RING THEORY

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ON MAXIMAL IDEALS IN ORE EXTENSIONS

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This article studies the relationship between the maximal (two-sided) ideals of a ring R and the ones of the Ore extension $S = R[x,\alpha]$ obtained by adjoining an indeterminate x subject to the equation

$$x a = a^{\alpha} x , \quad a \in R ,$$

where α is an assigned automorphism of the ring R.

As is well known, Ore extensions have applications to the study of group rings of polycyclic groups. Motivated by J.E. Roseblade's theorem ([5], Theorem A) on the irreducible representations of polycyclic groups over absolute fields (i.e. algebraic extensions of finite fields) or the integers $\mathbb{Z}$, we here restrict ourselves to finitely generated rings R, which are right Noetherian. Under these assumptions on R we show : If every simple homomorphic image of R is finite, and if every prime ideal of R is an intersection of maximal ideals, then both properties are inherited by S (Theorem 5).

Of course S is then a finitely generated right Noetherian ring too. Rings R meeting the hypothesis of the theorem are called <u>strong Jacobson rings</u>. By well known theorems of O.Goldman and W.Krull finitely generated commutative rings are examples.

However, we should like to point out that there are non commutative primitive strong Jacobson rings which are not simple. An example is given at the end of the paper; the particular ring R was used by R. Irving [4] for other purposes before.

As an immediate application of Theorem 5, we obtain an elementary proof of the following weak version of Roseblade's theorem: Let K be an absolute field and G a polycyclic group. Then every simple homomorphic image of the group algebra K[G] is finite dimensional, and every prime ideal of K[G] is an intersection of maximal ideals.

Concerning our notation and terminology, we refer to the article [3] by A.W.Goldie and the second author.

If Y is a right ideal in S, then $\tau(Y)$ denotes the set of elements of R which are leading coefficients of elements of Y, together with zero. Define $\tau_o(Y)$ to be the subset of $\tau(Y)$ corresponding to polynomials of minimal degree in Y. Both, $\tau(Y)$ and $\tau_o(Y)$ are right ideals in R. If Y is a two-sided or α-invariant ideal, then these properties are inherited by $\tau(Y)$ and by $\tau_o(Y)$.

The essential content of the following lemma is the Euclidian algorithm for S.

<u>LEMMA 1</u>. Let I be an α-invariant ideal of S with $I \cap R = (0)$, and let $T = \tau_o(I)$. For any right ideal Y of R such that $YS + I = S$ there exists an integer $k \geq 0$ such that $T^k \subseteq Y$.

<u>Proof</u>. If $I = (0)$, then $YS + I = S$ implies $Y = R \geq T$. Therefore we assume that $I \neq (0)$. Let n be the minimal degree of nonzero polynomials in I. Since $I \cap R = (0)$, we have $n > 0$, and $U = \sum_{i=0}^{n-1} R x^i$ is a right and left R-submodule of S with the following property. If $f(x)$, $g(x)$ are elements of U with $f(x) \equiv g(x) \pmod{I}$, then $f(x) = g(x)$.

We claim that for any $l \geq 0$ there exists $k(l) \geq 0$ such that $x^l T^{k(l)} \subseteq U \pmod{I}$. Suppose that $l < n$, then we may choose $k(l) = 0$, and if $x^l T^{k(l)} \subseteq U \pmod{I}$, then we obtain modulo I

$$x^{l+1} T^{k(l)+1} = x(x^l T^{k(l)}) T \subseteq x U T = \left(\sum_{i=1}^{n} R x^i \right) T$$

$$\subseteq U T + R x^n T \subseteq U \pmod{I},$$

because $x^n T = T x^n \subseteq U \pmod{I}$.

Since $YS + I = S$, there exist elements $r_i \in Y$, $(i = p, p+1, \dots, q; p, q \geq 0)$, and $f(x) \in I$ such that

$$\sum_{i=p}^{q} r_i x^i + f(x) = 1 .$$

Putting $k = k(p) + \dots + k(q)$, we obtain $x^i T^k \subseteq U \pmod{I}$ for all $i = p, \dots, q$. Hence $\left(\sum_{i=p}^{q} r_i x^i\right) T^k \subseteq \sum_{i=p}^{q} r_i U \subseteq \sum_{i=0}^{n-1} Y x^i \pmod{I}$. On the other hand, $\left(\sum_{i=p}^{q} r_i x^i\right) T^k = T^k \pmod{I}$, which implies that $T^k \subseteq Y$.

For any ideal N of R we put $\mathcal{D}(N) = \bigcap_{i \in \mathbb{Z}} N^{\alpha^i}$. Then $\mathcal{D}(N)$ is an α-invariant ideal of R, and $\mathcal{D}(N)S$ is an ideal of S.

<u>LEMMA 2</u>. Suppose every α-prime ideal in R is an intersection of maximal ideals. Then for any α-invariant maximal ideal M of S there exists a maximal ideal M_1 of R such that

$$\mathcal{D}(M_1) = M \cap R .$$

<u>Proof</u>. Using the isomorphism $S / (M \cap R)S \cong (R / M \cap R)[x, \alpha]$ (cf. [3], Lemma 1.1) we may assume that $M \cap R = (0)$. In particular, R is then an α-prime ring, by [3], Lemma 1.3. Since $T = \tau_0(M)$ is a nonzero ideal of R, there exists a maximal ideal M_1 of R such that $M_1 \not\supseteq T$ by hypothesis. We show that $\mathcal{D}(M_1) = (0)$, which will prove the lemma.

Suppose the contrary. Then $\mathcal{D}(M_1)S + M = S$ by the maximality of M. Using Lemma 1 we deduce that $\mathcal{D}(M_1) \supseteq T^k$ for some integer $k \geq 0$. But this implies $M_1 \supseteq T$, contradicting the choice of M_1.

The next lemma is stated in a slightly more general form than we actually need it.

<u>LEMMA 3</u>. Let R be any finitely generated ring and let n be a positive integer. Then there are at most finitely many right ideals Y of R such that $\mathrm{card}(R/Y) = n$.

<u>Proof</u>. Let Y be such a right ideal of R , and let V be the additive abelian group $(R/Y,+)$. The R-module structure on V gives rise to a ring homomorphism $f : R \to \mathrm{End}(V)$. Clearly, $R/\ker f$ is a finite ring, since $\mathrm{End}(V)$ is finite, and $Y/\ker f$ is one of the finitely many right ideals of $R/\ker f$. Therefore our assertion will follow as soon as we have shown that there are only finitely many possibilities for the ideal $\ker f$ in R .

Now there are only finitely many isomorphism types $V_1, V_2, \dots, V_r$ of abelian groups of order n , and for each V_i there exist at most finitely many ring homomorphisms of R into $\mathrm{End}(V_i)$, because such a homomorphism is determined by the images of the finitely many generators of the ring R . If $V \cong V_i$, say, then f yields a homomorphism $f_i : R \longrightarrow \mathrm{End}(V_i)$ having the same kernel as f . Hence there are only finitely many possibilities for $\ker f$.

We remark that the argument does not really require R to be finitely generated. It is enough to assume that there exists a categorical ring epimorphism $T \to R$ for some finitely generated ring T . All of the rest works for this slightly more general class of rings.

LEMMA 4. Let R be a finitely generated ring, and let M be a maximal ideal of R such that R/M is finite. Then

(i) $R/\mathcal{D}(M)$ is finite, and

(ii) the ideal $\mathcal{D}(M)S$ of S is an intersection of maximal ideals of S.

Proof. Since R/M^{α^i} is isomorphic to R/M, we conclude from Lemma 3 that there are only finitely many α-conjugates M^{α^i} of M. Hence assertion (i) follows.

For part (ii), we consider the ring $\bar{S} = S/\mathcal{D}(M)S$ which is isomorphic to $R/\mathcal{D}(M)[x,\alpha]$. Here the coefficient ring $\bar{R} = R/\mathcal{D}(M)$ is a finite α-prime and semiprimitive ring, by part (i). An easy argument shows that $\bar{S} = \bar{R}[x,\alpha]$ is semiprimitive (see f.i. [3], proof of Theorem 1.11). Furthermore, the finiteness of $\bar{R}$ implies that some power α^k, $k > 0$, acts trivially on $\bar{R}$. Therefore, the subring C of $\bar{S}$ which is generated by 1 and x^k is central. $\bar{S}$ is clearly a finitely generated module over C. By [2], Theorem 4, every simple right $\bar{S}$-module V is semisimple of finite length as a right C-module. But the simple C-modules obviously are finite. Therefore V is finite too. In particular, each primitive homomorphic image of $\bar{S}$ is finite. Hence it is simple. This proves the second assertion.

After these preparations we can prove the main result of this note.

THEOREM 5. Let R be a finitely generated right Noetherian ring. If R is a strong Jacobson ring, then $S = R[x,\alpha]$ is a strong Jacobson ring too.

Proof. We first show that every simple homomorphic image of S is finite. So let M be a maximal ideal of S. If $x \in M$, then $S/M \cong R/(M \cap R)$, by [3], Lemma 1.2. Thus S/M is finite. We may therefore assume that $x \notin M$. Then M is α-invariant and $M \cap R$ is an α-prime ideal of R, by [3], Lemmas 1.2 and 1.3. Passing from S to $S/(M \cap R)S \cong (R/M \cap R)[x,\alpha]$ we may assume that $M \cap R = (0)$. By assumption on R also every α-prime ideal of R is an intersection of maximal ideals. By application of Lemma 2 we therefore obtain a maximal ideal M_1 of R such that $\mathcal{D}(M_1) = (0)$. The assumption on R together with Lemma 4 imply that R is a finite α-prime ring, and its maximal ideals are precisely the finitely many conjugates of M_1. In particular, R contains no non-trivial α-invariant ideals. Consequently, since $\tau(M)$ is α-invariant and nonzero, we conclude that $1 \in \tau(M)$. Thus there exists an element of the form $f(x) = f_0 + f_1 x + \ldots + f_{n-1} x^{n-1} + x^n \in M$. The Euclidian algorithm now implies that

$$S/M = R + R(x+M) + \ldots + R(x+M)^{n-1} .$$

Therefore S/M is finite, as R is.

It remains to show that every prime ideal of S can be written as an intersection of maximal ideals. So let P be a prime ideal of S and denote the intersection of the maximal ideals of S containing P by $\mathcal{M}(P)$. As in the first part of the proof we may assume that P is α-invariant, $P \cap R = (0)$, and R is α-prime. We first consider the case $P = (0)$. Let M be a maximal ideal of R. By Lemma 4, $\mathcal{D}(M)S$ is an intersection of maximal ideals in S. Hence $\mathcal{M}(0) \subseteq \mathcal{D}(M)S$. As $\bigcap_{M \in \mathcal{M}ax\, R} \mathcal{D}(M) = (0)$, where $\mathcal{M}ax\, R$ denotes the set of all maximal ideals of R, it follows that

$$\mathcal{M}(0) \subseteq \bigcap_{M \in \mathcal{M}ax\, R} \mathcal{D}(M)S = \Big(\bigcap_{M \in \mathcal{M}ax\, R} \mathcal{D}(M)\Big)S = (0) .$$

This proves the assertion in the case $P = (0)$, and we are left with the case that $P \neq (0)$. We distinguish two types of maximal ideals M in R .

If $\mathcal{D}(M)S + P \neq S$, then there exists a maximal ideal M_1 of S such that $M_1 \supseteq P$ and $M_1 \cap R \subseteq M$. For we can certainly find a maximal ideal M' of S with $M' \supseteq \mathcal{D}(M)S + P$. Hence $M' \supseteq P$ and $M' \cap R \supseteq \mathcal{D}(M)$. The latter inclusion shows that $M' \cap R \subseteq M^{\alpha^i}$ for some i , because, by Lemma 4, the only maximal ideals of R sitting above $\mathcal{D}(M)$ are the α-conjugates of M . Put $M_1 = (M')^{\alpha^{-i}}$. Then M_1 has the desired properties. Fix such a maximal ideal M_1 for each $M \in \mathcal{M}ax\, R$ with $\mathcal{D}(M)S + P \neq S$.

If $\mathcal{D}(M)S + P = S$, then $M \supseteq \tau_o(P)$, by Lemma 1. Let $\mathcal{M}_1$ denote the set of maximal ideals M of R satisfying $\mathcal{D}(M)S + P \neq S$ and let $\mathcal{M}_2$ denote the set of the remaining maximal ideals of R . Then both $\mathcal{M}_1$ and $\mathcal{M}_2$ are α- invariant subsets of $\mathcal{M}ax\, R$. Hence $\bigcap_{M \in \mathcal{M}_1} M$ and $\bigcap_{M \in \mathcal{M}_2} M$ are α - invariant ideals of R with zero intersection by hypothesis on R . Since the latter ideal contains the nonzero ideal $\tau_o(P)$, the α - primeness of R forces $\bigcap_{M \in \mathcal{M}_1} M$ to be zero. Thus

$$\mathcal{M}(P) \cap R \subseteq \Big(\bigcap_{M \in \mathcal{M}_1} M_1 \Big) \cap R \subseteq \bigcap_{M \in \mathcal{M}_1} M = (0) .$$

Therefore $\mathcal{M}(P) = P$ by Lemma 1.10 of [3] . This completes the proof.

This result has an immediate application to group rings of polycyclic groups G . Recall that a group G is polycyclic if and only if it has a finite subnormal series with cyclic factors. Hence, one can construct the group ring $R[G]$ from R by a finite number of Ore extensions and by forming homomorphic images.

Thus an induction based on Theorem 5 immediately gives the first part of the following

COROLLARY 6. Let G be a polycyclic group.

(i) If R is a finitely generated right Noetherian strong Jacobson ring, then $R[G]$ inherits the same properties.

(ii) Let K be an absolute field. Then every simple homomorphic image of $K[G]$ is finite-dimensional over K, and every prime ideal of $K[G]$ is an intersection of maximal ideals.

Proof. Only part (ii) has to be proved. Assume that K is an absolute field. Let M be a maximal ideal of $K[G]$. Since M is finitely generated as a right ideal of $K[G]$, we can find elements $m_1, m_2, \ldots, m_r \in M$ such that $M = m_1 K[G] + \ldots + m_r K[G]$. Let K' denote the subfield of K generated by the coefficients involved in the m_i $(i = 1,2,\ldots, r)$. Put $M' = m_1K'[G] + \ldots + m_rK'[G]$. Then K' is a finite field, $M' = M \cap K'[G]$ and $M = M'K$. In particular, M' is a maximal ideal of $K'[G]$, and $K[G]/M \cong (K'[G]/M') \otimes_{K'} K$. Thus in order to show that $K[G]/M$ is finite-dimensional, we may restrict ourselves to the case of a finite field K. But then $K[G]$ is a strong Jacobson ring, by part (i). Thus $K[G]/M$ is finite-dimensional over K.

Now let P be a prime ideal of $K[G]$. As in the first part of the proof, we can find a finite subfield K' of K and an ideal P' of $K'[G]$ such that $P = P'K$ and $P' = P \cap K'[G]$. Clearly, P' is a prime ideal of $K'[G]$ and again by part (i) we conclude that P' can be written as an intersection of maximal ideals, $P' = \bigcap_{\alpha \in A} M'_\alpha$. The ideals $M_\alpha = M'_\alpha K$ of $K[G]$ have finite codimension in K[G]. Furthermore, since K/K' is a separable field extension, well-known arguments show that each M_α is a semiprime ideal in K[G] (cf. [1], p.111). Consequently, the ideals M_α are intersections of maximal ideals in K[G] and hence the same holds for $P = \bigcap_{\alpha \in A} M_\alpha$. Thus the corollary is proved.

Of course, it would be nice to study primitive homomorphic images of K[G] via Ore extensions and show that these are simple for K absolute. However, this certainly cannot be deduced from the strong Jacobson property alone as the following example of a primitive non-simple strong Jacobson ring shows.

<u>EXAMPLE</u> (cf. R.Irving [4]). Let A be the ring $A = \mathbb{Z}[\frac{1}{2}, y, y^{-1}]$ and let T be the multiplicatively closed subset generated by the set of elements $\{y + 2n : n \in \mathbb{Z}\}$. Set $R = A_T$, the localization of A at the elements of T. The ring R has an automorphism α defined by $\alpha(y) = y + 2$. We let $S = R[x, x^{-1}; \alpha]$ and show that S has all the required properties. First, we infer from the work of R.Irving, that S is a finitely generated Noetherian Jacobson ring and is primitive ([4], Theorem 3).

(1) The prime ideals of R and their α-orbits are easily described: One has $\dim R = 2$. If P is a prime of height 2 in R, then $P \cap \mathbb{Z}[y]$ is maximal in $\mathbb{Z}[y]$ and hence $\mathbb{Z}[y] / P \cap \mathbb{Z}[y]$ is finite and $P \cap \mathbb{Z}[y]$ has only finitely many α-conjugates. (Observe that α defines an automorphism of $\mathbb{Z}[y]$.) It follows that P has finite orbit under α and that R/P is finite. Now let P be a height 1 prime in R. Then $P \cap \mathbb{Z}[y]$ also has height 1.

If $P \cap \mathbb{Z} \neq (0)$, then $P = pR$ for some prime $p \neq 2$ in $\mathbb{Z}$ and P is α-invariant. If $P \cap \mathbb{Z} = (0)$, then $P = f(y)R$ for some irreducible polynomial $f(y) \in \mathbb{Z}[y]$. It follows easily, that P has an infinite α-orbit in this case. - Thus the prime ideals of R with finite orbit under α are (0), the ideals pR for $p \neq 2$ a prime in $\mathbb{Z}$ and the primes of height 2, the latter ones having finite factor rings.

(2) We now show that the set of primitive ideals in S consists of (0) together with the maximal ideals of S and that for each $M \in \mathcal{M}ax\, S$ the factor S/M is finite: As we remarked above, the zero ideal of S is in fact primitive. So let P be a nonzero primitive ideal of S. Then since α has infinite order, the intersection $P \cap R$ is nonzero and is an α-prime ideal of R. Therefore, $P \cap R$ can be written as $P \cap R = Q \cap Q^{\alpha} \cap \ldots \cap Q^{\alpha^r}$, where Q is a nonzero prime of R and the intersection is over the full α-orbit of Q. By (1), Q either has height 2 or $Q = pR$ for some prime $p \neq 2$ in $\mathbb{Z}$. In the former case, $R/P \cap R$ is finite and it follows that S/P is finite (Lemma 4). So assume now that $Q = pR$. Then, since Q is α-invariant, we obtain that $P \cap R = pR$ and $\overline{S} = S/(P \cap R)S \cong R/pR[x, x^{-1}; \alpha]$. Here $\overline{R} = R/pR$ has characteristic p and hence α^p acts trivially on $\overline{R}$. Thus $\overline{S}$ is finitely generated as a left (and right) module over the ordinary Laurent polynomial ring $\overline{S}_1 = \overline{R}[x^p, x^{-p}]$. It follows from [2], Theorem 4, that every irreducible right $\overline{S}$-module is completely reducible of finite length when considered as an $\overline{S}_1$-module. In particular, letting $\overline{P}$ denote the image of P in $\overline{S}$ we obtain that $\overline{P} \cap \overline{S}_1$ can be written as an intersection of finitely many maximal ideals of $\overline{S}_1$.

But it is not hard to show that every maximal ideal of $\overline{S}_1 = \overline{R}[x^p, x^{-p}]$ intersects $\overline{R}$ in a maximal ideal. Consequently $\overline{P} \cap \overline{R}$ is nonzero, a contradiction. So the case $Q = pR$ cannot occur and the assertions follow.

(3) It remains to show that every prime ideal of S is an intersection of maximal ideals. But since S is a Jacobson ring, it follows from (2), that at least every nonzero prime of S is an intersection of maximal ideals. Furthermore for each prime $p \neq 2$ in $\mathbb{Z}$, the ideal pS is prime in S, and $\bigcap_{p \neq 2} pS = (0)$.

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