Communications in Algebra®, 39: 4733–4750, 2011

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2011.617619

PROJECTIVE MODULES OVER FROBENIUS ALGEBRAS AND HOPF COMODULE ALGEBRAS

Martin Lorenz[1] **and Loretta Fitzgerald Tokoly**[2]
[1]*Department of Mathematics, Temple University, Philadelphia, Pennsylvania, USA*
[2]*Mathematics Division, Howard Community College, Columbia, Maryland, USA*

This note presents some results on projective modules and the Grothendieck groups K_0 and G_0 for Frobenius algebras and for certain Hopf Galois extensions. Our principal technical tools are the Higman trace for Frobenius algebras and a product formula for Hattori-Stallings ranks of projectives over Hopf Galois extensions.

Key Words: Cartan map; Character; Comodule algebra; Frobenius algebra; Grothendieck group; Higman trace; Hopf algebra; Hopf Galois extension; Projective module; (Hattori–Stallings) rank.

2000 Mathematics Subject Classification: 16W30; 16W35; 16G10; 20G42.

Dedicated to Mia Cohen on the occasion of her retirement.

INTRODUCTION

The aim of this article is to generalize certain results from [10] on projective modules and the Grothendieck groups K_0 and G_0 for finite-dimensional Hopf algebras. Here, we take a more ring theoretic approach and consider general Frobenius algebras and Hopf Galois extensions instead of finite-dimensional Hopf algebras. Moreover, for the most part, we work over a commutative base ring rather than a field.

Section 1 serves to review some fairly standard material, notably the relationship between Hattori–Stallings ranks and ordinary characters of projective modules. This relationship is stated in the context of the Grothendieck groups $G_0(A) = K_0(A\text{-mod})$ and $K_0(A) = K_0(A\text{-proj})$, where A-mod denotes the category of all finitely generated left modules over the ring A and A-proj is the full subcategory consisting of all finitely generated projective left A-modules. Bass [2] and Brown [3, IX.2] are excellent background references for this section.

The core material of this article consists of Sections 2 and 3 which are largely independent of each other. Section 2 deals with Frobenius algebras A over a commutative ring $\Bbbk$, the main theme being the *Higman trace*

$$\tau : A \longrightarrow A;$$

Received August 30, 2010; Revised February 1, 2011. Communicated by S. Montgomery.
Address correspondence to Martin Lorenz, Department of Mathematics, Temple University, Philadelphia, PA 19122, USA; E-mail: lorenz@temple.edu

see 2.4. The main result of Section 2 is Theorem 3, a generalization of [10, Theorem 3.4]. It concerns the so-called *Cartan map* $c : K_0(A) \to G_0(A)$ coming from the inclusion $A\text{-proj} \hookrightarrow A\text{-mod}$. Under the assumption that $\Bbbk$ is a splitting field for A, we show that the rank of the $\Bbbk$-linear map

$$c \otimes 1_{\Bbbk} : K_0(A) \otimes_{\mathbb{Z}} \Bbbk \longrightarrow G_0(A) \otimes_{\mathbb{Z}} \Bbbk$$

is identical to the rank of the Higman trace. With an additional technical hypothesis on the Higman trace, Theorem 3 also states that invertibility of $c \otimes 1_{\Bbbk}$ implies semisimplicity of A.

Section 3 is based on some results from the second author's Ph.D. thesis [14]. The main result of this section, Theorem 9, gives a condition on the possible ranks of finitely generated projectives over certain Hopf Galois extensions which generalizes [10, Theorem 2.3(b)]. The proof of Theorem 9 given here is different from the original one in [14], the essential new ingredient being a product formula for Hattori–Stallings ranks; see Lemma 7. This allowed for a more general version of the result.

1. RANKS AND CHARACTERS

1.1. Hattori-Stallings Ranks

Let A be any ring (associative with 1). We let $[A, A]$ denote the additive subgroup of A that is generated by all Lie commutators $[x, y] = xy - yx$ with $x, y \in A$ and consider the canonical group epimorphism

$$T : A \twoheadrightarrow T(A) = A/[A, A], \qquad a \mapsto T(a) = a + [A, A].$$

Now let P be a finitely generated projective (left) A-module. The *trace map* is defined by

$$\begin{array}{cccc} \mathrm{Tr}_{P/A} : \mathrm{End}_A(P) \xrightarrow{\sim} & \mathrm{Hom}_A(P, A) \otimes_A P & \longrightarrow & T(A) \\ & \cup\!\!\!| & & \cup\!\!\!| \\ & f \otimes v & \longmapsto & T(f(v)) \,. \end{array}$$

If $\{(f_i, v_i)\}_1^n \subseteq \mathrm{Hom}_A(P, A) \times P$ are dual bases for P, that is, $v = \sum_i f_i(v)v_i$ holds for all $v \in P$, then

$$\mathrm{Tr}_{P/A}(\phi) = \sum_i T(f_i(\phi(v_i))) \qquad (\phi \in \mathrm{End}_A(P)). \tag{1}$$

The *Hattori–Stallings rank* of P is defined by

$$r(P) = r_A(P) := \mathrm{Tr}_{P/A}(1_P) = \sum_i T(f_i(v_i)) \in T(A).$$

In particular, if $P \cong A^n$, then $r(P) = nT(1)$.

Hattori–Stallings ranks are additive, that is, $r(P \oplus Q) = r(P) + r(Q)$ holds for any two finitely generated projective A-modules P and Q. Thus we obtain a group homomorphism

$$r = r_A : K_0(A) \to T(A), \qquad [P] \mapsto r(P).$$

1.1.1. Functoriality. Given any ring homomorphism $f : A \to B$, the canonical group homomorphisms $K_0(f) = \mathrm{Ind}_A^B : K_0(A) \to K_0(B)$, $[P] \mapsto [B \otimes_A P]$ ("induction"), and $T(f) : T(A) \to T(B)$, $T(a) \mapsto T(f(a))$, fit into a commutative diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{K_0(f)} & K_0(B) \\ {\scriptstyle r_A}\downarrow & & \downarrow{\scriptstyle r_B} \\ T(A) & \xrightarrow[T(f)]{} & T(B)\,. \end{array} \tag{2}$$

1.1.2. Commutative Rings. For any commutative ring A, the Hattori–Stallings rank function $r : K_0(A) \to T(A) = A$ of Section 1.1 factors through the rank map

$$\mathrm{rank} : K_0(A) \longrightarrow H_0(A) := [\mathrm{Spec}\, A, \mathbb{Z}],$$

where $[\mathrm{Spec}\, A, \mathbb{Z}]$ denotes the collection of all continuous functions $\mathrm{Spec}\, A \to \mathbb{Z}$ with $\mathbb{Z}$ carrying the discrete topology. For any P in A-proj, the value of $\mathrm{rank}(P)$ on $\mathfrak{p} \in \mathrm{Spec}\, A$, denoted by $\mathrm{rank}_{\mathfrak{p}}(P)$, is defined to be the ordinary rank of the free $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A P$:

$$P_{\mathfrak{p}} \cong A_{\mathfrak{p}}^{\mathrm{rank}_{\mathfrak{p}}(P)};$$

Any continuous function $f : \mathrm{Spec}\, A \to \mathbb{Z}$ has only finitely many values, because $\mathrm{Spec}\, A$ is quasi-compact. If f has values $f_1, \ldots, f_c$, say, then we can write $A = \prod_{i=1}^{c} e_i A$ with orthogonal idempotents $e_i = e_i^2 \in A$ in such a way that the various $\mathrm{Spec}\, e_i A$ are exactly the fibres of f; see [1, IX.3] for all this. Defining $H_0(A) \to A$ by $f \mapsto \sum f_i e_i$ it is easy to see that the Hattori-Stallings rank function factors as

$$r : K_0(A) \xrightarrow{\mathrm{rank}} H_0(A) \to A;$$

see [15, Chapter II].

1.1.3. Algebras. If A is an algebra over some commutative base ring $\Bbbk$, then $T(A)$ is a $\Bbbk$-module and the homomorphism r extends canonically to a $\Bbbk$-module map

$$r_{\Bbbk} : K_0(A) \otimes_{\mathbb{Z}} \Bbbk \to T(A), \qquad [P] \otimes k \mapsto kr(P). \tag{3}$$

1.2. Characters

Assume that A is an algebra over some commutative base ring $\Bbbk$, and let $A\text{-mod}_{\Bbbk}$ denote the full subcategory of A-mod consisting of all A-modules that are finitely generated projective over $\Bbbk$. The *character* χ_V of a module V in $A\text{-mod}_{\Bbbk}$ is defined by

$$\chi_V(a) = \mathrm{Tr}_{V/\Bbbk}(a_V) \in \Bbbk \qquad (a \in A),$$

where $a_V \in \mathrm{End}_{\Bbbk}(V)$ is given by $a_V(v) = av$. Thus,

$$\chi_V \in T(A)^* \subseteq A^*,$$

where $.^* = \mathrm{Hom}_{\Bbbk}(., \Bbbk)$ denotes the $\Bbbk$-linear dual. Throughout, we will identify the $\Bbbk$-linear dual $T(A)^*$ of $T(A)$ with the $\Bbbk$-submodule of A^* consisting of all $\Bbbk$-linear maps $A \to \Bbbk$ that vanish on $[A, A]$; these maps will be referred to as *trace forms* on A. Following Swan [13], we let

$$G_0^{\Bbbk}(A) = K_0(A\text{-mod}_{\Bbbk})$$

denote the Grothendieck group of $A\text{-mod}_{\Bbbk}$. Thus $G_0^{\Bbbk}(A)$ is the abelian group with generators $[V]$ for each module V in $A\text{-mod}_{\Bbbk}$ and relations $[V] = [U] + [W]$ for each exact sequence $0 \to U \to V \to W \to 0$ in $A\text{-mod}_{\Bbbk}$. Since we also have the relation $\chi_V = \chi_U + \chi_W$ in $T(A)^*$, we obtain a well-defined group homomorphism

$$\chi : G_0^{\Bbbk}(A) \longrightarrow T(A)^*, \qquad [V] \mapsto \chi_V.$$

1.3. Characters of Projectives

Assume that the algebra A is finitely generated projective over $\Bbbk$. Then each finitely generated projective A-module P belongs to $A\text{-mod}_{\Bbbk}$, and hence both $r(P) \in T(A)$ and $\chi_P \in T(A)^*$ are defined. In fact, the Hattori–Stallings rank $r(P)$ determines the character χ_P. For finite group algebras $A = \Bbbk G$ this was spelled out explicitly by Hattori [5]; see also [2, 5.8]. The proposition below is taken from Bass [2, 4.7].

The inclusion $A\text{-proj} \hookrightarrow A\text{-mod}_{\Bbbk}$ gives rise to a group homorphism

$$c^{\Bbbk} : K_0(A) \to G_0^{\Bbbk}(A), \qquad [P] \mapsto [P].$$

If the base ring $\Bbbk$ is regular, then the inclusion $A\text{-mod}_{\Bbbk} \hookrightarrow A\text{-mod}$ gives rise to an isomorphism $G_0^{\Bbbk}(A) \xrightarrow{\sim} G_0(A)$; see [13, Theorem 1.2]. Thus, identifying $G_0^{\Bbbk}(A)$ and $G_0(A)$ for regular $\Bbbk$, the map $c^{\Bbbk}$ becomes the ordinary *Cartan map*

$$c : K_0(A) \to G_0(A).$$

A map $T(A) \to T(A)^*$ is obtained by sending $a \in A$ to the linear form $b \mapsto \mathrm{Tr}_{A/\Bbbk}(L_b \circ R_a)$ on A. Here, $R_a, L_b \in \mathrm{End}_{\Bbbk}(A)$ denote right and left multiplication by a and b, respectively. Note that if a or b belongs to $[A, A]$, then $L_b \circ R_a \in$

$[\mathrm{End}_{\Bbbk}(A), \mathrm{End}_{\Bbbk}(A)]$, and so $\mathrm{Tr}_{A/\Bbbk}(L_b \circ R_a) = 0$. Therefore, $\mathrm{Tr}_{A/\Bbbk}(L_b \circ R_a)$ depends only on $T(a)$ and $T(b)$, and we obtain a well-defined $\Bbbk$-linear map

$$.^t : T(A) \to T(A)^* \hookrightarrow A^*, \qquad T(a) \mapsto (b \mapsto \mathrm{Tr}_{A/\Bbbk}(L_b \circ R_a)). \tag{4}$$

Proposition 1 (Bass [2]). *Let A be a $\Bbbk$-algebra that is finitely generated projective over $\Bbbk$. Then the following diagram commutes*:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{c^{\Bbbk}} & G_0^{\Bbbk}(A) \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle \chi} \\ T(A) & \xrightarrow[.^t]{} & T(A)^* . \end{array}$$

1.4. Finite-Dimensional Algebras over a Field

Now let A be a finite-dimensional algebra over a field $\Bbbk$, and let $\mathrm{rad}\, A$ denote the Jacobson radical of A. Since $\mathrm{rad}\, A$ is nilpotent, the character χ_V of any V in A-mod vanishes on $\mathrm{rad}\, A$, and hence the character map $\chi : G_0(A) \to T(A)^*$ of Section 1.2 actually takes values in $T(A/\mathrm{rad}\, A)^* \subseteq T(A)^*$. By $\Bbbk$-linear extension of χ, we obtain a map

$$\widetilde{\chi} : G_0(A) \otimes_{\mathbb{Z}} \Bbbk \longrightarrow T(A/\mathrm{rad}\, A)^*.$$

For the same reason, the map $.^t$ in (4) takes values in $T(A/\mathrm{rad}\, A)^*$, and it factors through $T(A/\mathrm{rad}\, A)$. Hence $.^t$ factors through a $\Bbbk$-linear map

$$\widetilde{.^t} : T(A/\mathrm{rad}\, A) \longrightarrow T(A/\mathrm{rad}\, A)^*.$$

Finally, we have the composite

$$\tilde{r} : K_0(A) \otimes_{\mathbb{Z}} \Bbbk \xrightarrow{r_{\Bbbk}} T(A) \overset{\text{can.}}{\twoheadrightarrow} T(A/\mathrm{rad}\, A)$$

with $r_{\Bbbk}$ as in (3). By Proposition 1, these maps fit into a commutative diagram of $\Bbbk$-linear maps

$$\begin{array}{ccc} K_0(A) \otimes_{\mathbb{Z}} \Bbbk & \xrightarrow{c \otimes 1_{\Bbbk}} & G_0(A) \otimes_{\mathbb{Z}} \Bbbk \\ {\scriptstyle \widetilde{r}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\chi}} \\ T(A/\mathrm{rad}\, A) & \xrightarrow[\widetilde{.^t}]{} & T(A/\mathrm{rad}\, A)^* . \end{array} \tag{5}$$

where c is the Cartan map from Section 1.3. If $\Bbbk$ is a *splitting field* for A, that is, $\mathrm{End}_A(V) = \Bbbk$ holds for all irreducible A-modules V, then both $\widetilde{r}$ and $\widetilde{\chi}$ are isomorphisms; see [10, 1.6].

2. FROBENIUS ALGEBRAS

In this section, we give an alternative description of the map (4) in the special case where A is a Frobenius algebra over a commutative base ring $\Bbbk$ and derive various consequences for the Cartan map.

2.1. Frobenius Algebras

We briefly recall some basics concerning Frobenius algebras refering to [9] for additional information and some details that are omitted below.

The dual $A^* = \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$ carries a standard (A, A)-bimodule structure:

$$(afb)(x) = f(bxa) \qquad (a, b, x \in A, f \in A^*).$$

The $\Bbbk$-algebra A is said to be Frobenius if A is finitely generated projective over $\Bbbk$ and A is isomorphic to A^* as left A-module or, equivalently, as right A-module. These isomorphism amount to the existence of a nonsingular associative $\Bbbk$-bilinear form $\beta : A \times A \to \Bbbk$. Given such a form β, we obtain an isomorphism of left A-modules

$$\boldsymbol{l}_\beta : {}_AA \stackrel{\sim}{\longrightarrow} {}_AA^*, \qquad a \mapsto \beta(., a).$$

In place of β, one can equally well work with the so-called *Frobenius homomorphism*

$$\lambda = \lambda_\beta = \beta(., 1) = \beta(1, .) \in A^*.$$

Indeed, $\beta(a, b) = \lambda(ab)$ holds for all $a, b \in A$. The isomorphism $\boldsymbol{l}_\beta$ then takes the form

$$\boldsymbol{l}_\beta(a) = a\lambda. \tag{6}$$

The linear form λ is a free generator of A^* as both left and as right A-module; see, e.g., [9, 1.1.1]. The automorphism $\alpha = \alpha_\beta \in \mathrm{Aut}_{\Bbbk\text{-alg}}(A)$ that is given by

$$\lambda a = \alpha(a)\lambda \qquad (a \in A) \tag{7}$$

is called the *Nakayama automorphism* that is associated to β.

2.2. Change of Bilinear Form

If $\beta, \beta' : A \times A \to \Bbbk$ are two nonsingular associative $\Bbbk$-bilinear forms, then the isomorphism $\boldsymbol{l}_\beta^{-1} \circ \boldsymbol{l}_{\beta'} : {}_AA \stackrel{\sim}{\longrightarrow} {}_AA$ is given by right multiplication by some unit $u \in A^\times$. Hence, $\beta'(., .) = \beta(., .u)$. The Frobenius homomorphisms $\lambda' = \boldsymbol{l}_{\beta'}(1)$ and $\lambda = \boldsymbol{l}_\beta(1)$ are related by $\lambda' = u\lambda$, and the Nakayama automorphisms α and α' that are associated with β and β', respectively, differ by an inner automorphism: $\alpha'(a) = u\alpha(a)u^{-1}$.

2.3. Dual Bases

Let A be a Frobenius $\Bbbk$-algebra with nonsingular associative $\Bbbk$-bilinear form β as in 2.1. In view of the canonical isomorphism $\operatorname{End}_{\Bbbk}(A) \cong A \otimes_{\Bbbk} A^*$, the isomorphism $\boldsymbol{l}_\beta$ yields an isomorphism $\operatorname{End}_{\Bbbk}(A) \xrightarrow{\sim} A \otimes_{\Bbbk} A$. Writing the image of $1_A \in \operatorname{End}_{\Bbbk}(A)$ under this isomorphism as $\sum_i x_i \otimes y_i \in A \otimes_{\Bbbk} A$, we have

$$a = \sum_i \beta(a, y_i)x_i = \sum_i \lambda(ay_i)x_i \qquad (a \in A). \tag{8}$$

The elements $\{x_i\}, \{y_i\}$ of A are usually referred to as *dual bases* for β. The first equation in (8) is equivalent to

$$a = \sum_i \beta(x_i, a)y_i \qquad (a \in A); \tag{9}$$

see [9, equation (8)].

2.4. The Higman Trace

Let A be a Frobenius $\Bbbk$-algebra with nonsingular associative $\Bbbk$-bilinear form β as in 2.1. Since the element $\sum_i x_i \otimes y_i \in A \otimes_{\Bbbk} A$ is completely determined by β, the map

$$\tau = \tau_\beta : A \to A, \qquad a \mapsto \sum_i x_i a y_i \tag{10}$$

only depends on β and not on the choice of dual bases $\{x_i\}, \{y_i\}$ for β. Furthermore, τ is clearly $\mathscr{Z}(A)$-linear, where $\mathscr{Z}(A)$ denotes the center of A.

Part (a) of the following lemma gives the desired description of the map (4); part (b) will not be needed in this article but may be of independent interest.

Lemma 2. *Let (A, β) be a Frobenius algebra with Frobenius homomorphism $\lambda = \lambda_\beta \in A^*$ and Nakayama automorphism $\alpha = \alpha_\beta$, and let τ be as in* (10). *Then*:

(a) *$T(a)^t = \tau(a)\lambda$ holds or all $a \in A$. In particular, τ vanishes on* $[A, A]$;
(b) *$\beta(\tau(a), b) = \beta(a, \tau(b))$ and $a\tau(b) = \tau(b)\alpha(a)$ for all $a, b \in A$. Moreover, $\alpha\tau = \tau\alpha$.*

Proof. (a) Equation (1) gives

$$\operatorname{Tr}_{A/\Bbbk}(\phi) = \sum_i \beta(\phi(x_i), y_i) = \sum_i \lambda(\phi(x_i)y_i)$$

for any $\phi \in \operatorname{End}_{\Bbbk}(A)$. Applying this to the endomorphism $\phi = L_b \circ R_a$ in (4), we obtain

$$\operatorname{Tr}_{A/\Bbbk}(L_b \circ R_a) = \sum_i \lambda(bx_i a y_i) = \lambda(b\tau(a))$$

for $a, b \in A$. This proves the asserted formula for $T(a)^t$. Since $T([A, A])^t = 0$, it follows that $\tau(.)\lambda$ vanishes on $[A, A]$. Finally, since $\tau(.)\lambda = \boldsymbol{l}_\beta \circ \tau$ by (6), injectivity of $\boldsymbol{l}_\beta$ implies that τ vanishes on $[A, A]$.

(b) Equation (7) says that

$$\beta(a, b) = \beta(b, \alpha(a)) \qquad (a, b \in A).$$

Hence (9) can be written as $a = \sum_i \beta(a, \alpha(x_i))y_i$. It follows that

$$\sum_i x_i \otimes y_i = \sum_i y_i \otimes \alpha(x_i),$$

both elements being the image of $1_A \in \mathrm{End}_{\Bbbk}(A)$ under the isomorphism $1_A \otimes \boldsymbol{l}_\beta^{-1}$: $\mathrm{End}_{\Bbbk}(A) = A \otimes_{\Bbbk} A^* \xrightarrow{\sim} A \otimes_{\Bbbk} A$. Therefore,

$$\tau(a) = \sum_i y_i a\alpha(x_i). \tag{11}$$

Now we compute, for $a, b \in A$,

$$\begin{aligned}\beta(\tau(a), b) &= \sum_i \beta(x_i a y_i, b) = \sum_i \beta(x_i a, y_i b) = \sum_i \beta(y_i b, \alpha(x_i a)) \\ &= \sum_i \beta(y_i b\alpha(x_i), \alpha(a)) = \sum_i \beta(a, y_i b\alpha(x_i)) = \beta(a, \tau(b)).\end{aligned}$$

and

$$\begin{aligned}a\tau(b) &\underset{(11)}{=} \sum_i a y_i b\alpha(x_i) \underset{(9)}{=} \sum_{i,j} \beta(x_j, a y_i) y_j b\alpha(x_i) = \sum_{i,j} \beta(x_j a, y_i) y_j b\alpha(x_i) \\ &= \sum_{i,j} y_j b\beta(x_j a, y_i)\alpha(x_i) \underset{(8)}{=} \sum_j y_j b\alpha(x_j a) \underset{(11)}{=} \tau(b)\alpha(a).\end{aligned}$$

Finally,

$$\beta(b, \alpha\tau(a)) = \beta(\tau(a), b) = \beta(a, \tau(b)) = \beta(\tau(b), \alpha(a)) = \beta(b, \tau\alpha(a)),$$

which shows that $\alpha\tau(a) = \tau\alpha(a)$. □

We will refer to $\tau = \tau_\beta$ as the *Higman trace* that is associated to β. If $\beta' : A \times A \to \Bbbk$ is another nonsingular associative $\Bbbk$-bilinear form then $\tau_{\beta'}(a) = \tau(a)u^{-1}$ for some unit $u \in A^\times$; see 2.2.

2.4.1. In [6], Higman introduced the following *Casimir operator*[1] for a given nonsingular associative bilinear form β on A:

$$A \to \mathscr{Z}(A), \qquad a \mapsto \sum_i y_i a x_i;$$

see also [9, 3.1]. If A is a *symmetric* $\Bbbk$-algebra, that is, A and A^* are isomorphic as (A, A)-bimodules, then the form β can be chosen to be symmetric. The corresponding Nakayama automorphism α is the identity and by (11) the Higman trace τ coincides with the Casimir operator in this case.

[1]Called Gaschütz–Ikeda operator in [4].

2.4.2. Let A be a Frobenius algebra over a field $\Bbbk$. By Section 1.4 the map $.^t$ factors through $T(A/\operatorname{rad} A)$, and so Lemma 2(a) tells us that τ vanishes on $\operatorname{rad} A$. Moreover, since $.^t$ takes values in $T(A/\operatorname{rad} A)^*$, we also have $b\tau(a)\lambda = T(a)^t(.b) = 0$ for all $a \in A$, $b \in \operatorname{rad} A$. Hence, $(\operatorname{rad} A)\tau(a) = 0$ and so $\operatorname{Im} \tau \subseteq \operatorname{soc} A$, the socle of A (which is in fact the same for ${}_AA$ and for A_A by [4, 58.12]). This shows that the Higman trace τ factors through a map

$$\widetilde{\tau} : T(A/\operatorname{rad} A) \longrightarrow \operatorname{soc} A \hookrightarrow A. \tag{12}$$

2.5. Examples

2.5.1. If $A = \mathrm{M}_n(\Bbbk)$ is the $n \times n$-matrix algebra over $\Bbbk$, then we can take the ordinary trace $\lambda = \operatorname{trace} \in A^*$ as Frobenius homomorphism. Dual bases are given by $\sum_i x_i \otimes y_i = \sum_{j,k} e_{j,k} \otimes e_{k,j}$, where $e_{l,m}$ is the matrix with 1 in the (l, m)-position and 0s elsewhere. The Higman trace is

$$\tau(a) = \operatorname{trace}(a) 1_{n\times n}.$$

2.5.2. The group algebra $A = \Bbbk G$ of a finite group G has Frobenius homomorphism λ with $\lambda(\sum_{g\in G} k_g g) = k_1$. Dual bases for λ are $\sum_i x_i \otimes y_i = \sum_{g\in G} g \otimes g^{-1}$, and the Higman trace is

$$\tau(a) = \sum_{g\in G} gag^{-1}.$$

2.5.3. Generalizing 2.5.2, let H be a Hopf $\Bbbk$-algebra that is finitely generated projective over $\Bbbk$, with augmentation ε, antipode $\mathcal{S}$, and comultiplication Δ. By [12], H is Frobenius over $\Bbbk$ if and only if the $\Bbbk$-module $\int_H^r = \{a \in H \mid ab = \varepsilon(b)a\}$ of right integrals of H is free of rank 1 over $\Bbbk$. In this case, we may pick integrals $\Lambda \in \int_H^r$ and $\lambda \in \int_{H^*}^l$ satisfying $\lambda(\Lambda) = 1$. The form λ serves as Frobenius homomorphism and dual bases are given by $\sum x_i \otimes y_i = \sum \Lambda_2 \otimes \mathcal{S}(\Lambda_1)$. Here we have used the standard Sweedler notation $\Delta h = \sum h_1 \otimes h_2$. Thus, the Higman trace takes the form

$$\tau(a) = \sum \Lambda_2 a \mathcal{S}(\Lambda_1).$$

In the special case where $H = \Bbbk G$, we may take $\Lambda = \sum_{g\in G} g$ and λ as in 2.5.2 to obtain the previous formula.

2.6. Rank and Determinant of the Cartan Map

We now specialize to the case where A is a Frobenius algebra over a field $\Bbbk$. Our goal is to determine the rank (i.e., the dimension of the image) of the map

$$c \otimes 1_{\Bbbk} : K_0(A) \otimes_{\mathbb{Z}} \Bbbk \longrightarrow G_0(A) \otimes_{\mathbb{Z}} \Bbbk$$

in (5). Since $K_0(A)$ and $G_0(A)$ are free abelian groups of the same finite rank, the number of isomorphism classes of irreducible left A-modules, the Cartan map is given by a square integer matrix C, called the *Cartan matrix* of A. The rank of $c \otimes 1_{\Bbbk}$

is equal to the rank of the Cartan matrix C when $\operatorname{char} \Bbbk = 0$, and to the rank of the reduction of C modulo p in case $\operatorname{char} \Bbbk = p > 0$.

The following result generalizes [10, Theorem 3.4]. For part (b), recall from (12) that the Higman trace $\tau = \tau_\beta$ factors as $\tau = \widetilde{\tau} \circ \left(A \overset{\text{can}}{\twoheadrightarrow} T(A/\operatorname{rad} A)\right)$. Note that $\widetilde{\tau}^{-1}(A^\times \cup \{0\})$ only depends on A and not on the choice of β.

Theorem 3. *Let A be a Frobenius $\Bbbk$-algebra, where $\Bbbk$ is a splitting field for A. Then*:

(a) *$\operatorname{rank}(c \otimes 1_\Bbbk) = \operatorname{rank} \tau$*;
(b) *The following are equivalent*:

  (i) *A is semisimple*;
  (ii) *$C = Id$ and $\widetilde{\tau}^{-1}(A^\times \cup \{0\}) \neq 0$*;
  (iii) *$\operatorname{char} \Bbbk$ does not divide $\det C$ and $\widetilde{\tau}^{-1}(A^\times \cup \{0\}) \neq 0$.*

Proof. (a) Since $\tilde{r}$ and $\tilde{\chi}$ are isomorphisms by our hypothesis on $\Bbbk$, the commutative diagram (5) tells us that $c \otimes 1_\Bbbk$ and $\widetilde{.}^{t}$ have the same rank. Finally, by Lemma 2(a) and (6), the rank of $\widetilde{.}^{t}$ is the same as the rank of the Higman trace τ.

(b) For (i) $\Rightarrow$ (ii), note that $C = \text{Id}$ because (A-proj) = (A-mod) for semisimple A. Furthermore, A is a finite product of matrix algebras $\mathrm{M}_{n_i}(\Bbbk)$. Letting $a \in A$ denote the element whose ith component is the $n_i \times n_i$ diagonal matrix $\operatorname{diag}(1, 0, \ldots, 0)$, and choosing τ componentwise as in 2.5.1, we obtain $\tau(a) = 1$. Therefore, $\widetilde{\tau}^{-1}(A^\times \cup \{0\}) \neq 0$.

The implication (ii) $\Rightarrow$ (iii) being clear, it remains to prove (iii) $\Rightarrow$ (i). The hypothesis that $\operatorname{char} \Bbbk$ does not divide $\det C$ says that $c \otimes 1_\Bbbk$ is invertible; so we know that $\operatorname{rank}(c \otimes 1_\Bbbk) = \dim_\Bbbk G_0(A) \otimes_{\mathbb{Z}} \Bbbk = \dim_\Bbbk T(A/\operatorname{rad} A)^*$, where the second equality holds because $\tilde{\chi}$ is an isomorphism. Theorem 3 now gives $\operatorname{rank} \tau = \dim_\Bbbk T(A/\operatorname{rad} A)^*$, and hence the map $\widetilde{\tau}$ in (12) is injective. Therefore, our hypothesis $\widetilde{\tau}^{-1}(A^\times \cup \{0\}) \neq 0$ implies that there exists $a \in A$ such that $\tau(a)$ is a unit in A. Since $\tau(a) \in \operatorname{soc} A$ by (12), this forces A to be semisimple. □

Part (b) of Theorem 3 is most useful for algebras where the condition $\tilde{\tau}^{-1}(A^\times \cup \{0\}) \neq 0$ is *a priori* known to hold. One such class of algebras, taken from [10, Theorem 3.4], is as follows.

2.6.1. Let H be a finite-dimensional Hopf algebra over the splitting field $\Bbbk$. Assume that $\mathcal{S}^2$ is inner, say $\mathcal{S}^2 = u^{-1}(.)u$ for some unit $u \in H^\times$. Choosing τ as in 2.5.3, we obtain

$$u^{-1}\tau(u) = \sum u^{-1}\Lambda_2 u \mathcal{S}(\Lambda_1) = \mathcal{S}\left(\sum \Lambda_1 \mathcal{S}(\Lambda_2)\right) = \mathcal{S}\big(\varepsilon(\Lambda)1\big) \in \Bbbk.$$

Since H is augmented, the class of u in $T(H/\operatorname{rad} H)$ is nonzero, and the above computation shows that $\tau(u) \in \Bbbk u \subseteq H^\times \cup \{0\}$. Therefore, $\widetilde{\tau}^{-1}(H^\times \cup \{0\}) \neq 0$.

3. MODULES OVER HOPF COMODULE ALGEBRAS

In this section, H will denote a Hopf algebra over the commutative ring $\Bbbk$, with unit u, multiplication m, counit ε, comultiplication Δ, and antipode $\mathcal{S}$.

We will use the Sweedler notation $\Delta h = \sum h_1 \otimes h_2$. Throughout, $\otimes = \otimes_{\Bbbk}$ and $.^* = \mathrm{Hom}_{\Bbbk}(., \Bbbk)$ denotes $\Bbbk$-linear duals. Finally, $\langle ., . \rangle : H^* \times H \to \Bbbk$ is the evaluation pairing.

3.1. *H*-Comodule Algebras and *H*-Galois Extensions

A $\Bbbk$-algebra B is called a right H-comodule algebra if B is a right H-comodule, with structure map

$$\rho : B \to B \otimes H, \qquad b \mapsto \sum b_0 \otimes b_1,$$

such that $\rho(ab) = \sum a_0 b_0 \otimes a_1 b_1$ for all $a, b \in B$ and $\rho(1) = 1 \otimes 1$; see [11, 4.1.2]. The H-coinvariants in B, defined by

$$A = B^{co\,H} = \{a \in B \mid \rho(a) = a \otimes 1\},$$

form a $\Bbbk$-subalgebra of B.

The standard (B, B)-bimodule structure on B gives rise to a (B, B)-bimodule structure on $B \otimes H$. Since ρ is an (A, A)-bimodule map, we obtain a natural left B-module map

$$'\!\rho : B \otimes_A B \to B \otimes H, \qquad b \otimes \tilde{b} \mapsto (b \otimes 1)\rho(\tilde{b}) \tag{13}$$

and a corresponding right B-module map

$$\rho' : B \otimes_A B \to B \otimes H, \qquad b \otimes \tilde{b} \mapsto \rho(b)(\tilde{b} \otimes 1). \tag{14}$$

The extension B/A is said to be *H-Galois* if $'\!\rho$ or ρ' is bijective. In case the antipode $\mathcal{S}$ is bijective both conditions are equivalent: $'\!\rho$ is bijective if and only if ρ' is so; see [11, 8.1.1 and subsequent remarks]. In particular, this equivalence holds when H is finitely generated projective over $\Bbbk$, because $\mathcal{S}$ is known to be bijective in this case [12, Proposition 4]. Furthermore, if H is finitely generated projective over $\Bbbk$ and B/A is H-Galois then B is finitely generated projective as A-module both on the left and on the right; see [7, 1.7, 1.8].

Important examples of H-Galois extensions are provided by crossed products $B = A\#_\sigma H$. Over a field $\Bbbk$, crossed products are precisely those right H-Galois extensions that enjoy the so-called (right) normal basis property; cf. [11, Corollary 8.2.5].

3.2. Tensor Products of Modules

Let B be a right H-comodule algebra. Given M in B-Mod and V in H-Mod, the tensor product $M \otimes V$ becomes a left B-module with the "diagonal" action

$$b(m \otimes v) = \sum b_0 m \otimes b_1 v \qquad (b \in B, m \in M, v \in V).$$

We note the following basic properties.

3.2.1. Functoriality. If $f : M \to M'$ is a B-module map and $g : V \to V'$ is an H-module map, then $f \otimes g : M \otimes V \to M' \otimes V'$ is a B-module map.

3.2.2. Associativity. If V and V' are H-modules then, viewing $V \otimes V'$ as H-module via Δ, we have $M \otimes (V \otimes V') \cong (M \otimes V) \otimes V'$.

The following lemma describes some special cases of tensor products of B-modules with H-modules.

Lemma 4. *Let B be a right H-comodule algebra and let $A = B^{co\,H}$. Then:*

(a) *Suppose B/A is H-Galois, with* (14) *being bijective. Then, for any M be in B-*Mod,

$$M \otimes H \cong B \otimes_A M$$

as left B-modules.

(b) *Given L in A-*Mod *and V in H-*Mod, *there is a B-module isomorphism*

$$(B \otimes_A L) \otimes V \cong B \otimes_A (L \otimes V),$$

where $L \otimes V$ is viewed as A-module via $a(l \otimes v) = al \otimes v$.

Proof. (a) Note that ρ' in (14) is actually a (B, B)-bimodule map, where the (B, B)-bimodule structure on $B \otimes_A B$ is given by ${}_BB \otimes_A B_B$ and B acts diagonally on $B \otimes H$ from the left and via the right regular action on the factor B from the right. Therefore, we have left B-module isomorphisms

$$B \otimes_A M \cong (B \otimes_A B) \otimes_B M \cong (B \otimes H) \otimes_B M \cong M \otimes H$$

sending $b \otimes m \mapsto \sum b_0 m \otimes b_1$.

(b) First construct a map $\gamma : B \otimes_A (L \otimes V) \to (B \otimes_A L) \otimes V$ as follows. The canonical map $\phi : L \to B \otimes_A L$, $l \mapsto 1 \otimes_A l$, gives rise to a map of A-modules, $\psi = \phi \otimes 1_V : L \otimes V \to (B \otimes_A L) \otimes V$. Since $(B \otimes_A L) \otimes V$ is in fact a B-module, with the diagonal B-action, ψ in turn yields a map of B-modules

$$\begin{aligned} \gamma : B \otimes_A (L \otimes V) &\longrightarrow (B \otimes_A L) \otimes V \\ b \otimes_A (l \otimes v) &\longmapsto \sum (b_0 \otimes_A l) \otimes b_1 v \end{aligned} \tag{15}$$

For the inverse map, we define

$$\begin{aligned} \delta : (B \otimes_A L) \otimes V &\longrightarrow B \otimes_A (L \otimes V) \\ (b \otimes_A l) \otimes v &\longmapsto \sum b_0 \otimes_A (l \otimes \mathcal{S}(b_1) v) \end{aligned} \tag{16}$$

To see that this map is well-defined note that the formula is linear in l, v and b. Moreover, the map is $\Bbbk$-balanced, that is, $\delta((b \otimes_A l)k \otimes v) = \delta((b \otimes_A l) \otimes kv)$ holds for all $k \in \Bbbk$. To verify A-balancedness, we use the formula $a \otimes 1 = \sum a_0 \otimes a_1$. With this, we calculate for $a \in A$

$$\begin{aligned} \delta((ba \otimes_A l) \otimes v) &= \sum b_0 a_0 \otimes_A (l \otimes \mathcal{S}(b_1 a_1) v) \\ &= \sum b_0 a \otimes_A (l \otimes \mathcal{S}(b_1) v) \end{aligned}$$

and

$$\delta((b \otimes_A al) \otimes v) = \sum b_0 \otimes_A (al \otimes \mathcal{S}(b_1)v)$$
$$= \sum b_0 \otimes_A a(l \otimes \mathcal{S}(b_1)v)$$
$$= \sum b_0 a \otimes_A (l \otimes \mathcal{S}(b_1)v).$$

It remains to check that γ and δ are indeed inverse to each other. We will carry out the verification of the identity $\gamma \circ \delta = 1_{(B\otimes_A L)\otimes V}$; the check of $\delta \circ \gamma = 1_{B\otimes_A(L\otimes V)}$ can be handled in an entirely analogous fashion:

$$(\gamma \circ \delta)((b \otimes_A l) \otimes v) = \gamma\left(\sum b_0 \otimes_A (l \otimes \mathcal{S}(b_1)v)\right)$$
$$= \sum (b_0 \otimes_A l) \otimes b_1 \mathcal{S}(b_2)v$$
$$= \sum (b_0 \otimes_A l) \otimes \langle \varepsilon, b_1 \rangle v$$
$$= (b \otimes_A l) \otimes v,$$

as required. This completes the proof of the lemma. □

Proposition 5. *Let B be a right H-comodule algebra. Given M in B-*Mod *and V in H-*Mod, *view $M \otimes V$ as B-module with the diagonal B-action.*

(a) *If M is finitely generated and V is finitely generated over $\Bbbk$ then $M \otimes V$ is finitely generated.*
(b) *If M is projective and V is projective over $\Bbbk$ then $M \otimes V$ is projective.*

Proof. (a) Since M is finitely generated, $M \otimes V$ is a homomorphic image of $B^n \otimes V \cong (B \otimes V)^n$ for some n. By Lemma 4(b) with $L = A$, the B-module $B \otimes V$ with the diagonal B-action is isomorphic to $B \otimes V$ with B just acting on the left factor. Since V is assumed finite over $\Bbbk$, it follows that $B \otimes V$ is finitely generated as B-module, and hence so is $M \otimes V$.

(b) Since $. \otimes V$ commutes with direct sums, it suffices to consider the special case $M = B$. As above, Lemma 4(b) with $L = A$ implies that $B \otimes V$ is projective as B-module. □

3.3. Module Structure on Grothendieck Groups

Let B be a right H-comodule algebra.

3.3.1. Recall from Section 1.2 that H-mod$_{\Bbbk}$ denotes the category of all left H-modules that are finitely generated projective over $\Bbbk$. If V and V' belong to H-mod$_{\Bbbk}$, then so does the tensor product $V \otimes V'$, where H acts on $V \otimes V'$ via the comultiplication Δ as usual. Therefore, the Grothendieck group $G_0^{\Bbbk}(H) = K_0(H\text{-mod}_{\Bbbk})$, introduced in Section 1.2, is a ring with multiplication given by $[V][V'] = [V \otimes V']$. By Proposition 5, we may view $K_0(B) = K_0(B\text{-proj})$ as right module over $G_0^{\Bbbk}(H)$:

$$K_0(B) \otimes_{\mathbb{Z}} G_0^{\Bbbk}(H) \longrightarrow K_0(B), \qquad [M] \otimes [V] \mapsto [M \otimes V]. \tag{17}$$

3.3.2. Following [15, Chapter II, Example 7.1.4], we will also consider the Grothendieck group

$$G_0(B) := K_0(B\text{-}\mathsf{FPmod}_\infty),$$

where B-FPmod_∞ denotes the full subcategory of B-Mod consisting of all left B-modules of *type* FP_∞, that is, modules M that have a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

where all P_i are finitely generated projective B-modules. If the algebra B is left noetherian then B-$\mathsf{FPmod}_\infty = B$-$\mathsf{mod}$, the category of all finitely generated left B-modules. In general, it is not hard to see that, for any short exact exact sequence $0 \to M' \to M \to M'' \to 0$ in B-Mod , if two of $\{M', M, M''\}$ belong to B-FPmod_∞, then all three do.

Proposition 5 implies that, for any M in B-FPmod_∞ and any V in H-$\mathsf{mod}_{\Bbbk}$, the tensor product $M \otimes V$ belongs to B-FPmod_∞. Therefore, we also have a $G_0^{\Bbbk}(H)$-module structure on $G_0(B)$:

$$G_0(B) \otimes_{\mathbb{Z}} G_0^{\Bbbk}(H) \longrightarrow G_0(B), \qquad [M] \otimes [V] \mapsto [M \otimes V]. \tag{18}$$

The Cartan map $c : K_0(B) \to G_0(B)$, coming from the inclusion B-$\mathsf{proj} \hookrightarrow B$-$\mathsf{FPmod}_\infty$, is clearly a $G_0^{\Bbbk}(H)$-module map.

3.3.3. We now turn to the special case where the Hopf algebra H is finitely generated projective over $\Bbbk$ and B/A is a right H-Galois extension (so $A = B^{co\,H}$). As we remarked in Section 3.1, the algebra B is then finitely generated projective as left and right A-module. Therefore, we have well-defined restriction and induction maps

$$\mathrm{Res}_A^B : K_0(B) \to K_0(A), \qquad [P] \mapsto [{}_AP]$$
$$\mathrm{Ind}_A^B : K_0(A) \to K_0(B), \qquad [Q] \mapsto [B \otimes_A Q]$$

and similarly for G_0. Lemma 4(a) has the following immediate consequence.

Lemma 6. *Assume that H is finitely generated projective over $\Bbbk$. Then, for any right H-Galois extension B/A, the map $\mathrm{Ind}_A^B \circ \mathrm{Res}_A^B : G_0(B) \to G_0(B)$ is given by the action of $[H] \in G_0^{\Bbbk}(H)$ on $G_0(B)$:*

$$(\mathrm{Ind}_A^B \circ \mathrm{Res}_A^B)([M]) = [M][H].$$

It is similar for $K_0(B)$.

3.4. The Product Formula

For a given right H-comodule algebra B, we now describe how the Hattori–Stallings map $r_B : K_0(B) \to T(B) = B/[B, B]$ of Section 1.1 behaves with respect to the module action in Section 3.3. The lemma below generalizes [2, Prop. 5.5(c)], which treats the case where $B = H = \Bbbk G$ is the group algebra of a finite group G.

View B as a left H^*-module as in [11, 1.6.4]:

$$f \cdot b = \sum b_0 \langle f, b_1 \rangle \qquad (f \in H^*, b \in B). \tag{19}$$

The $\Bbbk$-submodule $[B, B]$ that is spanned by the Lie commutators in B is stable under the action of the subalgebra $T(H)^* \subseteq H^*$ consisting of all trace forms on H: $f \cdot [b, b'] = \sum [b_0, b'_0] \langle f, b_1 b'_1 \rangle$ holds for all $f \in T(H)^*$ and $b, b' \in B$. Therefore, $T(B) = B/[B, B]$ becomes a left $T(H)^*$-module via (19). It will be convenient to let H^* act from the right on B by defining

$$b \cdot f := \mathscr{S}^*(f) \cdot b = \sum b_0 \langle f, \mathscr{S}(b_1) \rangle \qquad (f \in H^*, b \in B). \tag{20}$$

Since $T(H)^*$ is stable under the antipode $\mathscr{S}^*$ of H^*, the $\Bbbk$-module $T(B)$ becomes a right $T(H)^*$-module in this way.

Lemma 7. *Let B be a right H-comodule algebra. Then, for any M in B-*proj *and V in H-*$\mathsf{mod}_{\Bbbk}$*, we have $r_B(M \otimes V) = r_B(M) \cdot \chi_V$. Thus the following diagram commutes:*

$$\begin{array}{ccc} K_0(B) \otimes_{\mathbb{Z}} G_0^{\Bbbk}(H) & \xrightarrow{(17)} & K_0(B) \\ {\scriptstyle r_B \otimes \chi} \downarrow & & \downarrow {\scriptstyle r_B} \\ T(B) \otimes_{\mathbb{Z}} T(H)^* & \xrightarrow[(20)]{} & T(B) \end{array} .$$

Proof. Write $M = e(F)$ with $F = B^r$ free over B and $e = e^2 \in \operatorname{End}_B(F)$. Then $M \otimes V$ is the image of $e \otimes 1_V \in \operatorname{End}_B(F \otimes V)$; see Section 3.2.1. By Lemma 4,

$$F \otimes V = (B \otimes_A A^r) \otimes V \underset{\delta}{\overset{\sim}{\to}} B \otimes_A (A^r \otimes V)$$

with δ as in (16). Under this isomorphism, $e \otimes 1_V \in \operatorname{End}_B(F \otimes V)$ is transformed into $e' = \delta \circ (e \otimes 1_V) \circ \gamma \in \operatorname{End}_B(B \otimes_A (A^r \otimes V))$ with $\gamma = \delta^{-1}$ as in (15). Fix an A-basis $\{x_i\}_1^r$ of A^r and write $e(1 \otimes_A x_i) = \sum_j e^{i,j} \otimes_A x_j$ with $e^{i,j} \in B$. Then

$$r_B(M) = \sum_i T(e^{i,i}).$$

Let $\{v_l, f_l\} \subseteq V \times V^*$ be dual bases for V as $\Bbbk$-module; so $1_V = \sum_l v_l \otimes f_l$ under the standard identification $\operatorname{End}_{\Bbbk}(V) = V \otimes V^*$. Then

$$\begin{aligned} e'(1 \otimes_A (x_i \otimes v_l)) &= (\delta \circ (e \otimes 1_V) \circ \gamma)(1 \otimes_A (x_i \otimes v_l)) \\ &\underset{(15)}{=} (\delta \circ (e \otimes 1_V))((1 \otimes_A x_i) \otimes v_l) \\ &= \sum_j \delta((e^{i,j} \otimes_A x_j) \otimes v_l) \\ &\underset{(16)}{=} \sum_j e_0^{i,j} \otimes_A (x_j \otimes \mathscr{S}(e_1^{i,j}) v_l) \end{aligned}$$

$$= \sum_{j,k} e_0^{i,j} \otimes_A (x_j \otimes \langle f_k, \mathscr{S}(e_1^{i,j}) v_l \rangle v_k)$$
$$= \sum_{j,k} e_0^{i,j} \langle f_k, \mathscr{S}(e_1^{i,j}) v_l \rangle \otimes_A (x_j \otimes v_k).$$

Therefore,

$$r_B(M \otimes V) = \sum_{i,l} T(e_0^{i,i}) \langle f_l, \mathscr{S}(e_1^{i,i}) v_l \rangle = \sum_i T(e_0^{i,i}) \langle \chi_V, \mathscr{S}(e_1^{i,i}) \rangle \overset{(20)}{=} r_B(M) \cdot \chi_V. \qquad \square$$

3.5. *H*-Galois Extensions with *H* Frobenius

In this section, we focus on right H-Galois extensions B/A such that H is involutory (i.e., $\mathscr{S}^2 = 1$) and a Frobenius algebra over $\Bbbk$. The Frobenius property for H is equivalent to H being finitely generated projective over $\Bbbk$ and the $\Bbbk$-module

$$\int_H^r = \{x \in H \mid xy = \langle \varepsilon, y \rangle x \text{ for all } y \in H\}$$

of right integrals of H being free of rank 1 over $\Bbbk$; see [12] or [9, Theorem 10]. If H is finitely generated projective over $\Bbbk$ then $\int_H^l = \mathscr{S}(\int_H^r)$, because $\mathscr{S}$ is an anti-automorphism of H, and $\int_H^r$ and $\int_H^l$ are finitely generated projective of constant rank 1 over $\Bbbk$ by [12, Proposition 3]. In particular, if $\Bbbk$ has trivial Picard group, then H is a Frobenius $\Bbbk$-algebra if and only if H is finitely generated projective over $\Bbbk$ [12, Corollary 1].

Define the ideal $\mathfrak{d}H$ of $\Bbbk$ by

$$\mathfrak{d}H := \left\langle \varepsilon, \int_H^l \right\rangle = \left\langle \varepsilon, \int_H^r \right\rangle.$$

For example, if $H = \Bbbk G$ is the group algebra of a finite group G, then $\int_H^l = \int_H^r = \Bbbk\Lambda$ with $\Lambda = \sum_{g \in G} g$ as in 2.5.3, and hence $\mathfrak{d}(\Bbbk G) = |G|1$. In general, by a classical result of Larson and Sweedler [8] (see also [9, Corollary 11] for the case of an arbitrary base ring $\Bbbk$), H is separable over $\Bbbk$ if and only if H is Frobenius over $\Bbbk$ with $\mathfrak{d}H = \Bbbk$.

Proposition 8. *Let B/A be a right H-Galois extension with H involutory and Frobenius over $\Bbbk$ and let $\imath : A \hookrightarrow B$ denote the inclusion map. Then the composite*

$$T(\imath) \circ r_A \circ \operatorname{Res}_A^B : K_0(B) \to K_0(A) \to T(A) \to T(B)$$

has image in $(\mathfrak{d}H)A \mod [B, B]$.

Proof. For any M in B-proj, we compute

$$(T(\imath) \circ r_A \circ \operatorname{Res}_A^B)([M]) \underset{(2)}{=} (r_B \circ K_0(\imath) \circ \operatorname{Res}_A^B)([M])$$
$$\underset{\text{Lemma 6}}{=} r_B([M][H])$$
$$\underset{\text{Lemma 7}}{=} r_B(M) \cdot \chi_H.$$

Since H involutory and Frobenius over $\Bbbk$, we further know that

$$\Bbbk\chi_H = (\eth H)\int_{H^*}^{l};$$

see, e.g., [9, Lemma 12(a)]. For any $x \in H$, let $L_x, R_x \in \mathrm{End}_{\Bbbk}(H)$ be given by left and right multiplication by x, respectively. Then $\chi_H(x) = \mathrm{Tr}_{H/\Bbbk}(L_x) = \mathrm{Tr}_{H/\Bbbk}(R_x)$, where the first equality holds by definition of χ_H and the second equality is [9, Equation (13)]. Thus,

$$\begin{aligned}\langle \mathcal{S}^*(\chi_H), x\rangle = \langle \chi_H, \mathcal{S}(x)\rangle &= \mathrm{Tr}_{H/\Bbbk}(L_{\mathcal{S}(x)})\\ &= \mathrm{Tr}_{H/\Bbbk}(\mathcal{S}\circ R_x \circ \mathcal{S}^{-1}) = \mathrm{Tr}_{H/\Bbbk}(R_x) = \chi_H(x),\end{aligned}$$

which shows that $\mathcal{S}^*(\chi_H) = \chi_H$. Finally, under the left H^*-action (19) on B, we have $\int_{H^*}^{l} \cdot B \subseteq B^{H^*} = A$; see [11, 1.7.2] for the latter equality. Therefore, the element $r_B(M)\cdot\chi_H$ belongs to $(\eth H)T(\imath)(T(A))$. □

Recall from Section 1.1.2 that if A is a commutative ring without idempotents $\neq 0, 1$, then $H_0(A) = [\mathrm{Spec}\,A, \mathbb{Z}]$ consists of constant functions. In particular, $\mathrm{rank}\,Q$ is constant for every Q in (A-proj); we will denote the constant value of this function by $\mathrm{rank}_A Q \in \mathbb{Z}$.

Theorem 9. *Let B/A be a right H-Galois extension such that H is involutory and Frobenius over $\Bbbk$. Assume further that*:

(i) *A is commutative without idempotents* $\neq 0, 1$; *and*
(ii) *There exists $f \in T(B)^*$ with $f(1) = 1$.*

*Then $\mathrm{rank}_A M \cdot 1 \in \eth H$ holds for each M in B-*proj.

Proof. By Proposition 8, $T(\imath)\circ r_A(M)$ belongs to $(\eth H)T(B)$ and by hypothesis (i), we have $r_A(M) = \mathrm{rank}_A M\cdot 1 \in A$. Applying the trace function $f: T(B)\to \Bbbk$ in (ii) to $\mathrm{rank}_A M \cdot 1 \mod [B, B] \in (\eth H)T(B)$ yields the result. □

In the special case where $B = H$ is a finite-dimensional Hopf algebra over a field $\Bbbk$ and $A = \Bbbk$, hypothesis (i) is trivially holds and we can take $f = \varepsilon$ in (ii). Theorem 9, therefore, implies [10, Theorem 2.3(b)]: if H is involutory and not semisimple, then $\mathrm{char}\,\Bbbk$ is positive and a divisor of $\dim_{\Bbbk} M$ for every M in H-proj.

ACKNOWLEDGMENTS

The first author thanks the organizers of the conference on the occasion of Mia Cohen's retirement, held at the Sde-Boker campus of Ben-Gurion University May during 24–27, 2010, for giving him the opportunity to present some related results on ring theoretic methods in the representation theory of Hopf algebras to an expert audience. Thanks are also due to the referee for a careful reading of this article.

Research of the first author supported in part by NSA Grant H98230-09-1-0026.

REFERENCES

[1] Bass, H. (1968). *Algebraic K-Theory*. New York–Amsterdam: W. A. Benjamin, Inc.
[2] Bass, H. (1976). Euler characteristics and characters of discrete groups. *Invent. Math.* 35:155–196.
[3] Brown, K. S. (1982). Cohomology of groups. *Graduate Texts in Mathematics*. Vol. 87, New York: Springer-Verlag.
[4] Curtis, C. W., Reiner, I. (1962). Representation theory of finite groups and associative algebras. Pure and Applied Mathematics. Vol. XI, New York–London: Interscience Publishers, a division of John Wiley & Sons.
[5] Hattori, A. (1965). Rank element of a projective module. *Nagoya Math. J.* 25:113–120.
[6] Higman, D. G. (1955). On orders in separable algebras. *Canad. J. Math.* 7:509–515.
[7] Kreimer, H. F., Takeuchi, M. (1981). Hopf algebras and Galois extensions of an algebra. *Indiana Univ. Math. J.* 30:675–692.
[8] Larson, R. G., Sweedler, M. E. (1969). An associative orthogonal bilinear form for Hopf algebras. *Amer. J. Math.* 91:75–94.
[9] Lorenz, M. (2011). Some applications of Frobenius algebras to Hopf algebras. In: *Groups, Algebras and Applications, Contemporary Mathematics*, Vol. 537. Providence, RI: Amer. Math. Soc., pp. 269–289.
[10] Lorenz, M. (1997). Representations of finite-dimensional Hopf algebras. *J. Algebra* 188(2):476–505.
[11] Montgomery, S. (1993). Hopf algebras and their actions on rings, CBMS Regional Conference Series in Mathematics, Vol. 82, Washington, DC: Published for the Conference Board of the Mathematical Sciences.
[12] Pareigis, B. (1971). When Hopf algebras are Frobenius algebras. *J. Algebra* 18:588–596.
[13] Swan, R. G., Graham Evans, E. (1970). K-theory of finite groups and orders. Berlin: Lecture Notes in Mathematics. Vol. 149, Springer-Verlag.
[14] Tokoly, L. F. (1999). *Frobenius reciprocity and Grothendieck groups of Hopf Galois extensions*. Ph.D. thesis, Temple University.
[15] Weibel, C. A. An introduction to algebraic K-theory, beta version of complete book (November 4, 2011). Available at http://www.math.rutgers.edu/~weibel/Kbook.html.