

Some applications of Frobenius algebras to Hopf algebras

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ABSTRACT. This expository article presents a unified ring theoretic approach, based on the theory of Frobenius algebras, to a variety of results on Hopf algebras. These include a theorem of S. Zhu on the degrees of irreducible representations, the so-called class equation, the determination of the semisimplicity locus of the Grothendieck ring, the spectrum of the adjoint class and a non-vanishing result for the adjoint character.

Introduction

0.1. It is a well-known fact that all finite-dimensional Hopf algebras over a field are Frobenius algebras. More generally, working over a commutative base ring R with trivial Picard group, any Hopf R -algebra that is finitely generated projective over R is a Frobenius R -algebra [22]. This article explores the Frobenius property, and some consequences thereof, for Hopf algebras and for certain algebras that are closely related to Hopf algebras without generally being Hopf algebras themselves: the Grothendieck ring $G_0(H)$ of a split semisimple Hopf algebra H and the representation algebra $R(H) \subseteq H^*$. Our principal goal is to quickly derive various consequences from the fact that the latter algebras are Frobenius, or even symmetric, thereby giving a unified ring theoretic approach to a variety of known results on Hopf algebras.

0.2. The first part of this article, consisting of four sections, is entirely devoted to Frobenius and symmetric algebras over commutative rings; its sole purpose is to deploy the requisite ring theoretical tools. The content of these sections is classical over fields and the case of general commutative base rings is easily derived along the same lines. In the interest of readability, we have opted for a self-contained development. The technical core of this part are the construction of certain central idempotents in Propositions 4 and 5 and the description of the separability locus of a Frobenius algebra in Proposition 6. The essence of the latter proposition goes back to D. G. Higman [11].

0.3. Applications to Hopf algebras are given in Part 2. We start by considering Hopf algebras that are finitely generated projective over a commutative ring. After reviewing some standard facts, due to Larson-Sweedler [16], Pareigis [22], and Oberst-Schneider [21], concerning the Frobenius property of Hopf algebras, we spell out the content of the

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aforementioned Propositions 4 and 5 in this context (Proposition 15). A generalization, due to Rumynin [24], of Frobenius' classical theorem on the degrees of irreducible complex representations of finite groups follows in a few lines from this result (Corollary 16).

The second section of Part 2 focuses on semisimple Hopf algebras H over fields, specifically their Grothendieck rings $G_0(H)$. As an application of Proposition 15, we derive a result of S. Zhu [27] on the degrees of certain irreducible representations of H (Theorem 18). The celebrated class equation for semisimple Hopf algebras is presented as an application of Proposition 4 in Theorem 19. The proof given here follows the outline of our earlier proof in [18], with a clearer separation of the purely ring theoretical underpinnings. Other applications concern a new proof of an integrality result, originally due to Sommerhäuser [26], for the eigenvalues of the “adjoint class” (Proposition 20), the determination of the semisimplicity locus of $G_0(H)$ (Proposition 22), and a non-vanishing result for the adjoint character (Proposition 23).

For the sake of simplicity, we have limited ourselves for the most part to base fields of characteristic 0. In some cases, this restriction can be removed with the aid of p -modular systems. For example, as has been observed by Etingof and Gelaki [8], any bi-semisimple Hopf algebra over an algebraically closed field of positive characteristic, along with all its irreducible representations, can be lifted to characteristic 0. We plan to address this aspect more fully in a sequel to this article.

In closing I wish to thank the referee for a careful reading of this article and for valuable comments.

Part 1. RING THEORY

Throughout this first part of the paper, R will denote a commutative ring and A will be an R -algebra that is finitely generated projective over R .

1. Preliminaries on Frobenius and symmetric algebras

1.1. Definitions.

1.1.1. *Frobenius algebras.* Put $A^\vee = \text{Hom}_R(A, R)$; this is an (A, A) -bimodule via

$$(1.1) \quad (afb)(x) = f(bxa) \quad (a, b, x \in A, f \in A^\vee).$$

The algebra A is called *Frobenius* if $A \cong A^\vee$ as left A -modules. This is equivalent to $A \cong A^\vee$ as right A -modules. Indeed, using the standard isomorphism $A_A \xrightarrow{\sim} (A^{\vee\vee})_A = \text{Hom}_R({}_A A^\vee, R)_A$ given by $a \mapsto (f \mapsto f(a))$, one deduces from ${}_A A^\vee \cong {}_A A$ that

$$(1.2) \quad A_A \cong \text{Hom}_R({}_A A^\vee, R)_A \cong \text{Hom}_R({}_A A, R)_A = A^\vee_A.$$

The converse is analogous.

More precisely, any isomorphism $L: {}_A A \xrightarrow{\sim} {}_A A^\vee$ has the form

$$L(a) = a\lambda \quad (a \in A),$$

where we have put $\lambda = L(1) \in A^\vee$. The linear form λ is called a *Frobenius homomorphism*. Tracing $1 \in A$ through (1.2) we obtain $1 \mapsto (a\lambda \mapsto (a\lambda)(1) = \lambda(a)) \mapsto (a \mapsto \lambda(a)) = \lambda$. Hence, the resulting isomorphism $R: A_A \xrightarrow{\sim} A^\vee_A$ is explicitly given by

$$R(a) = \lambda a \quad (a \in A).$$

In particular, λ is also a free generator of A^\vee as right A -module. The automorphism $\alpha \in \text{Aut}_{R\text{-alg}}(A)$ that is given by $\lambda a = \alpha(a)\lambda$ for $a \in A$ is called the *Nakayama automorphism*.

1.1.2. *Symmetric algebras.* If $A \cong A^\vee$ as (A, A) -bimodules then the algebra A is called *symmetric*. Any (A, A) -bimodule isomorphism $A \cong A^\vee$ restricts to an isomorphism of Hochschild cohomology modules $H^0(A, A) \cong H^0(A, A^\vee)$. Here, $H^0(A, A) = \mathcal{Z}(A)$ is the center of A , and $H^0(A, A^\vee) = A_{\text{trace}}^\vee$ consists of all *trace forms* on A , that is, R -linear forms $f \in A^\vee$ vanishing on all Lie commutators $[x, y] = xy - yx$ for $x, y \in A$. Thus, if A is symmetric then we obtain an isomorphism of $\mathcal{Z}(A)$ -modules

$$(1.3) \quad \mathcal{Z}(A) \xrightarrow{\sim} A_{\text{trace}}^\vee .$$

1.2. Bilinear forms.

1.2.1. *Nonsingularity.* Let $\text{Bil}(A; R)$ denote the R -module consisting of all R -bilinear forms $\beta: A \times A \rightarrow R$. Putting $r_\beta(a) = \beta(a, \cdot)$ for $a \in A$, we obtain an isomorphism of R -modules

$$(1.4) \quad r: \text{Bil}(A; R) \xrightarrow{\sim} \text{Hom}_R(A, A^\vee), \quad \beta \mapsto r_\beta .$$

The bilinear form β is called *left nonsingular* if r_β is an isomorphism. Inasmuch as A and A^\vee are locally isomorphic projectives over R , it suffices to assume that r_β is surjective; see [6, Cor. 4.4(a)].

Similarly, there is an isomorphism

$$(1.5) \quad l: \text{Bil}(A; R) \xrightarrow{\sim} \text{Hom}_R(A, A^\vee), \quad \beta \mapsto l_\beta$$

with $l_\beta(a) = \beta(\cdot, a)$, and β is called *right nonsingular* if l_β is an isomorphism. Right and left nonsingularity are in fact equivalent. After localization, this follows from [14, Prop. XIII.6.1]. In the following, we will therefore call such forms simply *nonsingular*.

1.2.2. *Dual bases.* Fix a nonsingular bilinear form $\beta: A \times A \rightarrow R$. Since A is finitely generated projective over R , we have a canonical isomorphism $\text{End}_R(A) \cong A \otimes_R A^\vee$; see [2, II.4.2]. Thus, the isomorphism l_β in (1.5) yields an isomorphism $\text{End}_R(A) \xrightarrow{\sim} A \otimes_R A$. Writing the image of $\text{Id}_A \in \text{End}_R(A)$ as $\sum_{i=1}^n x_i \otimes y_i \in A \otimes_R A$, we have

$$(1.6) \quad a = \sum_i \beta(a, y_i) x_i \quad \text{for all } a \in A.$$

Conversely, assume that $\beta: A \times A \rightarrow R$ is such that there are elements $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ satisfying (1.6). Then any $f \in A^\vee$ satisfies

$$(1.7) \quad f = \sum_i l_\beta(y_i) f(x_i) = l_\beta\left(\sum_i f(x_i) y_i\right) .$$

This shows that $l_\beta: A \rightarrow A^\vee$ is surjective, and hence β is nonsingular. To summarize, a bilinear form $\beta \in \text{Bil}(A; R)$ is nonsingular if and only if there exist “dual bases” $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ satisfying (1.6).

Note also that, for a nonsingular R -bilinear form $\beta: A \times A \rightarrow R$ and elements $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$, condition (1.6) is equivalent to

$$(1.8) \quad b = \sum_i \beta(x_i, b) y_i \quad \text{for all } b \in A.$$

Indeed, both (1.6) and (1.8) are equivalent to $\beta(a, b) = \sum_i \beta(a, y_i) \beta(x_i, b)$ for all $a, b \in A$.

1.2.3. *Associative bilinear forms.* An R -bilinear form $\beta: A \times A \rightarrow R$ is called *associative* if $\beta(xy, z) = \beta(x, yz)$ for all $x, y, z \in A$. Let $\text{Bil}_{\text{assoc}}(A; R)$ denote the R -submodule of $\text{Bil}(A; R)$ consisting of all such forms. Under the isomorphism (1.4), $\text{Bil}_{\text{assoc}}(A; R)$ corresponds to the submodule $\text{Hom}(A_A, A^{\vee}_A) \subseteq \text{Hom}_R(A, A^{\vee})$. Similarly, (1.5) yields an isomorphism of R -modules $\text{Bil}_{\text{assoc}}(A; R) \cong \text{Hom}({}_A A, {}_A A^{\vee})$. Therefore:

The algebra A is Frobenius if and only if there exists a nonsingular associative R -bilinear form $\beta: A \times A \rightarrow R$.

1.2.4. *Symmetric forms.* The form β is called *symmetric* if $\beta(x, y) = \beta(y, x)$ for all $x, y \in A$. The isomorphisms r and l in (1.4), (1.5) agree on the submodule consisting of all symmetric bilinear forms, and they yield an isomorphism between the R -module consisting of all associative symmetric bilinear forms on A and the submodule $\text{Hom}({}_A A A, {}_A A^{\vee}_A) \subseteq \text{Hom}_R(A, A^{\vee})$ consisting of all (A, A) -bimodule maps $A \rightarrow A^{\vee}$. Thus:

The algebra A is symmetric if and only if there exists a nonsingular associative symmetric R -bilinear form $\beta: A \times A \rightarrow R$.

1.2.5. *Change of bilinear form.* Given two nonsingular forms $\beta, \gamma \in \text{Bil}_{\text{assoc}}(A; R)$, we obtain an isomorphism of left A -modules $l_{\beta}^{-1} \circ l_{\gamma}: {}_A A \xrightarrow{\sim} {}_A A$. Since this isomorphism has the form $a \mapsto au$ ($a \in A$) for some unit $u \in A$, we see that

$$\gamma(\cdot, \cdot) = \beta(\cdot, \cdot u).$$

If β and γ are both symmetric then $l_{\beta}^{-1} \circ l_{\gamma}$ is an isomorphism of (A, A) -bimodules, and hence $u \in \mathcal{Z}(A)$, the center of A .

2. Characters

Throughout this section, M will denote a left A -module that is assumed to be finitely generated projective over R . For $a \in A$, we let $a_M \in \text{End}_R(M)$ denote the endomorphism given by the action of a on M :

$$a_M(m) = am \quad (a \in A, m \in M).$$

2.1. Trace and rank. The trace map

$$\text{Tr}: \text{End}_R(M) \cong M \otimes_R M^{\vee} \rightarrow R;$$

it is defined via evaluation of $M^{\vee} = \text{Hom}_R(M, R)$ on M ; see [2, II.4.3]. The image of the trace map complements the annihilator $\text{ann}_R M = \{r \in R \mid rm = 0 \forall m \in M\}$:

$$(2.1) \quad \text{Im}(\text{Tr}) \oplus \text{ann}_R M = R;$$

see [5, Proposition I.1.9]. The *Hattori-Stallings rank* of M is defined by

$$\text{rank}_R M = \text{Tr}(1_M) \in R.$$

If M is free of rank n over R then $\text{rank}_R M = n \cdot 1$. The following lemma is standard and easy.

LEMMA 1. *The R -algebra $\text{End}_R(M)$ is symmetric, with nonsingular associative symmetric R -bilinear form $\text{End}_R(M) \times \text{End}_R(M) \rightarrow R$, $(x, y) \mapsto \text{Tr}(xy)$. Identifying $\text{End}_R(M)$ with $M \otimes_R M^{\vee}$ and writing $1_M = \sum m_i \otimes f_i$, dual bases for this form are given by*

$$\{x_{i,j} = m_i \otimes f_j\}, \{y_{i,j} = m_j \otimes f_i\}.$$

2.2. The character of M . The *character* of M is the trace form $\chi_M \in A_{\text{trace}}^\vee$ that is defined by

$$\chi_M(a) = \text{Tr}(a_M) \quad (a \in A).$$

If $e = e^2 \in A$ is an idempotent then $e_M = 1_{eM} \oplus 0_{(1-e)M}$, and so

$$(2.2) \quad \chi_M(e) = \text{rank}_R eM.$$

Now assume that A is Frobenius with associative nonsingular bilinear form β , and let $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ be dual bases for β as in (1.6). Then the preimage of $\chi_M \in A_{\text{trace}}^\vee \subseteq A^\vee$ under the isomorphism $\mathbf{l}_\beta: {}_A A \xrightarrow{\sim} {}_A A^\vee$ in (1.5) is the element

$$(2.3) \quad z(M) = z_\beta(M) := \sum_i \chi_M(x_i) y_i \in A;$$

see (1.7). So

$$(2.4) \quad \chi_M(\cdot) = \beta(\cdot, z(M)).$$

In particular, $z(M)$ is independent of the choice of dual bases $\{x_i\}, \{y_i\}$.

If β is symmetric then $z(M) \in \mathcal{Z}(A)$ by (1.3).

2.3. The regular character. The left regular representation of A is defined by $A \rightarrow \text{End}_R(A), a \mapsto (a_A: x \mapsto ax)$. Similarly, the right regular representation is given by $A \rightarrow \text{End}_R(A), a \mapsto ({}_A a: x \mapsto xa)$.

If A is Frobenius, with associative nonsingular bilinear form β and dual bases $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ for β , then equations (1.6) and (1.8) give the following expression

$$a_A = \sum_i x_i \otimes \beta(a \cdot, y_i) = \sum_i y_i \otimes \beta(x_i, a \cdot) \in A \otimes_R A^\vee \cong \text{End}_R(A).$$

Similarly, ${}_A a = \sum_i x_i \otimes \beta(\cdot, a, y_i) = \sum_i y_i \otimes \beta(x_i, \cdot, a)$. Taking traces, we obtain

$$(2.5) \quad \text{Tr}(a_A) = \sum_i \beta(x_i, a y_i) = \sum_i \beta(x_i a, y_i) = \text{Tr}({}_A a).$$

This trace is called the *regular character* of A ; it will be denoted by $\chi_{\text{reg}} \in A^\vee$. Since $\text{Tr}(a_A) = \sum_i \beta(a x_i, y_i)$ and $\text{Tr}({}_A a) = \sum_i \beta(x_i, y_i a)$, we have

$$(2.6) \quad \chi_{\text{reg}} = \beta(\cdot, z) = \beta(z, \cdot) \quad \text{with} \quad z = z_\beta := \sum_i x_i y_i.$$

Thus, the element z is associated to the regular character as in (2.4).

EXAMPLE 2. We compute the regular character for the algebra $\text{End}_R(M)$ using the trace form and the dual bases $\{x_{i,j}\}, \{y_{i,j}\}$ from Lemma 1. The element z in (2.6) evaluates to

$$z = \sum_{i,j} (m_i \otimes f_j)(m_j \otimes f_i) = \sum_{i,j} m_i f_j(m_j) \otimes f_i = (\text{rank}_R M) 1.$$

Therefore, the regular character of $\text{End}_R(M)$ is equal to $\text{Tr}(\cdot, z) = (\text{rank}_R M) \text{Tr}$.

2.4. Central characters. Assume that $\text{End}_A(M) \cong R$ as R -algebras. Then, for each $x \in \mathcal{Z}(A)$, we have $x_M = \omega_M(x)1_M$ with $\omega_M(x) \in R$. This yields an R -algebra homomorphism

$$\omega_M: \mathcal{Z}(A) \rightarrow R,$$

called the *central character* of M . Since $\text{Tr}(x_M) = \omega_M(x) \text{Tr}(1_M)$, we have

$$(2.7) \quad \chi_M(x) = \omega_M(x) \text{rank}_R M \quad (x \in \mathcal{Z}(A)).$$

Now assume that A is Frobenius with associative nonsingular bilinear form β , and let $z(M) = z_\beta(M) \in A$ be as in (2.3). Then

$$(2.8) \quad xz(M) = \omega_M(x)z(M) \quad (x \in \mathcal{Z}(A));$$

this follows from the computation $\beta(a, xz(M)) = \beta(ax, z(M)) = \chi_M(ax) = \omega_M(x)\chi_M(a) = \omega_M(x)\beta(a, z(M)) = \beta(a, \omega_M(x)z(M))$ for $a \in A$.

If β is symmetric then $z(M) \in \mathcal{Z}(A)$ and we can define the *index*¹ of M by

$$(2.9) \quad [A : M]_\beta := \omega_M(z(M)) \in R.$$

2.5. Integrality. Let A be a Frobenius R -algebra, with associative nonsingular bilinear form β and dual bases $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$. Assume that we are given a subring $S \subseteq R$. An S -subalgebra $B \subseteq A$ will be called a *weak S -form of (A, β)* if the following conditions are satisfied:

- (i) B is a finitely generated S -module, and
- (ii) $\sum_i x_i \otimes y_i \in A \otimes_R A$ belongs to the image of $B \otimes_S B$ in $A \otimes_R A$.

Recall from Section 1.2.2 that the element $\sum_i x_i \otimes y_i$ only depends on β . Note also that (ii) implies that $BR = A$. Indeed, for any $a \in A$, the map $\text{Id}_A \otimes \beta(a, \cdot): A \otimes_R A \rightarrow A \otimes_R R = A$ sends $\sum_i x_i \otimes y_i$ to a by (1.6), and it sends the image of $B \otimes_S B$ in $A \otimes_R A$ to BR .

LEMMA 3. *Let A be a symmetric R -algebra with form β . Assume that $\text{End}_A(M) \cong R$, and let $z(M) \in \mathcal{Z}(A)$ be as in (2.3). If there exists a weak S -form of (A, β) for some subring $S \subseteq R$, then $\chi_M(z(M))$ and $[A : M]_\beta = \omega_M(z(M))$ are integral over S .*

PROOF. All $b \in B$ are integral over S by condition (i). Hence the endomorphisms $b_M \in \text{End}_R(M)$ are integral over S , and so are their traces $\chi_M(b)$; see Prop. 17 in [1, V.1.6] for the latter. By (ii), it follows that $\chi_M(z(M)) = \sum_i \chi_M(x_i)\chi_M(y_i)$ is integral over S . Moreover, the subring $S' = S[\chi_M(B)] \subseteq R$ is finite over S by (i). Thus, all elements of $BS' \subseteq A$ are integral over S . In particular, this holds for $z(M)$, whence $\omega_M(z(M))$ is integral over S . \square

2.6. Idempotents.

PROPOSITION 4. *Let (A, β) be a symmetric R -algebra. Let $e = e^2 \in A$ be an idempotent and assume that the A -module $M = Ae$ satisfies $\text{End}_A(M) \cong R$ as R -algebras. Let $\omega_M: \mathcal{Z}(A) \rightarrow R$ be the central character of M and let $z(M) \in \mathcal{Z}(A)$ be as in (2.3). Then:*

- (a) $[A : M]_\beta$ is invertible in R , with inverse $\beta(e, 1)$.
- (b) $e(M) := [A : M]_\beta^{-1}z(M) \in \mathcal{Z}(A)$ is an idempotent satisfying $\omega_M(e(M)) = 1$ and

$$xe(M) = \omega_M(x)e(M) \quad (x \in \mathcal{Z}(A)).$$

¹The index $[A : M]$ is also called the *Schur element* of M ; see, e.g., [3, Lemma 4.6].

(c) Let $z = z_\beta = \sum_i x_i y_i \in \mathcal{Z}(A)$ be as in (2.6). Then

$$\omega_M(z) \cdot \text{rank}_R M = [A : M]_\beta \cdot \text{rank}_R e(M)A .$$

PROOF. Note that $eM = eAe \cong \text{End}_A(M) \cong R$. By (2.2), it follows that $\chi_M(e) = \text{rank}_R eM = 1$. Since $xe = \omega_M(x)e$ for $x \in \mathcal{Z}(A)$, we obtain

$$1 = \chi_M(e) \stackrel{(2.4)}{=} \beta(e, z(M)) = \beta(z(M)e, 1) = \omega_M(z(M))\beta(e, 1) .$$

This proves (a). In (b), $\omega_M(e(M)) = 1$ is clear by definition of $e(M)$, and (2.8) gives the identity $xe(M) = \omega_M(x)e(M)$ for all $x \in \mathcal{Z}(A)$. Together, these facts imply that $e(M)$ is an idempotent. Finally, the following computation proves (c):

$$\begin{aligned} \omega_M(z) \text{rank}_R M &\stackrel{(2.7)}{=} \chi_M(z) \stackrel{(2.4)}{=} \beta(z, z(M)) \stackrel{(2.6)}{=} \chi_{\text{reg}}(z(M)) \\ &= \omega_M(z(M))\chi_{\text{reg}}(e(M)) \stackrel{(2.2)}{=} \omega_M(z(M)) \text{rank}_R e(M)A . \end{aligned}$$

□

We now specialize the foregoing to separable algebras. For background, see DeMeyer and Ingraham [5]. We mention that, by a theorem of Endo and Watanabe [7, Theorem 4.2], any faithful separable R -algebra is symmetric.

PROPOSITION 5. Assume that the algebra A is separable and that the A -module M is cyclic and satisfies $\text{End}_A(M) \cong R$. Let $e(M) \in \mathcal{Z}(A)$ be the idempotent in Proposition 4(b). Then:

- (a) $e(M)A \cong \text{End}_R(M)$ and $\text{rank}_R e(M)A = (\text{rank}_R M)^2$.
- (b) $\chi_{\text{reg}} e(M) = (\text{rank}_R M)\chi_M$, where χ_{reg} is the regular character of A .

PROOF. We first note that M , being assumed projective over R , is in fact projective over A by [5, Proposition II.2.3]. Since M is cyclic, we have $M \cong Ae$ with $e = e^2 \in A$; so Proposition 4 applies.

(a) It suffices to show that $e(M)A \cong \text{End}_R(M)$, because the rank of $\text{End}_R(M) \cong M \otimes_R M^\vee$ equals $(\text{rank}_R M)^2$. Since M is finitely generated projective and faithful over R , the R -algebra $\text{End}_R(M)$ is Azumaya; see [5, Proposition II.4.1]. The Double Centralizer Theorem [5, Proposition II.1.11 and Theorem II.4.3] and our hypothesis $\text{End}_A(M) \cong R$ together imply that the map $A \rightarrow \text{End}_R(M)$, $a \mapsto a_M$, is surjective. Letting $I = \text{ann}_A M$ denote the kernel of this map, we further know by [5, Corollary II.3.7 and Theorem II.3.8] that $I = (I \cap \mathcal{Z}(A))A$. Finally, Proposition 4 tells us that $I \cap \mathcal{Z}(A)$ is generated by $1 - e(M)$, which proves (a).

(b) In view of the isomorphism $e(M)A \cong \text{End}_R(M)$, $e(M)a \mapsto a_M$ from part (a) and Example 2, we have $\chi_{\text{reg}}(e(M)a) = (\text{rank}_R M) \text{Tr}(a_M) = (\text{rank}_R M)\chi_M(a)$. □

3. Separability

The R -algebra A is assumed to be Frobenius throughout this section. We fix a non-singular associative R -bilinear form $\beta: A \times A \rightarrow R$ and dual bases $\{x_i\}_1^n, \{y_i\}_1^n \subseteq A$ for β .

3.1. The Casimir operator. Define a map, called the *Casimir operator*² of (A, β) , by

$$(3.1) \quad c = c_\beta: A \rightarrow \mathcal{Z}(A), \quad a \mapsto \sum_i y_i a x_i.$$

In order to check that $c(a) \in \mathcal{Z}(A)$ we calculate, for $a, b \in A$,

$$bc(a) \stackrel{(1.8)}{=} \sum_{i,j} \beta(x_j, by_i) y_j a x_i = \sum_{i,j} y_j a \beta(x_j b, y_i) x_i \stackrel{(1.6)}{=} c(a)b.$$

The map c is independent of the choice of dual bases $\{x_i\}, \{y_i\}$, because $\sum_i x_i \otimes y_i \in A \otimes_R A$ only depends on β ; see Section 1.2.2.

In case β is symmetric, $\{y_i\}, \{x_i\}$ are also dual bases for β , and hence

$$c(a) = \sum_i x_i a y_i.$$

In particular, the element $z = z_\beta$ in (2.6) arises as $z_\beta = c(1)$ if β is symmetric. We will refer to the element $z = z_\beta$ as the *Casimir element*³ of the symmetric algebra (A, β) ; it depends on β only up to a central unit (see 3.2 below).

3.2. The Casimir ideal. Since c is $\mathcal{Z}(A)$ -linear, the image $c(A)$ of the map $c = c_\beta$ in (3.1) is an ideal of $\mathcal{Z}(A)$. This ideal will be called the *Casimir ideal*⁴ of A ; it does not depend on the choice of the bilinear form β . Indeed, recall from Section 1.2.5 that any two nonsingular forms $\beta, \gamma \in \text{Bil}_{\text{assoc}}(A; R)$ are related by $\gamma(\cdot, a) = \beta(\cdot, au)$ for some unit $u \in A$. Hence, if $\{x_i\}, \{y_i\} \subseteq A$ are dual bases for β then $\{x_i\}, \{y_i u^{-1}\} \subseteq A$ are dual bases for γ . Therefore,

$$c_\gamma(a) = c_\beta(u^{-1}a) \quad (a \in A).$$

If β and γ are both symmetric then $u \in \mathcal{Z}(A)$ and so $c_\gamma(a) = u^{-1}c_\beta(a)$.

3.3. The separability locus. For a given Frobenius algebra A , we will now determine the set of all primes $\mathfrak{p} \in \text{Spec } R$ such that the $Q(R/\mathfrak{p})$ -algebra $A \otimes_R Q(R/\mathfrak{p})$ is separable or, equivalently, the $R_\mathfrak{p}$ -algebra $A \otimes_R R_\mathfrak{p}$ is separable [5, Theorem II.7.1]. The collection of these primes is called the separability locus of A .

PROPOSITION 6. *The separability locus of a Frobenius R -algebra A is*

$$\text{Spec } R \setminus V(c(A) \cap R) = \{\mathfrak{p} \in \text{Spec } R \mid \mathfrak{p} \not\supseteq c(A) \cap R\}.$$

PROOF. The case of a base field R is covered by Higman's Theorem which states that a Frobenius algebra A over a field R is separable if and only if $c(A) = \mathcal{Z}(A)$ or, equivalently, $1 \in c(A)$; see [11, Theorem 1] or [4, 71.6].

Now let R be arbitrary and let $\mathfrak{p} \in \text{Spec } R$ be given. Put $F = Q(R/\mathfrak{p})$ and $A_F = A \otimes_R F$. We know that A_F is Frobenius, with form $\bar{\beta} = \beta \otimes_R \text{Id}_F$ and corresponding dual bases $\{\bar{x}_i\}, \{\bar{y}_i\}$, where $\bar{\cdot}: A \rightarrow A_F, \bar{x} = x \otimes 1$, denotes the canonical map. By Higman's Theorem, we know that A_F is separable if and only if $1 \in c(A_F) = \overline{c(A)}F$. Thus:

The F -algebra A_F is separable if and only if $(A\mathfrak{p} + c(A)) \cap R \not\subseteq \mathfrak{p}$.

²We follow the terminology of Higman's original article [11]; the map c is called the *Gaschütz-Ikeda operator* in [4].

³In [3], the Casimir element z is referred to as the *central Casimir element*, while $\sum_i x_i \otimes y_i \in A \otimes_R A$ is called the Casimir element. The latter element is referred to as the *Frobenius element* in [13].

⁴The Casimir ideal $c(A)$ of a symmetric algebra A coincides with the *projective center* of A in the terminology of [3, Prop. 3.13]. It is also called the *Higman ideal* of A ; see [10].

If $\mathfrak{p} \not\supseteq c(A) \cap R$ then clearly $(A\mathfrak{p} + c(A)) \cap R \not\supseteq \mathfrak{p}$, and hence A_F is separable. Conversely, assume that $c(A) \cap R \subseteq \mathfrak{p}$. Since A is integral over its center $\mathcal{Z}(A)$, we have $c(A)A \cap \mathcal{Z}(A) \subseteq \sqrt{c(A)}$, the radical of the ideal $c(A)$; see Lemma 1 in [1, V.1.1]. Therefore, $c(A)A \cap R \subseteq \sqrt{c(A)} \cap R \subseteq \mathfrak{p}$. By Going Up [19, 13.8.14], there exists a prime ideal P of A with $c(A)A \subseteq P$ and $P \cap R = \mathfrak{p}$. But then $(A\mathfrak{p} + c(A)) \cap R \subseteq P \cap R = \mathfrak{p}$, and hence A_F is not separable. This proves the proposition. \square

3.4. Norms. Assume that the algebra A is free of rank n over R . Then the *norm* of an element $a \in A$ is defined by

$$N(a) = \det a_A \in R,$$

where $(a_A: x \mapsto ax) \in \text{End}_R(A) = M_n(R)$ is the left regular representation of A as in Section 2.3. The norm map $N: A \rightarrow R$ satisfies $N(ab) = N(a)N(b)$ and $N(r) = r^n$ for $a, b \in A$ and $r \in R$. Up to sign, $N(a)$ is the constant term of the characteristic polynomial of a_A . Since a satisfies this polynomial by the Cayley-Hamilton Theorem, we see that a divides $N(a)$ in $R[a] \subseteq A$. Putting

$$N(c(A)) = \sum_{a \in c(A)} RN(a),$$

we obtain $N(c(A)) \subseteq c(A) \cap R$. Moreover, since $N(r) = r^n$ for $r \in R$, we further conclude that, for any $\mathfrak{p} \in \text{Spec } R$, we have

$$\mathfrak{p} \supseteq N(c(A)) \iff \mathfrak{p} \supseteq c(A) \cap R.$$

4. Additional structure: augmentations, involutions, positivity

4.1. Augmentations and integrals. Let (A, β) be a Frobenius algebra and suppose that A has an augmentation, that is, an algebra homomorphism

$$\varepsilon: A \rightarrow R.$$

Put $\Lambda_\beta = r_\beta^{-1}(\varepsilon) \in A$, where r_β is as in Section 1.2; so $\beta(\Lambda_\beta, \cdot) = \varepsilon$. From (1.8), we obtain the following expression in terms of dual bases $\{x_i\}, \{y_i\}$ for β :

$$(4.1) \quad \Lambda_\beta = \sum_i \varepsilon(y_i)x_i.$$

The computation $\beta(\Lambda_\beta a, \cdot) = \beta(\Lambda_\beta, a \cdot) = \varepsilon(a)\varepsilon = \varepsilon(a)\beta(\Lambda_\beta, \cdot)$ for all $a \in A$ shows that $\Lambda_\beta a = \varepsilon(a)\Lambda_\beta$. Conversely, if $t \in A$ satisfies $ta = \varepsilon(a)t$ for all $a \in A$ then $\beta(t, a) = \beta(ta, 1) = \varepsilon(a)\beta(t, 1)$, whence $t = \beta(t, 1)\Lambda_\beta$. We put

$$\int_A^r = \{t \in A \mid ta = \varepsilon(a)t \text{ for all } a \in A\}$$

and call the elements of \int_A^r *right integrals* in A . The foregoing shows that $\int_A^r = R\Lambda_\beta$. Moreover, $rr_\beta^{-1}(\varepsilon) = 0$ implies $r\varepsilon = 0$ and hence $r = 0$. Thus:

$$\int_A^r = R\Lambda_\beta \cong R.$$

Similarly, one can define the R -module \int_A^l of *left integrals* in A and show that

$$\int_A^l = R\Lambda'_\beta \cong R \quad \text{with } \Lambda'_\beta = \sum_i \varepsilon(x_i)y_i = l_\beta^{-1}(\varepsilon).$$

Define the ideal $\text{Dim}_\varepsilon A$ of R by

$$(4.2) \quad \text{Dim}_\varepsilon A := \varepsilon(\int_A^r) = \varepsilon(\int_A^l) = \varepsilon(c(A)) = (\varepsilon(z)),$$

where $c(A)$ is the Casimir ideal and $z = z_\beta \in \mathcal{Z}(A)$ is as in (2.6). Note that always $c(A) \cap R \subseteq \varepsilon(c(A))$; so

$$(4.3) \quad c(A) \cap R \subseteq \text{Dim}_\varepsilon A.$$

If β is symmetric then $\mathbf{r}_\beta = \mathbf{l}_\beta$ and hence $\Lambda_\beta = \Lambda'_\beta$ and $\int_A^r = \int_A^l =: \int_A$. For further information on the material in this section, see [13, 6.1].

4.2. Involutions. Let A be a symmetric algebra with symmetric associative bilinear form $\beta: A \times A \rightarrow R$. Suppose further that A has an involution $*$, that is, an R -linear endomorphism of A satisfying $(xy)^* = y^*x^*$ and $x^{**} = x$ for all $x, y \in A$. If A is R -free with basis $\{x_i\}_1^n$ satisfying

$$(4.4) \quad \beta(x_i, x_j^*) = \delta_{i,j},$$

then we will call A a *symmetric $*$ -algebra*. The Casimir operator $c = c_\beta: A \rightarrow \mathcal{Z}(A)$ takes the form

$$c(a) = \sum_i x_i^* a x_i = \sum_i x_i a x_i^*,$$

and the Casimir element $z = z_\beta = c(1)$ is

$$(4.5) \quad z = \sum_i x_i^* x_i = \sum_i x_i x_i^*.$$

LEMMA 7. *Let $(A, \beta, *)$ be a symmetric $*$ -algebra. Then:*

- (a) β is $*$ -invariant: $\beta(x, y) = \beta(x^*, y^*)$ for all $x, y \in A$.
- (b) The Casimir operator c is $*$ -equivariant: $c(a)^* = c(a^*)$ for all $a \in A$. In particular, $z^* = z$.
- (c) If $a = a^* \in \mathcal{Z}(A)$ then the matrix of $(a_A: x \mapsto ax) \in \text{End}_R(A)$ with respect to the R -basis $\{x_i\}$ of A is symmetric.

PROOF. Part (a) follows from $\beta(x_i, x_j^*) = \delta_{i,j} = \beta(x_j, x_i^*) = \beta(x_i^*, x_j)$, and (b) follows from $c_\beta(a)^* = \sum_i (x_i^* a x_i)^* = \sum_i x_i^* a^* x_i = c_\beta(a^*)$.

(c) Let $(a_{i,j}) \in M_n(R)$ be the matrix of a_A ; so $ax_j = \sum_i a_{i,j} x_i$. We compute using associativity, symmetry and $*$ -invariance of β :

$$a_{i,j} = \beta(x_i^*, ax_j) = \beta(ax_i^*, x_j) = \beta(x_j, ax_i^*) = \beta(x_j^*, ax_i) = a_{j,i}.$$

□

4.3. Positivity. Let $(A, \beta, *)$ be a symmetric $*$ -algebra with R -basis $\{x_i\}_1^n$ satisfying (4.4). Assume that $R \subseteq \mathbb{R}$ and put $R_+ = \{r \in R \mid r \geq 0\}$. If

$$(4.6) \quad A_+ := \bigoplus_i R_+ x_i \text{ is closed under multiplication and stable under } *$$

then we will then say that A has a *positive structure* and call A_+ the *positive cone* of A .

We now consider the endomorphism $(z_A: x \mapsto zx) \in \text{End}_R(A)$ for the Casimir element $z = z_\beta$ in (4.5). By Lemma 7, we know that the matrix of z_A with respect to the basis $\{x_i\}$ is symmetric. The following proposition gives further information.

PROPOSITION 8. *Let $(A, \beta, *)$ be a symmetric $*$ -algebra over the ring $R \subseteq \mathbb{R}$, and let $z = z_\beta$ be the Casimir element.*

- (a) *The matrix of z_A with respect to the basis $\{x_i\}$ is symmetric and positive definite. In particular, all eigenvalues of z_A are positive real numbers that are integral over R .*

(b) If A has a positive structure and an augmentation $\varepsilon: A \rightarrow R$ satisfying $\varepsilon(a) > 0$ for all $0 \neq a \in A_+$. Then the largest eigenvalue of z_A is $\varepsilon(z)$.

PROOF. (a) Let $Z = (z_{i,j})$ be the matrix of z_A ; so $z_{i,j} = \beta(x_i^*, zx_j)$. Extending $*$ and β to $A_{\mathbb{R}} = A \otimes_{\mathbb{R}} \mathbb{R}$ by linearity, one computes for $x = \sum_i \xi_i x_i \in A_{\mathbb{R}}$:

$$\sum_l \beta((x_l x)^*, x_l x) = \beta(x^*, zx) = (\xi_1, \dots, \xi_n) Z (\xi_1, \dots, \xi_n)^t.$$

The sum on the left is positive if $x \neq 0$, because $\beta(y^*, y) = \sum \eta_i^2$ for $y = \sum_i \eta_i x_i \in A_{\mathbb{R}}$. This shows that Z is positive definite. The assertion about the eigenvalues of Z is a standard fact about positive definite symmetric matrices over the reals.

(b) By hypothesis on A_+ , the matrix of a_A with respect to the basis $\{x_i\}$ has non-negative entries for any $a \in A_+$. Moreover, the Casimir element $z = \sum_i x_i^* x_i$ belongs to A_+ , and so z_A is non-negative. Now let $\Lambda = \sum_i \varepsilon(x_i^*) x_i \in \int_A$ be the integral of A that is associated to the augmentation ε ; see (4.1). Then Λ is an eigenvector for z_A with eigenvalue $\varepsilon(z)$. Since all $\varepsilon(x_i^*) > 0$, it follows that $\varepsilon(z)$ is in fact the largest (Frobenius-Perron) eigenvalue of z_A is $\dim_{\mathbb{k}} H$; see [9, Chapter XIII, Remark 3 on p. 63/4]. \square

COROLLARY 9. If A is a symmetric $*$ -algebra over the ring $R \subseteq \mathbb{R}$, then the Casimir element z is a regular element of A . Furthermore, $A \otimes_R Q(R)$ is separable.

PROOF. Regularity of z is clear from Proposition 8(a). Since z is integral over R , it follows that $z\mathcal{Z}(A) \cap R \neq 0$. Therefore, $c(A) \cap R \neq 0$ and Proposition 6 gives that $A \otimes_R Q(R)$ is separable. \square

Part 2. HOPF ALGEBRAS

Throughout this part, H will denote a finitely generated projective Hopf algebra over the commutative ring R (which will be assumed to be a field in Section 6), with unit u , multiplication m , counit ε , comultiplication Δ , and antipode \mathcal{S} . We will use the Sweedler notation $\Delta h = \sum h_1 \otimes h_2$.

In addition to the bimodule action of H on H^\vee in (1.1), we now also have an analogous bimodule action of the dual algebra H^\vee on $H = H^{\vee\vee}$. In order to avoid confusion, it is customary to indicate the target of the various actions by \rightarrow or \leftarrow :

$$(4.7) \quad \begin{aligned} \langle a \rightarrow f \leftarrow b, c \rangle &= \langle f, bca \rangle & (a, b, c \in H, f \in H^\vee), \\ \langle e, f \rightarrow a \leftarrow g \rangle &= \langle gef, a \rangle & (e, f, g \in H^\vee, a \in H). \end{aligned}$$

Here and for the remainder of this article, $\langle \cdot, \cdot \rangle: H^\vee \times H \rightarrow R$ denotes the evaluation pairing.

5. Frobenius Hopf algebras over commutative rings

5.1. The following result is due to Larson-Sweedler [16], Pareigis [22], and Oberst-Schneider [21].

- THEOREM 10. (a) The antipode \mathcal{S} is bijective. Consequently, $\int_H^l = \mathcal{S}(\int_H^r)$.
 (b) H is a Frobenius R -algebra if and only if $\int_H^r \cong R$. This always holds if $\text{Pic } R = 1$. Furthermore, if H is Frobenius then so is the dual algebra H^\vee .
 (c) Assume that H is Frobenius. Then H is symmetric if and only if
 (i) H is unimodular (i.e., $\int_H^l = \int_H^r$), and
 (ii) \mathcal{S}^2 is an inner automorphism of H .

PROOF. Part (a) is [22, Proposition 4] and (c) is [21, 3.3(2)]. For necessity of the condition $\int_H^r \cong R$ in (b), in the more general context of augmented Frobenius algebras, see Section 4.1. Conversely, if $\int_H^r \cong R$ holds then [22, Theorem 1] asserts that the dual algebra H^\vee is Frobenius. This forces $\int_{H^\vee}^r$ to be free of rank 1 over R , and hence H is Frobenius by [22, Theorem 1]. The statement about $\text{Pic } R = 1$ is a consequence of the fact that the R -module \int_H^r is invertible (i.e., locally free of rank 1) for any finitely generated projective Hopf R -algebra H ; see [22, Proposition 3]. \square

5.2. We spell out some of the data associated with a Frobenius Hopf algebra H referring the reader to the aforementioned references [16], [22], [21] for complete details.

Fix a generator $\Lambda \in \int_H^r$. There is a unique $\lambda \in \int_{H^\vee}^l$ satisfying $\lambda \leftarrow \Lambda = \varepsilon$ or, equivalently, $\langle \lambda, \Lambda \rangle = 1$. Note that this equation implies that $\int_{H^\vee}^l = R\lambda$, because $\int_{H^\vee}^l$ is an invertible R -module. A nonsingular associative bilinear form $\beta = \beta_\lambda$ for H is given by

$$(5.1) \quad \beta(a, b) = \langle \lambda, ab \rangle \quad (a, b \in H).$$

Dual bases $\{x_i\}, \{y_i\}$ for β are given by $\sum x_i \otimes y_i = \sum \Lambda_2 \otimes \mathcal{S}(\Lambda_1)$:

$$(5.2) \quad a = \sum \langle \lambda, a\mathcal{S}(\Lambda_1) \rangle \Lambda_2 = \sum \langle \lambda, \Lambda_2 a \rangle \mathcal{S}(\Lambda_1) \quad (a \in H).$$

By [16, p. 83], the form β is *orthogonal* for the right action \leftarrow of H^\vee on H :

$$(5.3) \quad \beta(a, b \leftarrow f) = \beta(a \leftarrow \mathcal{S}^\vee(f), b) \quad (a, b \in H, f \in H^\vee),$$

where $\mathcal{S}^\vee = \cdot \circ \mathcal{S}$ is the antipode of H^\vee .

5.3. By (5.2) the Casimir operator has the form

$$c = c_\Lambda: H \rightarrow \mathcal{Z}(H), \quad a \mapsto \sum \mathcal{S}(\Lambda_1) a \Lambda_2.$$

In particular, $c(1) = \langle \varepsilon, \Lambda \rangle \in R$. Therefore, equality holds in (4.3):

$$(5.4) \quad \text{Dim}_\varepsilon H = \langle \varepsilon, \int_H^r \rangle = \langle \varepsilon, \int_H^l \rangle = c(H) \cap R.$$

Proposition 6 now gives the following classical result of Larson and Sweedler [16].

COROLLARY 11. *The separability locus of a Frobenius Hopf algebra H over R is*

$$\text{Spec } R \setminus V(\text{Dim}_\varepsilon H).$$

In particular, H is separable if and only if $\langle \varepsilon, \Lambda \rangle = 1$ for some right or left integral $\Lambda \in H$.

The equality $\langle \varepsilon, \Lambda \rangle = 1$ implies that Λ is an idempotent two-sided integral such that $\int_H^r = \int_H^l = R\Lambda$, because \int_H^r and \int_H^l are invertible R -modules.

5.4. Let H be a Frobenius Hopf algebra over R , and let $\Lambda \in \int_H^r$ and $\lambda \in \int_{H^\vee}^l$ be as in 5.2; so $\langle \lambda, \Lambda \rangle = 1$. The isomorphisms r_β and l_β in (1.4) and (1.5) for the the bilinear form $\beta = \beta_\lambda$ in (5.1) will now be denoted by r_λ and l_λ , respectively:

$$(5.5) \quad \begin{aligned} r_\lambda: H_H &\xrightarrow{\sim} H_H^\vee, & a &\mapsto (\beta(a, \cdot) = \lambda \leftarrow a), \\ l_\lambda: {}_H H &\xrightarrow{\sim} {}_H H^\vee, & a &\mapsto (\beta(\cdot, a) = a \rightarrow \lambda). \end{aligned}$$

Equation (5.3) states that

$$(5.6) \quad r_\lambda(a \leftarrow \mathcal{S}^\vee(f)) = f r_\lambda(a) \quad \text{and} \quad l_\lambda(a \leftarrow f) = \mathcal{S}^\vee(f) l_\lambda(a).$$

for $a \in H, f \in H^\vee$. Since $r_\lambda(\Lambda) = \varepsilon$ is the identity of H^\vee , we obtain the following expression for the inverse of r_λ :

$$(5.7) \quad r_\lambda^{-1}: H_H^\vee \xrightarrow{\sim} H_H, \quad f \mapsto (\Lambda \leftarrow \mathcal{S}^\vee(f)).$$

5.5. In contrast with the Frobenius property, symmetry does not generally pass from H to H^\vee ; see [17, 2.5]. We will call H *bi-symmetric* if both H and H^\vee are symmetric.

LEMMA 12. Assume that H is Frobenius and fix integrals $\Lambda \in \int_H^r$ and $\lambda \in \int_{H^\vee}^l$ such that $\langle \lambda, \Lambda \rangle = 1$, as in Sections 5.2, 5.4. Then:

- (a) If H is involutory (i.e., $S^2 = 1$) then the regular character χ_{reg} of H is given by $\chi_{\text{reg}} = \langle \varepsilon, \Lambda \rangle \lambda$.
- (b) H is symmetric and involutory if and only if H is unimodular and all left and right integrals in H^\vee belong to H_{trace}^\vee .
- (c) Let H be separable and involutory. Then H is bi-symmetric. Furthermore,

$$\int_{H^\vee} = R \chi_{\text{reg}} \quad \text{and} \quad \text{Dim}_{u^\vee} H^\vee = (\text{rank}_R H) ,$$

where $u^\vee = \langle \cdot, 1 \rangle$ the counit of H^\vee .

PROOF. (a) Equations (2.6), (5.1) and (5.2), with $z = \sum_i x_i y_i = \sum \Lambda_2 \mathcal{S}(\Lambda_1) = \langle \varepsilon, \Lambda \rangle$ (using $S^2 = 1$), give $\chi_{\text{reg}} = \langle \lambda, \cdot z \rangle = \langle \varepsilon, \Lambda \rangle \lambda$.

(b) First assume that H is symmetric and involutory. Then H is unimodular by Theorem 10(c). By [21, 3.3(1)] we further know that the Nakayama automorphism of H is equal to S^2 , and hence it is the identity. Thus, $\lambda \leftarrow a = a \rightarrow \lambda$ for all $a \in H$, which says that λ is a trace form. Hence, $\int_{H^\vee}^l \subseteq H_{\text{trace}}^\vee$. Since H_{trace}^\vee is stable under the antipode S^\vee of H^\vee , it also contains $\int_{H^\vee}^r$. The converse follows by retracing these steps.

(c) Now let H be separable and involutory. By Corollary 11 and the subsequent remark, H is unimodular and we may choose $\Lambda \in \int_H$ such that $\langle \varepsilon, \Lambda \rangle = 1$. Part (a) gives $\int_{H^\vee}^l = R \chi_{\text{reg}}$. The computation

$$\langle \mathcal{S}^\vee(\chi_{\text{reg}}), a \rangle = \text{Tr}(\mathcal{S}(a)_A) = \text{Tr}(\mathcal{S} \circ_A a \circ \mathcal{S}^{-1}) = \text{Tr}(Aa) \stackrel{(2.5)}{=} \langle \chi_{\text{reg}}, a \rangle$$

for $a \in A$ shows that $\mathcal{S}^\vee(\chi_{\text{reg}}) = \chi_{\text{reg}}$. Therefore, we also have $\int_{H^\vee}^r = R \chi_{\text{reg}}$. In view of Theorem 10(c), this shows that H is bi-symmetric. Finally, since $\langle \chi_{\text{reg}}, 1 \rangle = \text{rank}_R H$, equation (5.4) yields $\text{Dim}_{u^\vee} H^\vee = (\text{rank}_R H)$. \square

As was observed in the proof of (b), the maps r_λ and l_λ in (5.5) coincide if and only if λ is a trace form. In this case, we will denote the (H, H) -bimodule isomorphism $r_\lambda = l_\lambda$ by b_λ :

$$(5.8) \quad b_\lambda : {}_H H_H \xrightarrow{\sim} {}_H H^\vee_H, \quad a \mapsto (\lambda \leftarrow a = a \rightarrow \lambda) .$$

We also remark that the formula $\text{Dim}_{u^\vee} H^\vee = (\text{rank}_R H)$ in (c) is a special case of the following formula which holds for any involutory H ; see [21, 3.6]:

$$(5.9) \quad \text{Dim}_\varepsilon H \cdot \text{Dim}_{u^\vee} H^\vee = (\text{rank}_R H) .$$

5.6. We let

$$C(H) = \{a \in H \mid \sum a_1 \otimes a_2 = \sum a_2 \otimes a_1\}$$

denote the R -subalgebra of H consisting of all cocommutative elements. Thus, $C(H^\vee) = H_{\text{trace}}^\vee$. Recall from Lemma 12(b) that all integrals in H^\vee are cocommutative if H is symmetric and involutory.

LEMMA 13. Let H be bi-symmetric and involutory. Fix a generator $\lambda \in \int_{H^\vee}$. Then the (H, H) -bimodule isomorphism b_λ in (5.8) restricts to an isomorphisms

$$\mathcal{Z}(H) \xrightarrow{\sim} H_{\text{trace}}^\vee \quad \text{and} \quad C(H) \xrightarrow{\sim} \mathcal{Z}(H^\vee) .$$

PROOF. The isomorphism $\mathcal{Z}(H) \xrightarrow{\sim} H_{\text{trace}}^\vee = C(H^\vee)$ is (1.3). By the same token, fixing a (necessarily cocommutative) generator $\Lambda \in \int_H$ such that $\langle \lambda, \Lambda \rangle = 1$, we obtain that \mathbf{b}_Λ is an (H^\vee, H^\vee) -bimodule isomorphism $H^\vee \xrightarrow{\sim} H^{\vee\vee} = H$ that restricts to an isomorphism $\mathcal{Z}(H^\vee) \xrightarrow{\sim} C(H)$. By equation (5.7) we have

$$(5.10) \quad \mathbf{b}_\Lambda(\mathcal{S}^\vee(f)) = \mathbf{b}_\Lambda^{-1}(f)$$

for $f \in H^\vee$. Since $\mathcal{Z}(H^\vee)$ is stable under the antipode \mathcal{S}^\vee of H^\vee , we conclude that \mathbf{b}_Λ^{-1} restricts to an isomorphism $\mathcal{Z}(H^\vee) \xrightarrow{\sim} C(H)$, and hence \mathbf{b}_Λ restricts to an isomorphism $C(H) \xrightarrow{\sim} \mathcal{Z}(H^\vee)$. \square

5.7. Turning to modules now, we review some standard constructions and facts. For any two left H -modules M and N , the tensor product $M \otimes_R N$ becomes an H -module via Δ , and $\text{Hom}_R(M, N)$ carries the following H -module structure:

$$(a\varphi)(m) = \sum a_1\varphi(\mathcal{S}(a_2)m)$$

for $a \in H, m \in M, \varphi \in \text{Hom}_R(M, N)$. In particular, viewing R as H -module via ε , the H -action on the dual $M^\vee = \text{Hom}_R(M, R)$ takes the following form:

$$\langle af, m \rangle = \langle f, \mathcal{S}(a)m \rangle$$

for $a \in H, m \in M, f \in M^\vee$. The H -invariants in $\text{Hom}_R(M, N)$ are exactly the H -module maps:

$$\text{Hom}_R(M, N)^H = \{\varphi \in \text{Hom}_R(M, N) \mid a\varphi = \langle \varepsilon, a \rangle \varphi \forall a \in A\} = \text{Hom}_H(M, N);$$

see, e.g., [28, Lemma 1]. Moreover, it is easily checked that the canonical map

$$N \otimes_R M^\vee \rightarrow \text{Hom}_R(M, N), \quad n \otimes f \mapsto (m \mapsto \langle f, m \rangle n)$$

is a homomorphism of H -modules. This map is an isomorphism if M or N is finitely generated projective over R ; see [2, II.4.2].

Finally, we consider the trace map $\text{Tr}: \text{End}_R(M) \cong M \otimes_R M^\vee \rightarrow R$ of Section 2.1.

LEMMA 14. *Let H be involutory and let M be a left H -module that is finitely generated projective over R . Then the trace map Tr is an H -module map.*

PROOF. In view of the foregoing, it suffices to check H -equivariance of the evaluation map $M \otimes_R M^\vee \rightarrow R$. Using the identity $\sum \mathcal{S}(a_2)a_1 = \langle \varepsilon, a \rangle$ for $a \in H$ (from $\mathcal{S}^2 = 1$), we compute

$$a \cdot (m \otimes f) = \sum a_1 m \otimes a_2 f \mapsto \sum \langle a_2 f, a_1 m \rangle = \sum \langle f, \mathcal{S}(a_2)a_1 m \rangle = \langle \varepsilon, a \rangle \langle f, m \rangle,$$

as desired. \square

5.8. We now focus on modules over a separable involutory Hopf algebra H . In particular, we will compute the image of the central idempotents $e(M)$ from Proposition 4 under the isomorphism $\mathcal{Z}(H) \xrightarrow{\sim} H_{\text{trace}}^\vee$ in Lemma 13 and the β -index $[H : M]_\beta$ of (2.9). Recall that H is bi-symmetric by Lemma 12(c).

PROPOSITION 15. *Assume that H is separable and involutory. Fix $\Lambda \in \int_H, \lambda \in \int_{H^\vee}$ such that $\langle \lambda, \Lambda \rangle = 1$ and let β denote the form (5.1). Then, for every cyclic left H -module M that is finitely generated projective over R and satisfies $\text{End}_H(M) \cong R$,*

- (a) $\text{rank}_R M$ is invertible in R ;
- (b) $[H : M]_\beta = \langle \varepsilon, \Lambda \rangle (\text{rank}_R M)^{-1}$ is invertible in R ;

(c) $\mathbf{b}_\lambda(e(M)) = [H : M]_\beta^{-1} \chi_M$. In particular, $\mathbf{b}_\lambda(e(M)) = (\text{rank}_R M) \chi_M$ holds for $\lambda = \chi_{\text{reg}}$.

PROOF. (a) Our hypothesis $\text{End}_H(M) \cong R$ implies that M is faithful as R -module. Hence, the trace map $\text{Tr} : \text{End}_R(M) \cong M \otimes_R M^\vee \rightarrow R$ is surjective by (2.1), and it is an H -module map by Lemma 14. Moreover, for each $\varphi \in \text{End}_R(M)$, we have $\Lambda\varphi = r_\varphi 1_M$ for some $r_\varphi \in R$, since $\Lambda \text{End}_R(M) \subseteq \text{End}_H(M) \cong R$. Therefore, $\text{Tr}(\Lambda\varphi) = r_\varphi \text{rank}_R M$ and $\text{Tr}(\Lambda\varphi) = \Lambda \text{Tr}(\varphi) = \langle \varepsilon, \Lambda \rangle \text{Tr}(\varphi)$. Choosing φ with $\text{Tr}(\varphi) = 1$, we obtain from Corollary 11 that $\text{Tr}(\Lambda\varphi)$ is a unit in R . Hence so is $\text{rank}_R M$, proving (a).

(b) By Propositions 4(c) and 5(a), we have

$$\omega_M(z) \text{rank}_R M = [H : M]_\beta (\text{rank}_R M)^2,$$

where $z = \sum x_i y_i = \langle \varepsilon, \Lambda \rangle$ is as in the proof of Lemma 12(a). In view on part (a), the above equality amounts to the asserted formula for $[H : M]_\beta$. Finally, invertibility of $[H : M]_\beta$ is Proposition 4(a) (and it also follows from Corollary 11).

(c) Proposition 5(b) gives $\chi_{\text{reg}} \leftarrow e(M) = (\text{rank}_R M) \chi_M$, which is the asserted formula for $\mathbf{b}_\lambda(e(M))$ with $\lambda = \chi_{\text{reg}}$. For general λ , we have $\langle \varepsilon, \Lambda \rangle \mathbf{b}_\lambda(e(M)) = \chi_{\text{reg}} \leftarrow e(M)$ by Lemma 12(a). The formula for $\mathbf{b}_\lambda(e(M))$ now follows from (b). \square

5.9. Assume that, for some subring $S \subseteq R$, there is an S -subalgebra $B \subseteq H$ such that

- (i) B is finitely generated as S -module, and
- (ii) $((S \otimes 1_H) \circ \Delta)(\Lambda) = \sum S(\Lambda_1) \otimes \Lambda_2 \in H \otimes H$ belongs to the image of $B \otimes_S B$ in $H \otimes H$.

Adapting the terminology of Section 2.5, we will call A a *weak R -form* of (H, Λ) . The following corollary is a consequence of Proposition 15(b) and Lemma 3; it is due to Rumynin [24] over fields of characteristic 0.

COROLLARY 16. *Let H be separable and involutory. Assume that, for some generating integral $\Lambda \in \int_H$, there is a weak S -form for (H, Λ) for some subring $S \subseteq R$. Then, for every left H -module M that is finitely generated projective over R and satisfies $\text{End}_H(M) \cong R$, the index $[H : M]_\beta = \langle \varepsilon, \Lambda \rangle (\text{rank}_R M)^{-1}$ is integral over S .*

EXAMPLE 17. The group algebra RG of a finite group G has generating (right and left) integral $\Lambda = \sum_{g \in G} g$. The corresponding integral $\lambda \in \int_{(RG)^\vee}$ with $\langle \lambda, \Lambda \rangle = 1$ is the trace form given by $\langle \lambda, \sum_{g \in G} r_g g \rangle = r_1$. Note that $\langle \varepsilon, \Lambda \rangle = |G|$. Thus, Corollary 11 tells us that RG is semisimple if and only if $|G|$ is a unit in R ; this is Maschke’s classical theorem. Assuming $\text{char } R = 0$, a weak \mathbb{Z} -form for (RG, Λ) is given by the integral group ring $B = \mathbb{Z}G$. Therefore, Corollary 16 yields the following version of Frobenius’ Theorem: The rank of every R -free RG -module M such that $\text{End}_{RG}(M) \cong R$ divides the order of G .

6. Grothendieck rings of semisimple Hopf algebras

From now on, we will focus on the case where $R = \mathbb{k}$ is a field. Throughout, we will assume that H is a split semisimple Hopf algebra over \mathbb{k} . In particular, H is finite-dimensional over \mathbb{k} and hence Frobenius. We will write $\otimes = \otimes_{\mathbb{k}}$ and \mathbb{k} -linear duals will now be denoted by $(\cdot)^*$. Finally, $\text{Irr } H$ will denote a full set of non-isomorphic irreducible H -modules.

6.1. The Grothendieck ring. We review some standard material; for details, see [17].

6.1.1. The *Grothendieck ring* $G_0(H)$ of H is the abelian group that is generated by the isomorphism classes $[V]$ of finite-dimensional left H -modules V modulo the relations $[V] = [U] + [W]$ for each short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$. Multiplication in $G_0(H)$ is given by $[V] \cdot [W] = [V \otimes W]$. The subset $\{[V] \mid V \in \text{Irr } H\} \subseteq G_0(H)$ forms a \mathbb{Z} -basis of $G_0(H)$, and the positive cone

$$G_0(H)_+ := \bigoplus_{V \in \text{Irr } H} \mathbb{Z}_+[V] = \{[V] \mid V \text{ a finite-dimensional } H\text{-module}\}$$

is closed under multiplication. The Grothendieck ring $G_0(H)$ has the *dimension augmentation*,

$$\dim: G_0(H) \rightarrow \mathbb{Z}, \quad [V] \mapsto \dim_{\mathbb{k}} V,$$

and an involution given by $[V]^* = [V^*]$, where the dual $V^* = \text{Hom}_{\mathbb{k}}(V, \mathbb{k})$ has H -action as in Section 5.7. The basis $\{[V] \mid V \in \text{Irr } H\}$ is stable under the involution $*$, and hence so is the positive cone $G_0(H)_+$.

6.1.2. The Grothendieck ring $G_0(H)$ is a symmetric $*$ -algebra over \mathbb{Z} . A suitable bilinear form β is given by

$$(6.1) \quad \beta([V], [W]) = \dim_{\mathbb{k}} \text{Hom}_H(V, W^*).$$

Using the standard isomorphism $(W \otimes V^*)^H \cong \text{Hom}_H(V, W)$, where $(\cdot)^H$ denotes the space of H -invariants, this form is easily seen to be \mathbb{Z} -bilinear, associative, symmetric, and $*$ -invariant. Dual \mathbb{Z} -bases of $G_0(H)$ are provided by $\{[V] \mid V \in \text{Irr } H\}$ and $\{[V^*] \mid V \in \text{Irr } H\}$:

$$\beta([V'], [V^*]) = \delta_{[V'], [V]} \quad (V, V' \in \text{Irr } H).$$

The integral in $G_0(H)$ that is associated to the dimension augmentation of $G_0(H)$ as in Section §4.1 is the class $[H]$ of the regular representation of H : $\beta([H], [V]) = \dim_{\mathbb{k}} V$. Thus,

$$(6.2) \quad \int_{G_0(H)} = \mathbb{Z} [H].$$

6.1.3. The *character map*

$$\chi: G_0(H) \rightarrow H^*, \quad [V] \mapsto \chi_V$$

is a ring homomorphism that respects augmentations:

$$\begin{array}{ccc} G_0(H) & \xrightarrow{\chi} & H^* \\ \text{dim} \downarrow & & \downarrow u^* \\ \mathbb{Z} & \longrightarrow & \mathbb{k} \end{array}$$

Moreover,

$$\chi_{V^*} = \mathcal{S}^*(\chi_V) = \chi_V \circ \mathcal{S}.$$

Thus, if H is involutory then χ also commutes with the standard involutions on $G_0(H)$ and H^* . The class of the regular representation $[H] \in \int_{G_0(H)}$ is mapped to the regular character $\chi_{\text{reg}} \in H^*$. If H is involutory then χ_{reg} is a nonzero integral of H^* ; see Lemma 12. Thus, in this case, we have

$$\mathbb{k} \chi(\int_{G_0(H)}) = \mathbb{k} \chi_{\text{reg}} = \int_{H^*}.$$

6.1.4. The \mathbb{k} -algebra $R(H) := G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}$ is called the *representation algebra* of H . The map $[V] \otimes 1 \mapsto \chi_V$ gives an algebra embedding $R(H) \hookrightarrow H^*$ whose image is the subalgebra $H^*_{\text{trace}} = (H/[H, H])^*$ of all trace forms on H :

$$R(H) \xrightarrow{\sim} H^*_{\text{trace}} \subseteq H^* .$$

6.2. As an application of Proposition 15, we prove the following elegant generalization of Frobenius’ Theorem (see Example 17) in characteristic 0 due to S. Zhu [27, Theorem 8].

THEOREM 18. *Let H be a split semisimple Hopf algebra over a field \mathbb{k} of characteristic 0 and let $V \in \text{Irr } H$ be such that $\chi_V \in \mathcal{Z}(H^*)$. Then $\dim_{\mathbb{k}} V$ divides $\dim_{\mathbb{k}} H$.*

PROOF. By [15, Theorem 4] H is involutory and cosemisimple. Let $\Lambda \in C(H)$ denote the character of the regular representation of H^* ; this is an integral of H by Lemma 12 and, clearly, $\langle \varepsilon, \Lambda \rangle = \dim_{\mathbb{k}} H$. Let $\lambda \in \int_{H^*}$ be such that $\langle \lambda, \Lambda \rangle = 1$ and consider the isomorphism $\mathbf{b}_\lambda: {}_H H H \xrightarrow{\sim} {}_H H^V H$ in (5.8). By Proposition 15, we have $\mathbf{b}_\lambda(e(V)) = \frac{\dim_{\mathbb{k}} V}{\dim_{\mathbb{k}} H} \chi_V$ and (5.10) gives

$$\frac{\dim_{\mathbb{k}} H}{\dim_{\mathbb{k}} V} e(V) = \mathbf{b}_\Lambda(\mathcal{S}^*(\chi_V)) .$$

Therefore, it suffices to show that $\mathbf{b}_\Lambda(\mathcal{S}^*(\chi_V))$ is integral over \mathbb{Z} .

By hypothesis, $\mathcal{S}^*(\chi_V) \in \mathcal{Z}(H^*)$. Furthermore, $\mathcal{S}^*(\chi_V) \in \chi(G_0(H))$ is integral over \mathbb{Z} . Hence $\mathcal{S}^*(\chi_V) \in \mathcal{Z}(H^*)^{\text{cl}}$, the integral closure of \mathbb{Z} in $\mathcal{Z}(H^*)$. Passing to an algebraic closure of \mathbb{k} , as we may, we can assume that H^* and $\mathcal{Z}(H^*)$ are split semisimple. Thus, $\mathcal{Z}(H^*) = \bigoplus_{M \in \text{Irr } H^*} \mathbb{k}e(M)$ and $\mathcal{Z}(H^*)^{\text{cl}} = \bigoplus_{M \in \text{Irr } H^*} \mathcal{O}e(M)$, where we have put $\mathcal{O} := \{\text{algebraic integers in } \mathbb{k}\}$. Proposition 15(c), with H^* in place of H , gives $\mathbf{b}_\Lambda(e(M)) = (\dim_{\mathbb{k}} M)\chi_M$. Thus,

$$\mathbf{b}_\Lambda(\mathcal{Z}(H^*)^{\text{cl}}) \subseteq \chi(G_0(H^*))\mathcal{O} \subseteq C(H) .$$

Finally, all elements of $G_0(H^*)$ are integral over \mathbb{Z} , and hence the same holds for the elements of $\chi(G_0(H^*))\mathcal{O}$. In particular, $\mathbf{b}_\Lambda(\mathcal{S}^*(\chi_V))$ is integral over \mathbb{Z} , as desired. \square

6.3. The class equation. We now derive the celebrated class equation, due to Kac [12, Theorem 2] and Zhu [28, Theorem 1], from Proposition 4. Frobenius’ Theorem (Example 17) in characteristic 0 also follows from this result applied to $H = (\mathbb{k}G)^*$. We also mentioned here that the class equation was used by Schneider in [25] to prove the following strong version of the Frobenius property for quasitriangular semisimple Hopf algebras H over a field \mathbb{k} of characteristic 0: if V is an absolutely irreducible H -module, then $(\dim_{\mathbb{k}} V)^2$ divides $\dim_{\mathbb{k}} H$.

THEOREM 19 (Class equation). *Let H be a split semisimple Hopf algebra over a field \mathbb{k} of characteristic 0. Then $\dim_{\mathbb{k}}(H^* \otimes_{R(H)} M)$ divides $\dim_{\mathbb{k}} H^*$ for every absolutely irreducible $R(H)$ -module M .*

PROOF. Inasmuch as $R(H)$ is semisimple by Corollary 9, we have $M \cong R(H)e$ for some idempotent $e = e^2 \in R(H)$ with $eR(H)e \cong \mathbb{k}$. Thus, $H^* \otimes_{R(H)} M \cong H^*e$ and the assertion of the theorem is equivalent to the statement that $\dim_{\mathbb{k}} H^*e$ divides $\dim_{\mathbb{k}} H^*$.

The bilinear form β in (6.1) can be written as $\beta([V], [W]) = \tau([V][W])$, where $\tau: G_0(H) \rightarrow \mathbb{Z}$ is the trace form given by

$$\tau([V]) = \dim_{\mathbb{k}} V^H .$$

Now let $\Lambda \in \int_H$ denote the regular character of the dual Hopf algebra H^* , as in the proof of Theorem 18. Thus,

$$(6.3) \quad \langle e, \Lambda \rangle \stackrel{(2.2)}{=} \dim_{\mathbb{k}} eH^* \stackrel{(2.5)}{=} \dim_{\mathbb{k}} H^*e .$$

Being an integral of H , Λ annihilates all $V \in \text{Irr } H \setminus \{\mathbb{k}_\varepsilon\}$ and so $\langle \chi_V, \Lambda \rangle = 0$. On the other hand, $\langle \chi_{\mathbb{k}_\varepsilon}, \Lambda \rangle = \langle \varepsilon, \Lambda \rangle = \dim_{\mathbb{k}} H^*$. This shows that $\Lambda|_{R(H)} = \dim_{\mathbb{k}} H^* \cdot \tau'$, where we have put $\tau' = \tau \otimes_{\mathbb{Z}} \text{Id}_{\mathbb{k}} : R(H) \rightarrow \mathbb{k}$. Therefore, (6.3) becomes

$$\tau'(e) = \frac{\dim_{\mathbb{k}} H^*e}{\dim_{\mathbb{k}} H^*} .$$

Now, $\tau'(e) = \beta'(e, 1)$, where $\beta' = \beta \otimes_{\mathbb{Z}} \text{Id}_{\mathbb{k}}$, and by Proposition 4(a), we have $\beta'(e, 1)^{-1} = [R(H) : M]_{\beta'}$. Thus,

$$(6.4) \quad [R(H) : M]_{\beta'} = \frac{\dim_{\mathbb{k}} H^*}{\dim_{\mathbb{k}} H^*e} = \frac{\dim_{\mathbb{k}} H^*}{\dim_{\mathbb{k}}(H^* \otimes_{R(H)} M)} .$$

Finally, Lemma 3 with $A = G_0(H)$ and $A' = R(H)$ tells us that this rational number is integral over \mathbb{Z} . Hence it is an integer, proving the theorem. \square

6.4. The adjoint class.

6.4.1. *The adjoint representation.* The left adjoint representation of H is given by

$$\text{ad} : H \longrightarrow \text{End}_{\mathbb{k}} H, \quad \text{ad}(h)(k) = \sum h_1 k \mathcal{S}(h_2)$$

for $h, k \in H$. There is an H -isomorphism

$$(6.5) \quad H_{\text{ad}} \cong \bigoplus_{V \in \text{Irr } H} V \otimes V^* .$$

This follows from standard H -isomorphism $V \otimes V^* \cong \text{End}_{\mathbb{k}}(V)$ (see Section 5.7) combined with the Artin-Wedderburn isomorphism, $H \cong \bigoplus_{V \in \text{Irr } H} \text{End}_{\mathbb{k}}(V)$, which is equivariant for the adjoint H -action on H .

6.4.2. *The adjoint class.* Equation (6.5) gives the following description of the Casimir element $z = z_{\beta}$ of the symmetric \mathbb{Z} -algebra $G_0(H)$:

$$(6.6) \quad z = \sum_{V \in \text{Irr } H} [V][V^*] = [H_{\text{ad}}] \in \mathcal{Z}(G_0(H)) .$$

Therefore, we will refer to the Casimir element z as the *adjoint class* of H .

We now consider the left regular action of z on $G_0(H)$, that is, the endomorphism $z_{G_0(H)} \in \text{End}_{\mathbb{Z}}(G_0(H))$ that is given by

$$z_{G_0(H)} : G_0(H) \rightarrow G_0(H), \quad x \mapsto zx .$$

By Proposition 8, we know that the eigenvalues of $z_{G_0(H)}$ are positive real algebraic integers and that the largest eigenvalue is $\dim(z) = \dim_{\mathbb{k}} H$. The following proposition gives more precise information; the result was obtained by Sommerhäuser [26, 3.11] using a different method.

PROPOSITION 20. *Let H be a split semisimple Hopf algebra over a field \mathbb{k} of characteristic 0. Then the eigenvalues of $z_{G_0(H)}$ are positive integers $\leq \dim_{\mathbb{k}} H$. If $G_0(H)$ or H is commutative then all eigenvalues of $z_{G_0(H)}$ divide $\dim_{\mathbb{k}} H$.*

PROOF. We may pass to the algebraic closure of \mathbb{k} ; this changes neither $G_0(H)$ nor z . Then the representation algebra $R(H) = G_0(H) \otimes_{\mathbb{Z}} \mathbb{k}$ is split semisimple by Corollary 9. Since $z \in \mathcal{Z}(R(H))$, the eigenvalues of $z_{G_0(H)}$ are exactly the $\omega_M(z) \in \mathbb{k}$, where M runs over the irreducible $R(H)$ -modules and ω_M denotes the central character of M as in 2.4. We know by Propositions 4(c) and 5(a) and equation (6.4) that

$$\omega_M(z) = \dim_{\mathbb{k}} M \cdot \frac{\dim_{\mathbb{k}} H^*}{\dim_{\mathbb{k}}(H^* \otimes_{R(H)} M)},$$

and this is a positive integer by Theorem 19. Since $M \subseteq H^* \otimes_{R(H)} M$, we have $\omega_M(z) \leq \dim_{\mathbb{k}} H$. If $G_0(H)$ is commutative then $\dim_{\mathbb{k}} M = 1$, and hence $\omega_M(z)$ divides $\dim_{\mathbb{k}} H$. If H is commutative then $R(H) = H^*$, and so $\omega_M(z) = \dim_{\mathbb{k}} H$. (Alternatively, if H is commutative then $H_{\text{ad}} \cong \mathbb{k}_{\varepsilon}^{\dim_{\mathbb{k}} H}$ and $z = [H_{\text{ad}}] = (\dim_{\mathbb{k}} H)1$.) \square

We mention that the Grothendieck ring $G_0(H)$ is commutative whenever the Hopf algebra H is almost commutative. In particular, this holds for all quasi-triangular Hopf algebras; see Montgomery [20, Section 10.1].

EXAMPLE 21. Let $H = \mathbb{k}G$ be the group algebra of the finite group G over a splitting field \mathbb{k} of characteristic 0. The representation algebra $R(H) \cong \bigoplus_{V \in \text{Irr } H} \mathbb{k}\chi_V \subseteq H^*$ is isomorphic to the algebra of \mathbb{k} -valued class functions on G , that is, functions $G \rightarrow \mathbb{k}$ that are constant on conjugacy classes of G . For any finite-dimensional $\mathbb{k}G$ -module V , the character values $\chi_V(g)$ ($g \in G$) are the eigenvalues of the endomorphisms $[V]_{G_0(H)} \in \text{End}_{\mathbb{Z}}(G_0(H))$. Specializing to the adjoint representation $V = H_{\text{ad}}$ we obtain the eigenvalues of $z_{G_0(H)}$: they are the integers $\chi_{H_{\text{ad}}}(g) = |C_G(g)|$ with $g \in G$.

6.5. The semisimple locus of $G_0(H)$. Let H be a split semisimple Hopf algebra over a field \mathbb{k} . Recall that $G_0(H) \otimes_{\mathbb{Z}} \mathbb{Q}$ is semisimple by Corollary 9. We will now describe the primes p for which the algebra $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple.

PROPOSITION 22. *Let H be a split semisimple Hopf algebra over a field \mathbb{k} . Then:*

- (a) *If p divides $\dim_{\mathbb{k}} H$ then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is not semisimple.*
- (b) *Assume that $\text{char } \mathbb{k} = 0$. Then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple for all $p > \dim_{\mathbb{k}} H$.*
- (c) *Assume that $\text{char } \mathbb{k} = 0$ and that $G_0(H)$ or H is commutative. Then $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple if and only if p does not divide $\dim_{\mathbb{k}} H$.*

PROOF. Semisimplicity is equivalent to separability over \mathbb{F}_p ; see [23, 10.7 Corollary b]. Therefore, we may apply Proposition 6. In detail, consider the Casimir operator that is associated with the bilinear form β of Section 6.1.2,

$$c: G_0(H) \rightarrow \mathcal{Z}(G_0(H)), \quad x \mapsto \sum_{V \in \text{Irr } H} [V^*]x[V].$$

By Proposition 6, $G_0(H) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is semisimple if and only if $(p) \not\subseteq \mathbb{Z} \cap \text{Im } c$.

(a) Consider the dimension augmentation $\dim: G_0(H) \rightarrow \mathbb{Z}$, $[V] \mapsto \dim_{\mathbb{k}} V$. The composite $\dim \circ c$ is equal to $\dim_{\mathbb{k}} H \cdot \dim$. Hence, $\mathbb{Z} \cap \text{Im } c \subseteq \text{Im}(\dim \circ c) \subseteq (\dim_{\mathbb{k}} H)$ holds in \mathbb{Z} , which implies (a).

(b) In view of Proposition 20, our hypothesis on p implies that the norm $N(z) = \det z_{G_0(H)}$ is not divisible by p . Since $z = c(1)$, it follows that $(p) \not\subseteq N(\text{Im } c)$, and hence $(p) \not\subseteq \mathbb{Z} \cap \text{Im } c$; see Section 3.4.

(c) Necessity of the condition on p follows from (a) and sufficiency follows from Proposition 20 as in (b). \square

6.6. Traces of group-like elements. Let H be a split semisimple Hopf algebra over a field \mathbb{k} and let $\chi_{\text{ad}} \in R(H) \subseteq H^*$ denote the character of the adjoint representation. Equation (6.6) gives

$$\chi_{\text{ad}} = \sum_{V \in \text{Irr } H} \chi_{V^*} \chi_V.$$

PROPOSITION 23. *Let H be a split semisimple Hopf algebra over a field \mathbb{k} . If $R(H)$ is semisimple then $\chi_{\text{ad}}(g) \neq 0$ for every group-like element $g \in H$.*

PROOF. By Proposition 6, semisimplicity of $R(H)$ is equivalent to surjectivity of the Casimir operator $c: R(H) \rightarrow \mathcal{Z}(R(H))$, $\chi \mapsto \sum_{V \in \text{Irr } H} \chi_{V^*} \chi \chi_V$. Fixing χ with $\sum_V \chi_{V^*} \chi \chi_V = 1$ we obtain

$$1 = \sum_V \chi_{V^*}(g) \chi(g) \chi_V(g) = \chi(g) \chi_{\text{ad}}(g),$$

which shows that $\chi_{\text{ad}}(g) \neq 0$. □

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