langle \chi_S, a\rangle$$

Thus, our goal is to show that

$$\langle \lambda, \gamma_\lambda(1)\, \boldsymbol{e}(S)\, a\rangle = d(S)\, \langle \chi_S, a\rangle \qquad (a \in A) \tag{7}$$

For this, we use the regular character:

$$\langle \chi_{\mathrm{reg}}, \boldsymbol{e}(S)\, a\rangle \underset{(5)}{=} \langle \lambda, \gamma_\lambda(1)\, \boldsymbol{e}(S)\, a\rangle$$

On the other hand, by Wedderburn's Structure Theorem, the regular representation of A has the form $A_{\mathrm{reg}} \cong \bigoplus_{T \in \mathrm{Irr}\, A} T^{\oplus d(T)}$, whence $\chi_{\mathrm{reg}} = \sum_{T \in \mathrm{Irr}\, A} d(T) \chi_T$. Since $\boldsymbol{e}(S) \rightharpoonup \chi_T = \chi_T \leftharpoonup \boldsymbol{e}(S) = \delta_{S,T} \chi_S$, we obtain

$$\boldsymbol{e}(S) \rightharpoonup \chi_{\mathrm{reg}} = \chi_{\mathrm{reg}} \leftharpoonup \boldsymbol{e}(S) = d(S) \chi_S \tag{8}$$

Therefore, $\langle \chi_{\mathrm{reg}}, \boldsymbol{e}(S) a\rangle = d(S) \langle \chi_S, a\rangle$, proving (7). Finally, (7) also shows that $\gamma_\lambda(1)\, \boldsymbol{e}(S) = 0$ if and only if $d(S)\chi_S = 0$, which implies the last assertion in the proposition. □

1.7. The Casimir Square. Continuing to assume that A be a finite-dimensional semisimple $\Bbbk$-algebra, we now describe the Casimir square $c_\lambda^2 \in \mathscr{Z}A \otimes \mathscr{Z}A$ (Lemma 1) in terms of the following isomorphism coming from the Wedderburn isomorphism (6):

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\sim} & \prod_{S,T \in \mathrm{Irr}\, A} \mathrm{End}_{D(S)}(S) \otimes \mathrm{End}_{D(T)}(T) \\ \cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid \\ a \otimes b & \longmapsto & \big(a_S \otimes b_T\big) \end{array} \tag{9}$$

We will write $t_{S,T} \in \mathrm{End}_{D(S)}(S) \otimes \mathrm{End}_{D(T)}(T)$ for the (S,T)-component of the image of $t \in A \otimes A$ under the above isomorphism; so $(a \otimes b)_{S,T} = a_S \otimes b_T$. Recall that $S \in \mathrm{Irr}\, A$ is absolutely irreducible if and only if $D(S) = \Bbbk$.

THEOREM 5. *Let A be a finite-dimensional semisimple $\Bbbk$-algebra with Frobenius trace form λ. Then $(c_\lambda)_{S,T} = 0$ for $S \neq T \in \mathrm{Irr}\, A$. If S is absolutely irreducible, then* $(\dim_\Bbbk S)^2\, (c_\lambda^2)_{S,S} = \gamma_\lambda(1)_S^2$.

PROOF. For $S \neq T$, we have

$$(c_\lambda)_{S,T} = \big((\boldsymbol{e}(S) \otimes \boldsymbol{e}(T)) c_\lambda\big)_{S,T} \underset{\text{Lemma 1}}{=} \big(c_\lambda (\boldsymbol{e}(T) \otimes \boldsymbol{e}(S))\big)_{S,T} = (c_\lambda)_{S,T} (0_S \otimes 0_T) = 0$$

It remains to consider $(c_\lambda^2)_{S,S}$. First,

$$c_\lambda^2 = \textstyle\sum_i (x_i \otimes y_i) c_\lambda \underset{\text{Lemma 1}}{=} \sum_i (x_i \otimes 1) c_\lambda (y_i \otimes 1) = (\gamma_\lambda \otimes \mathrm{Id})(c_\lambda) \tag{10}$$

Next, for $c \in \mathscr{Z}A$, the operator $c_S \in D(S)$ is a scalar, since S absolutely irreducible, and $\chi_S(c) = d(S) c_S$ with $d(S) = \dim_\Bbbk S$. Therefore, writing $\rho_S(a) = a_S$ for $a \in A$, we calculate

$$\begin{aligned} d(S)(\rho_S \circ \gamma_\lambda)(a) &= (\chi_S \circ \gamma_\lambda)(a) = \chi_S(\textstyle\sum_i x_i a y_i) = \chi_S(\sum_i a y_i x_i) \\ &= \chi_S(a\, \gamma_\lambda(1)) = \chi_S(a)\, \gamma_\lambda(1)_S \end{aligned} \tag{11}$$

and further

$$\begin{aligned} d(S)^2\,(c_\lambda^2)_{S,S} &\underset{(10)}{=} d(S)^2\,(\rho_S\otimes\rho_S)\big((\gamma_\lambda\otimes\mathrm{Id})(c_\lambda)\big)\\ &= d(S)^2\,\big((\rho_S\circ\gamma_\lambda)\otimes\rho_S\big)(c_\lambda)\\ &\underset{(11)}{=} d(S)\,(\chi_S\otimes\rho_S)(c_\lambda)\,\gamma_\lambda(1)_S\\ &= (\mathrm{Id}_{\Bbbk}\otimes\rho_S)\big(d(S)\,(\chi_S\otimes\mathrm{Id})(c_\lambda)\big)\,\gamma_\lambda(1)_S\\ &\underset{\text{Proposition 4}}{=} \rho_S\big(\boldsymbol{e}(S)\gamma_\lambda(1)\big)\,\gamma_\lambda(1)_S=\gamma_\lambda(1)_S^2 \end{aligned}$$

which completes the proof of the theorem. □

1.8. Integrality and Divisibility. We recall some standard facts about integrality. Let R be a ring and let S be a subring of the center $\mathscr{Z}R$. An element $r\in R$ is said to be ***integral*** over S if r satisfies some monic polynomial over S. The following basic facts will be used repeatedly below:

- An element $r\in R$ is integral over S if and only if $r\in R'$ for some subring $R'\subseteq R$ such that R' contains S and is finitely generated as an S-module.
- If R is commutative, then the elements of R that are integral over S form a subring of R containing S, called the ***integral cosure*** of S in R.
- The integral closure of $\mathbb{Z}$ in $\mathbb{Q}$ is $\mathbb{Z}$: an element of $\mathbb{Q}$ that is integral over $\mathbb{Z}$ must belong to $\mathbb{Z}$.

The last fact above reduces the problem of showing that a given nonzero integer s divides another integer t to proving that the fraction $\frac{t}{s}$ is merely integral over $\mathbb{Z}$.

A semisimple $\Bbbk$-algebra A is said to be ***split*** if $D(S)=\Bbbk$ holds for all $S\in\operatorname{Irr}A$.

Corollary 6. *Let A be a split semisimple $\Bbbk$-algebra with Frobenius trace form λ. Assume that* $\operatorname{char}\Bbbk=0$ *and that* $\gamma_\lambda(1)\in\mathbb{Z}$. *Then the following are equivalent:*

(i) *The degree of every irreducible representation of A divides $\gamma_\lambda(1)$;*
(ii) *the Casimir element c_λ is integral over $\mathbb{Z}$.*

Proof. Theorem 5 gives the formula $(c_\lambda^2)_{S,S}=\big(\frac{\gamma_\lambda(1)}{\dim_{\Bbbk}S}\big)^2$. If (i) holds, then the isomorphism (9) maps $\mathbb{Z}[c_\lambda^2]$ into $\prod_{S\in\operatorname{Irr}H}\mathbb{Z}$, because $(c_\lambda)_{S,T}=0$ for $S\neq T$ by Theorem 5. Thus, $\mathbb{Z}[c_\lambda]$ is a finitely generated $\mathbb{Z}$-module and (ii) follows. Conversely, (ii) implies that c_λ^2 also satisfies a monic polynomial over $\mathbb{Z}$ and all $(c_\lambda^2)_{S,S}$ satisfy the same polynomial. Therefore, the fractions $\frac{\gamma_\lambda(1)}{\dim_{\Bbbk}S}$ must be integers, proving (i). □

Next, for a given homomorphism $(A,\lambda)\to(B,\mu)$ of symmetric algebras, we may consider the induced module $\operatorname{Ind}_A^B S=B\otimes_A S$ for each $S\in\operatorname{Irr}A$

Corollary 7. *Let A be a split semisimple algebra over a field $\Bbbk$ of characteristic 0 and let λ be a Frobenius trace form for A. Furthermore, let (B,μ) be a symmetric $\Bbbk$-algebra such that $\gamma_\mu(1)\in\Bbbk$ and let $(A,\lambda)\to(B,\mu)$ be a homomorphism of symmetric algebras. If the Casimir element c_λ is integral over $\mathbb{Z}$, then so is the scalar $\frac{\gamma_\mu(1)}{\dim_{\Bbbk}\operatorname{Ind}_A^B S}$ for each $S\in\operatorname{Irr}A$.*

Proof. It suffices to show that

$$\frac{\gamma_\mu(1)}{\dim_{\Bbbk}\operatorname{Ind}_A^B S}=\frac{\gamma_\lambda(1)_S}{\dim_{\Bbbk}S}$$

Indeed, by Theorem 5, the square of the fraction on the right equals $(c_\lambda^2)_{S,S}$, which is integral over $\mathbb{Z}$ if c_λ is. To check the above equality, let us put $e := \boldsymbol{e}(S)$ for brevity. Then $S^{\oplus \dim_{\Bbbk} S} \cong Ae$ and so $\operatorname{Ind}_A^B S^{\oplus \dim_{\Bbbk} S} \cong B\phi(e)$, where ϕ denotes the given homomorphism $(A, \lambda) \to (B, \mu)$. Since $\phi(e) \in B$ is an idempotent, $\dim_{\Bbbk} B\phi(e) = \operatorname{trace}({}_B\phi(e))$. Therefore,

$$\begin{aligned}\dim_{\Bbbk} \operatorname{Ind}_A^B S^{\oplus \dim_{\Bbbk} S} &= \operatorname{trace}({}_B\phi(e)) \underset{(5)}{=} \langle \mu, \phi(e)\, \gamma_\mu(1)\rangle = \langle \mu, \phi(e)\rangle\, \gamma_\mu(1) \\ &= \langle \lambda, e\rangle\, \gamma_\mu(1) \underset{(7)}{=} \frac{(\dim_{\Bbbk} S)^2}{\gamma_\lambda(1)_S} \gamma_\mu(1)\end{aligned}$$

The desired equality is immediate from this. □

1.9. A First Application: Frobenius' Divisibility Theorem for Finite Group Algebras. Returning to the setting of §1.2.1, consider the group algebra $\Bbbk G$ of any finite group G and assume that $\Bbbk$ is a splitting field for $\Bbbk G$ with $\operatorname{char} \Bbbk = 0$; so $\Bbbk G$ is split semisimple. The Frobenius form λ of §1.2.1 satisfies $\langle \lambda, gh^{-1}\rangle = \delta_{g,h}$ for $g, h \in G$. Hence, the Casimir element is

$$c_\lambda = \sum_{g \in G} g \otimes g^{-1}$$

Since $c_\lambda \in \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G$, a subring of $\Bbbk G \otimes \Bbbk G$ that is finitely generated over $\mathbb{Z}$, condition (ii) in Corollary 6 is satisfied. Moreover, $\gamma_\lambda(1) = |G| \in \mathbb{Z}$ as also required in Corollary 6. Thus, the corollary yields that the degrees of all irreducible representations of $\Bbbk G$ divide $\gamma_\lambda(1) = |G|$, as stated in Frobenius' classical theorem.

2. Hopf Algebras

2.1. Preliminaries: Semisimplicity, Integrals, and Frobenius Forms. We begin with a few reminders on semisimple Hopf algebras H over a field $\Bbbk$ of characteristic 0; the reader is referred to [**10**], [**11**] and [**12**] for details. First, semisimplicity of H amounts to H being finite-dimensional and involutory, that is, the antipode of H satisfies $\mathcal{S}^2 = \mathrm{Id}_H$. Both properties pass to H^*; so H^* is semisimple as well. By Maschke's Theorem for Hopf algebras, H is unimodular and $\langle \varepsilon, \Lambda\rangle \neq 0$ holds for any nonzero integral $\Lambda \in \int_H$, where ε is the counit of H. Furthermore, each such Λ serves as Frobenius form for H^*. In this section, we will fix the unique $\Lambda \in \int_H$ such that

$$\langle \varepsilon, \Lambda\rangle = \dim_{\Bbbk} H \tag{12}$$

We also fix the following normalized version of the regular character χ_{reg} of H; see §1.5:

$$\lambda := (\dim_{\Bbbk} H)^{-1} \chi_{\mathrm{reg}} \tag{13}$$

Then λ is a nonzero integral of H^* satisfying $\langle \lambda, \Lambda\rangle = \langle \lambda, 1\rangle = 1$. Taking λ as Frobenius form for H, the associated Casimir element is

$$\begin{aligned} c_\lambda &= \mathcal{S}(\Lambda_{(1)}) \otimes \Lambda_{(2)} = \Lambda_{(2)} \otimes \mathcal{S}(\Lambda_{(1)}) \\ &= \mathcal{S}(\Lambda_{(2)}) \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes \mathcal{S}(\Lambda_{(2)})\end{aligned} \tag{14}$$

Thus, the Casimir trace $\gamma_\lambda \colon H \to \mathscr{Z}H$ is given by $\gamma_\lambda(h) = \mathcal{S}(\Lambda_{(1)})h\Lambda_{(2)}$ for $h \in H$. Therefore,

$$\gamma_\lambda(1) = \langle \varepsilon, \Lambda\rangle = \dim_{\Bbbk} H \tag{15}$$

Reversing the roles of H and H^* and taking Λ as a Frobenius form for H^*, the value of the Casimir trace γ_Λ at $\varepsilon = 1_{H^*}$ is

$$\gamma_\Lambda(\varepsilon) = \langle \lambda, 1 \rangle = 1 \tag{16}$$

EXAMPLE. For a finite group algebra $H = \Bbbk G$, the integral Λ in (12) is $\Lambda = \sum_{g \in G} g$. The normalized regular character λ in (13) is given by $\langle \lambda, g \rangle = \delta_{g,1}$ for $g \in G$; so λ is identical to the Frobenius form of §1.2.1. The Casimir element c_λ in (14) is $\sum_{g \in G} g \otimes g^{-1}$ as in §1.9. So $\gamma_\lambda(h) = \sum_{g \in G} ghg^{-1}$ for $h \in \Bbbk G$.

2.2. Frobenius Divisibility for Hopf Algebras. The following special case of Corollary 6 is due to Cuadra and Meir [**1**, Theorem 3.4]. Note that (14) gives a formula for the Casimir element to be tested for integrality. For $H = \Bbbk G$, the theorem gives Frobenius' original result (§1.9).

THEOREM 8 (Cuadra and Meir). *Then the following are equivalent for a split semisimple Hopf algebra H over a field $\Bbbk$ of characteristic* 0.

(i) *Frobenius divisibility for H: the degrees of all irreducible representations of H divide* $\dim_\Bbbk H$*;*
(ii) *the Casimir element* (14) *is integral over* $\mathbb{Z}$.

PROOF. Choosing the Frobenius form λ for H as in (13), the Casimir element c_λ is given by (14) and $\gamma_\lambda(1) = \dim_\Bbbk H$ by (15). Thus, the theorem is a consequence of Corollary 6. □

2.3. More Preliminaries: The Representation Algebra and the Character Map. We continue to let H denote a split semisimple Hopf algebra over a field $\Bbbk$ of characteristic 0. Our remaining applications of the material of Section 1 all involve the ***representation ring*** $\mathscr{R}(H)$ of H. We remind the reader that $\mathscr{R}(H)$, by definition, is the abelian group with generators the isomorphism classes $[V]$ of representations $V \in \mathsf{rep}\, H$ and with relations $[V \oplus W] = [V] + [W]$ for $V, W \in \mathsf{rep}\, H$. The multiplication of $\mathscr{R}(H)$ comes from the tensor product of representations: $[V][W] = [V \otimes W]$. As a group, $\mathscr{R}(H)$ is free abelian of finite rank, with $\mathbb{Z}$-basis given by the classes $[S]$ with $S \in \operatorname{Irr} H$; so all elements of $\mathscr{R}(H)$ are integral over $\mathbb{Z}$.

We shall also consider the $\Bbbk$-algebra $\mathscr{R}_\Bbbk(H) := \mathscr{R}(H) \otimes_\mathbb{Z} \Bbbk$, which can be thought of as a subalgebra of H^* via the ***character map***

$$\begin{array}{rcl} \chi_\Bbbk \colon \mathscr{R}_\Bbbk(H) & \hookrightarrow & H^* \\ \cup & & \cup \\ [V] \otimes 1 & \longmapsto & \chi_V \end{array}$$

The image of this map is the algebra $(H/[H,H])^*$ of all trace forms on H or, equivalently, the algebra of all cocommutative elements of H^*. We remind the reader of some standard facts about $\mathscr{R}_\Bbbk(H)$; for more details, see [**8**] for example. The algebra $\mathscr{R}_\Bbbk(H)$ is finite-dimensional semisimple. A Frobenius form for $\mathscr{R}_\Bbbk(H)$ is given by the dimension of H-invariants:

$$\delta \colon \mathscr{R}_\Bbbk(H) \to \Bbbk, \qquad [V] \otimes 1 \mapsto (\dim_\Bbbk V^H) 1_\Bbbk$$

The Casimir element c_δ is the image of the element $\sum_{S \in \operatorname{Irr} H} [S] \otimes [S^*] \in \mathscr{R}(H)^{\otimes 2}$ in $\mathscr{R}_\Bbbk(H)^{\otimes 2}$. Consequently, c_δ is integral over $\mathbb{Z}$, because the ring $\mathscr{R}(H)^{\otimes 2}$ is finitely generated as a $\mathbb{Z}$-module and so all its elements are integral over $\mathbb{Z}$. Finally, the character map does in fact give an embedding of symmetric $\Bbbk$-algebras,

$$\chi_{\Bbbk} = \chi \otimes \Bbbk \colon (\mathscr{R}_{\Bbbk}(H), \delta) \hookrightarrow (H^*, \Lambda_0) \tag{17}$$

where $\Lambda_0 = (\dim_{\Bbbk} H)^{-1}\Lambda$ is the unique integral of H satisfying $\langle \varepsilon, \Lambda_0 \rangle = 1$.

2.4. Characters that are Central in H^*. As an application of Proposition 4, we offer the following elegant generalization of Frobenius' Divisibility Theorem due to S. Zhu [**14**, Theorem 8]. Note that the hypothesis $\chi_S \in \mathscr{Z}(H^*)$ is of course automatic for finite group algebras $H = \Bbbk G$, because H^* is commutative in this case.

THEOREM 9 (S. Zhu). *Let H be a semisimple Hopf algebra over a field $\Bbbk$ of characteristic 0. Then $\dim_{\Bbbk} S$ divides $\dim_{\Bbbk} H$ for every absolutely irreducible $S \in \operatorname{Irr} H$ satisfying $\chi_S \in \mathscr{Z}(H^*)$.*

PROOF. Since semisimple Hopf algebras are separable, we may assume that $\Bbbk$ is algebraically closed. Thus, H and H^* are both split semisimple. Choose $\Lambda \in \int_H$ as in (12); so Λ is the character of the regular representation of H^*. Then, with λ as in (13), we have $\gamma_\lambda(1) = \dim_{\Bbbk} H$ and c_λ is given by (14). Thus, Proposition 4 gives the following formula for the primitive central idempotent $\boldsymbol{e}(S) \in \mathscr{Z}H$:

$$\boldsymbol{e}(S)\,\frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} S} \underset{\text{Prop. 4}}{=} \chi_S(\mathcal{S}(\Lambda_{(1)}))\Lambda_{(2)} = \Lambda \leftharpoonup \mathcal{S}^*(\chi_S) = \Lambda \leftharpoonup \chi_{S^*}$$

It suffices to show that the element $\Lambda \leftharpoonup \chi_{S^*} \in \mathscr{Z}H$ is integral over $\mathbb{Z}$. First, note that χ_{S^*} is integral over $\mathbb{Z}$, because this holds for $[S^*] \in \mathscr{R}(H)$. Furthermore, by hypothesis, $\chi_{S^*} = \mathcal{S}^*(\chi_S) \in \mathscr{Z}(H^*)$ and so χ_{S^*} belongs to the integral closure $\mathscr{Z}(H^*)^{\text{int}} := \{f \in \mathscr{Z}(H^*) \mid f \text{ is integral over } \mathbb{Z}\}$. Thus, it suffices to show that all elements of $\Lambda \leftharpoonup \mathscr{Z}(H^*)^{\text{int}}$ are integral over $\mathbb{Z}$. But $\mathscr{Z}(H^*) = \sum_{M \in \operatorname{Irr} H^*} \Bbbk \boldsymbol{e}(M) \cong \Bbbk \times \Bbbk \times \cdots \times \Bbbk$ and so $\mathscr{Z}(H^*)^{\text{int}} = \sum_{M \in \operatorname{Irr} H^*} \mathcal{O}\boldsymbol{e}(M)$, where $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}$ in $\Bbbk$. Furthermore, $\Lambda \leftharpoonup \boldsymbol{e}(M) = (\dim_{\Bbbk} M)\chi_M$ by (8). Therefore, $\Lambda \leftharpoonup \mathscr{Z}(H^*)^{\text{int}} \subseteq \chi(\mathscr{R}(H^*))\mathcal{O}$. Finally, since the ring $\chi(\mathscr{R}(H^*))\mathcal{O}$ is a finitely generated $\mathcal{O}$-module, all its elements are integral over $\mathbb{Z}$, completing the proof. □

2.5. The Class Equation. We now prove the celebrated class equation due to Kac [**6**, Theorem 2] and Y. Zhu [**15**, Theorem 1]; the proof given here is based on [**8**]. Recall that the representation algebra $\mathscr{R}_{\Bbbk}(H)$ embeds into H^* via the character map (17). Thus, for any M in $\mathsf{rep}\,\mathscr{R}_{\Bbbk}(H)$, we may consider the induced module $\operatorname{Ind}_{\mathscr{R}_{\Bbbk}(H)}^{H^*} M$.

THEOREM 10 (Kac, Y. Zhu). *Let H be a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Then $\dim_{\Bbbk} \operatorname{Ind}_{\mathscr{R}_{\Bbbk}(H)}^{H^*} M$ divides $\dim_{\Bbbk} H$ for every M in $\operatorname{Irr} \mathscr{R}_{\Bbbk}(H)$.*

PROOF. This is an application of Corollary 7 to the the character map (17). The main hypotheses have been checked in §2.3, in particular integrality of the Casimir element c_δ over $\mathbb{Z}$. In addition, note that, $\gamma_{\Lambda_0}(\varepsilon) = \dim_{\Bbbk} H$ by (16). Therefore, Corollary 7 applies and yields that the fraction $\frac{\dim_{\Bbbk} H}{\dim_{\Bbbk} \operatorname{Ind}_{\mathscr{R}_{\Bbbk}(H)}^{H^*} M}$ is integral over $\mathbb{Z}$, proving the theorem. □

Frobenius' Divisibility Theorem for finite group algebras $\Bbbk G$ also follows from Theorem 10 applied to $H = (\Bbbk G)^*$, because $\chi_{\Bbbk} \colon \mathscr{R}_{\Bbbk}(H) \xrightarrow{\sim} H^* = \Bbbk G$ in this case.

2.6. Factorizable Hopf Algebras. We remind the reader of some facts about factorizable Hopf algebras. Let H be a Hopf algebra, which need not be finite-dimensional for now. Following Drinfeld [**2**], H is called ***almost cocommutative*** if there is an $R \in (H \otimes H)^{\times}$ satisfying the condition

$$\tau(\Delta(h))R = R\Delta(h) \qquad (h \in H)$$

where $\tau \in \mathrm{Aut}_{\mathsf{Alg}_{\Bbbk}}(H \otimes H)$ is the switch map as in Lemma 1. An almost cocommutative Hopf algebra (H, R) is called ***quasitriangular*** if:

$$\Delta(R^1) \otimes R^2 = R^1 \otimes r^1 \otimes R^2 r^2$$
$$R^1 \otimes \Delta(R^2) = R^1 r^1 \otimes r^2 \otimes R^2$$

Here, elements $t \in H \otimes H$ are symbolically written as $t = t^1 \otimes t^2$, with summation over the superscript being assumed, and we have written $R = R^1 \otimes R^2 = r^1 \otimes r^2$ to indicate two different summation indices. Put $b := \tau(R)R = r^2 R^1 \otimes r^1 R^2 \in H \otimes H$ and define a $\Bbbk$-linear map $\Phi = \Phi_R$ by

$$\begin{array}{cccc} \Phi\colon & H^* & \longrightarrow & H \\ & \cup\!\!\!\!\mid & & \cup\!\!\!\!\mid \\ & f & \longmapsto & b^1\langle f, b^2\rangle \end{array}$$

The quasitriangular Hopf algebra (H, R) is called ***factorizable*** if Φ is bijective. Note that this forces H to be finite-dimensional. An important example of a factorizable Hopf algebra is the Drinfeld double of any finite-dimensional Hopf algebra; see [**11**, Theorem 13.2.1].

Now let us again focus on the case where H is a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ with $\operatorname{char} \Bbbk = 0$. Consider the representation algebra $\mathscr{R}_{\Bbbk}(H)$ and the embedding of symmetric algebras $(\mathscr{R}_{\Bbbk}(H), \delta) \hookrightarrow (H^*, \Lambda_0)$ given by the character map (17) and recall that the image of this map is the algebra $(H/[H,H])^*$ of all trace forms on H. Recall also from (13) that $\lambda = (\dim_{\Bbbk} H)^{-1}\chi_{\mathrm{reg}} \in (H/[H,H])^*$ is a nonzero integral of H^*.

PROPOSITION 11. *Let (H, R) be a factorizable Hopf algebra over an algebraically closed field $\Bbbk$ with $\operatorname{char} \Bbbk = 0$ and assume that H is semisimple. Then the map $\Psi = \Phi \circ \chi_{\Bbbk}\colon (\mathscr{R}_{\Bbbk}(H), \delta) \hookrightarrow (H, \lambda)$ is an embedding of symmetric algebras with image $\operatorname{Im} \Psi = \mathscr{Z} H$.*

PROOF. By [**13**, Theorem 2.3], the restriction of Φ to $(H/[H,H])^*$ is an algebra isomorphism with $\mathscr{Z} H$. So we just need to check that $\lambda \circ \Psi = \delta$ or, equivalently, $\langle \lambda, \Phi(c)\rangle = \langle c, \Lambda_0\rangle$ for all $c \in (H/[H,H])^*$. Since λ and c are trace forms, we compute

$$\begin{aligned}\langle \lambda, \Phi(c)\rangle &= \langle \lambda, b^1\langle c, b^2\rangle\rangle = \langle \lambda, b^1\rangle\langle c, b^2\rangle = \langle \lambda, r^2 R^1\rangle\langle c, r^1 R^2\rangle = \langle \lambda, R^1 r^2\rangle\langle c, R^2 r^1\rangle \\ &= \langle c, b^1\rangle\langle \lambda, b^2\rangle = \langle c, b^1\langle \lambda, b^2\rangle\rangle = \langle c, \Phi(\lambda)\rangle\end{aligned}$$

Thus, it suffices to show that $\Phi(\lambda) = \Lambda_0$, where $\Lambda_0 \in H$ is as in §2.3; so Λ_0 is the unique integral of H satisfying $\langle \varepsilon, \Lambda_0\rangle = 1$.

But $\langle \varepsilon, b^1\rangle b^2 = \langle \varepsilon, r^2\rangle\langle \varepsilon, R^1\rangle r^1 R^2 = 1$, because $R^1\langle \varepsilon, R^2\rangle = 1 = \langle \varepsilon, R^1\rangle R^2$ by [**10**, Proposition 10.1.8]. Hence, for any $f \in H^*$,

$$\langle \varepsilon, \Phi(f)\rangle = \langle \varepsilon, b^1\langle f, b^2\rangle\rangle = \langle \varepsilon, b^1\rangle\langle f, b^2\rangle = \langle f, \langle \varepsilon, b^1\rangle b^2\rangle = \langle f, 1\rangle$$

Using this and the identity $\Phi(fc) = \Phi(f)\Phi(c)$ for $f \in H^*$ and $c \in (H/[H,H])^*$ from [**13**, Theorem 2.1], we obtain, for any $h \in H$,

$$h\Phi(\lambda) = \Phi(\Phi^{-1}(h)\lambda) = \Phi(\langle \Phi^{-1}(h), 1\rangle\lambda) = \langle \Phi^{-1}(h), 1\rangle\Phi(\lambda) = \langle \varepsilon, h\rangle\Phi(\lambda)$$

Thus, $\Phi(\lambda)$ is an integral of H and, since $\langle \varepsilon, \Phi(\lambda)\rangle = \langle \lambda, 1\rangle = 1$, we must have $\Phi(\lambda) = \Lambda_0$ as desired. $\square$

We are now ready to prove the following result of Schneider [**13**, Theorem 3.2].

THEOREM 12 (Schneider). *Let H be a semisimple factorizable Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Then $(\dim_{\Bbbk} S)^2$ divides $\dim_{\Bbbk} H$ for every $S \in \operatorname{Irr} H$.*

PROOF. Consider the primitive central idempotent $\boldsymbol{e}(S) \in \mathscr{Z}H$. The preimage $\Psi^{-1}(\boldsymbol{e}(S))$ is a primitive idempotent in the semisimple algebra $\mathscr{R}_{\Bbbk}(H)$; so $S' := \mathscr{R}_{\Bbbk}(H)\Psi^{-1}(\boldsymbol{e}(S))$ is an irreducible representation of $\mathscr{R}_{\Bbbk}(H)$, with $\operatorname{Ind}^H_{\mathscr{R}_{\Bbbk}(H)}(S') \cong H\boldsymbol{e}(S) \cong \operatorname{End}_{\Bbbk}(S)$. By Proposition 11 and our remarks in §2.3, we can apply Corollary 7 to the map Ψ to get that $\dim_{\Bbbk} \operatorname{Ind}^H_{\mathscr{R}_{\Bbbk}(H)}(S') = (\dim_{\Bbbk} S)^2$ divides $\gamma_\lambda(1) = \dim_{\Bbbk} H$. $\square$

We mention in closing that Schneider's Theorem implies an earlier result of Etingof and Gelaki [**3**], which confirms Kaplansky's conjecture FD for any quasitriangular semisimple Hopf algebra H over an algebraically closed field of characteristic 0. Indeed, the Drinfeld double $D(H)$ is a semisimple factorizable Hopf algebra of dimension $(\dim_{\Bbbk} H)^2$ [**11**, Corollary 13.2.3] which maps onto H [**3**]. Therefore, any $S \in \operatorname{Irr} H$ can be viewed as an irreducible representation of $D(H)$. Theorem 12 gives that $(\dim_{\Bbbk} S)^2$ divides $\dim_{\Bbbk} D(H)$; so $\dim_{\Bbbk} S$ divides $\dim_{\Bbbk} H$.

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