

# ALGEBRAIC GROUP ACTIONS ON NONCOMMUTATIVE SPECTRA

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**Abstract.** Let  $G$  be an affine algebraic group and let  $R$  be an associative algebra with a rational action of  $G$  by algebra automorphisms. We study the induced  $G$ -action on the set  $\text{Spec } R$  of all prime ideals of  $R$ , viewed as a topological space with the Jacobson–Zariski topology, and on the subspace  $\text{Rat } R \subseteq \text{Spec } R$  consisting of all rational ideals of  $R$ . Here, a prime ideal  $P$  of  $R$  is said to be rational if the extended centroid  $\mathcal{C}(R/P)$  is equal to the base field. Our results generalize the work of Mœglin and Rentschler and of Vonessen to arbitrary associative algebras while also simplifying some of the earlier proofs.

The map  $P \mapsto \bigcap_{g \in G} g.P$  gives a surjection from  $\text{Spec } R$  onto the set  $G\text{-Spec } R$  of all  $G$ -prime ideals of  $R$ . The fibers of this map yield the so-called  $G$ -stratification of  $\text{Spec } R$  which has played a central role in the recent investigation of algebraic quantum groups, in particular, in the work of Goodearl and Letzter. We describe the  $G$ -strata of  $\text{Spec } R$  in terms of certain commutative spectra. Furthermore, we show that if a rational ideal  $P$  is locally closed in  $\text{Spec } R$  then the orbit  $G.P$  is locally closed in  $\text{Rat } R$ . This generalizes a standard result on  $G$ -varieties. Finally, we discuss the situation where  $G\text{-Spec } R$  is a finite set.

## Introduction

### 0.1.

This paper continues our investigation [15] of the action of an affine algebraic group  $G$  on an arbitrary associative algebra  $R$ . Our focus will now be on some topological aspects of the induced action on the set  $\text{Spec } R$  of all prime ideals of  $R$ , the main themes being local closedness of  $G$ -orbits in  $\text{Spec } R$  and the stratification of  $\text{Spec } R$  by means of suitable commutative spectra. The stratification in question plays a central role in the theory of algebraic quantum groups; see Brown and Goodearl [7] for a panoramic view of this area. Our goal here is to develop the principal results in a context that is free of the standard finiteness conditions, Noetherianness or the Goldie property, that underlie the pioneering works of Mœglin and Rentschler [19], [20], [21], [22] and of Vonessen [27], [28].

Throughout, we work over an algebraically closed base field  $\mathbb{k}$  and we assume

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that the action of  $G$  on  $R$  is rational; the definition will be recalled in Section 3.1. The action will generally be written as

$$G \times R \rightarrow R, \quad (g, r) \mapsto g.r.$$

**0.2.**

The set  $\text{Spec } R$  carries the familiar Jacobson–Zariski topology; see Section 1.1 for details. Since the  $G$ -action on  $R$  sends prime ideals to prime ideals, it induces an action of  $G$  by homeomorphisms on  $\text{Spec } R$ . In the following,  $G \backslash \text{Spec } R$  will denote the set of all  $G$ -orbits in  $\text{Spec } R$ . We will also consider the set  $G\text{-Spec } R$  consisting of all  $G$ -prime ideals of  $R$ . Recall that a proper  $G$ -stable ideal  $I$  of  $R$  is called  $G$ -prime if  $AB \subseteq I$  for  $G$ -stable ideals  $A$  and  $B$  of  $R$  implies that  $A \subseteq I$  or  $B \subseteq I$ .

There are surjective maps

$$\text{Spec } R \xrightarrow{\text{can.}} G \backslash \text{Spec } R, \quad P \mapsto G.P = \{g.P \mid g \in G\}, \quad (1)$$

$$\gamma: \text{Spec } R \longrightarrow G\text{-Spec } R, \quad P \mapsto P:G = \bigcap_{g \in G} g.P. \quad (2)$$

See [15, Prop. 8] for surjectivity of  $\gamma$ . We will give  $G \backslash \text{Spec } R$  and  $G\text{-Spec } R$  the final topologies for these maps: closed subsets are those whose preimage in  $\text{Spec } R$  is closed [5, I.2.4]. Since (2) factors through (1), we obtain a surjection

$$G \backslash \text{Spec } R \longrightarrow G\text{-Spec } R, \quad G.P \mapsto P:G. \quad (3)$$

This map is continuous and closed; see Section 1.3.

**0.3.**

As in [15], we will be particularly concerned with the subsets  $\text{Rat } R \subseteq \text{Spec } R$  and  $G\text{-Rat } R \subseteq G\text{-Spec } R$  consisting of all rational and  $G$ -rational ideals of  $R$ , respectively. Recall that rationality and  $G$ -rationality is defined in terms of the extended centroid  $\mathcal{C}(\cdot)$  of the corresponding factor algebra [15]. Specifically,  $P \in \text{Spec } R$  is said to be *rational* if  $\mathcal{C}(R/P) = \mathbb{k}$ , and  $I \in G\text{-Spec } R$  is  $G$ -*rational* if the  $G$ -invariants  $\mathcal{C}(R/I)^G \subseteq \mathcal{C}(R/I)$  coincide with  $\mathbb{k}$ . For the definition and basic properties of the extended centroid, the reader is referred to [15]. Here, we just recall that  $\mathcal{C}(R/P)$  and  $\mathcal{C}(R/I)^G$  are always extension fields of  $\mathbb{k}$ , for any  $P \in \text{Spec } R$  and any  $I \in G\text{-Spec } R$ . The extended centroid of a semiprime Noetherian (or Goldie) algebra is identical to the center of the classical ring of quotients. In the context of enveloping algebras of Lie algebras and related Noetherian algebras, the field  $\mathcal{C}(R/P)$  is commonly called the *heart* (cœur, Herz) of the prime  $P$  (e.g., [10], [3], [4]). We will follow this tradition here.

The sets  $\text{Rat } R$  and  $G\text{-Rat } R$  will be viewed as topological spaces with the topologies that are induced from  $\text{Spec } R$  and  $G\text{-Spec } R$ : closed subsets of  $\text{Rat } R$  are the intersections of closed subsets of  $\text{Spec } R$  with  $\text{Rat } R$ , and similarly for  $G\text{-Rat } R$  [5, I.3.1]. The  $G$ -action on  $\text{Spec } R$  stabilizes  $\text{Rat } R$ . Hence we may consider the set  $G \backslash \text{Rat } R \subseteq G \backslash \text{Spec } R$  consisting of all  $G$ -orbits in  $\text{Rat } R$ . We endow  $G \backslash \text{Rat } R$  with the topology that is induced from  $G \backslash \text{Spec } R$ ; this turns out to be identical

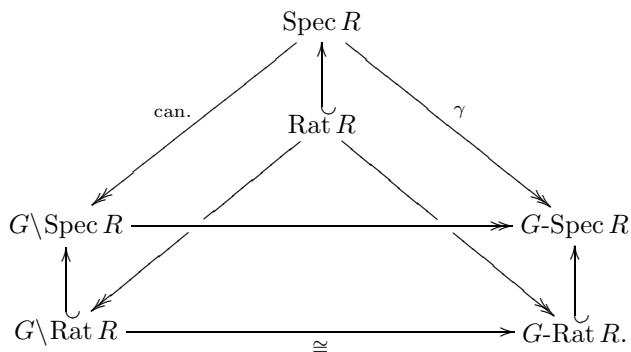
to the final topology for the canonical surjection  $\text{Rat } R \rightarrow G \backslash \text{Rat } R$  [5, III.2.4, Prop. 10]. By [15, Theorem 1], the surjection (3) restricts to a bijection

$$G \backslash \text{Rat } R \xrightarrow{\text{bij.}} G\text{-Rat } R. \tag{4}$$

This map is in fact a homeomorphism; see Section 1.5.

**0.4.**

The following diagram summarizes the various topological spaces under consideration and their relations to each other:



Here,  $\rightarrow$  indicates a surjection whose target space carries the final topology,  $\hookrightarrow$  indicates an inclusion whose source has the induced topology, and  $\cong$  is the homeomorphism (4).

**0.5.**

The technical core of the paper is Theorem 9 which describes the  $\gamma$ -fiber over a given  $I \in G\text{-Spec } R$ . This fiber will be denoted by

$$\text{Spec}_I R = \{P \in \text{Spec } R \mid P:G = I\}$$

as in [7]. The partition

$$\text{Spec } R = \bigsqcup_{I \in G\text{-Spec } R} \text{Spec}_I R \tag{5}$$

is called the *G-stratification* of  $\text{Spec } R$  in [7, II.2]. In the special case where  $R$  is Noetherian and  $G$  is an algebraic torus, a description of the  $G$ -strata  $\text{Spec}_I R$  in terms of suitable commutative spectra was given in [7, II.2.13], based on the work of Goodearl and Letzter [11]. For general  $R$  and  $G$ , the intersection

$$\text{Rat}_I R = \text{Spec}_I R \cap \text{Rat } R$$

was treated in [15, Theorem 22]. Our proof of Theorem 9, to be given in Section 3, elaborates on the one in [15].

Assuming  $G$  to be connected for simplicity, we put

$$T_I = \mathcal{C}(R/I) \otimes_{\mathbb{k}} \mathbb{k}(G),$$

where  $\mathbb{k}(G)$  denotes the field of rational functions on  $G$ . The algebra  $T_I$  is a tensor product of two commutative fields and  $T_I$  has no zero divisors. The given  $G$ -action on  $R$  and the right regular  $G$ -action on  $\mathbb{k}(G)$  naturally give rise to an action of  $G$  on  $T_I$ . Letting  $\text{Spec}^G(T_I)$  denote the collection of all  $G$ -stable primes of  $T_I$ , Theorem 9 establishes a bijection

$$c: \text{Spec}_I R \xrightarrow{\text{bij.}} \text{Spec}^G(T_I)$$

which is very well behaved: the map  $c$  is equivariant with respect to suitable  $G$ -actions, it is an order isomorphism for inclusion, and it allows one to control hearts and rationality. For the precise formulation of Theorem 9, we refer to Section 3.

**0.6.**

Theorem 9 and the tools developed for its proof will be used in Section 4 to investigate local closedness of rational ideals. Recall that a subset  $A$  of an arbitrary topological space  $X$  is said to be *locally closed* if  $A$  is closed in some neighborhood of  $A$  in  $X$ . This is equivalent to  $A$  being open in its topological closure  $\bar{A}$  in  $X$  or, alternatively,  $A$  being an intersection of an open and a closed subset of  $X$  [5, I.3.3]. A point  $x \in X$  is locally closed if  $\{x\}$  is locally closed. For  $X = \text{Spec } R$ , this amounts to the following familiar condition: a prime ideal  $P$  is locally closed in  $\text{Spec } R$  if and only if  $P$  is distinct from the intersection of all primes of  $R$  that properly contain  $P$ . A similar formulation holds for  $X = G\text{-Spec } R$ ; see Section 1.4. We remark that “locally closed in  $G\text{-Spec } R$ ” is referred to as “ $G$ -locally closed” in [21] and [28].

The second main result of this paper is the following theorem which will be proved in Section 4. Earlier versions assuming additional finiteness hypotheses are due to Mœglin and Rentschler [19, Théorème 3.8], [21, Théorème 3] and to Vonessen [28, Theorem 2.6].

**Theorem 1.** *The following are equivalent for a rational ideal  $P$  of  $R$ :*

- (a)  $P$  is locally closed in  $\text{Spec } R$ ;
- (b)  $\gamma(P) = P:G$  is locally closed in  $G\text{-Spec } R$ .

Theorem 1 in conjunction with (4) has the following useful consequence. The corollary below extends [28, Cor. 2.7] and a standard result on  $G$ -varieties [14, Satz II.2.2].

**Corollary 2.** *If  $P \in \text{Rat } R$  is locally closed in  $\text{Spec } R$  then the  $G$ -orbit  $G.P$  is open in its closure in  $\text{Rat } R$ .*

*Proof.* The point  $P:G \in G\text{-Spec } R$  is locally closed by Theorem 1. Applying the easy fact [5, I.3.3] that preimages of locally closed sets under continuous maps are again locally closed to  $f: \text{Rat } R \hookrightarrow \text{Spec } R \xrightarrow{\gamma} G\text{-Spec } R$ , we conclude from (4) that  $f^{-1}(P:G) = G.P$  is locally closed in  $\text{Rat } R$ .  $\square$

**0.7.**

In order to put Theorem 1 and Corollary 2 into perspective, we mention that rational ideals are oftentimes locally closed in  $\text{Spec } R$ . In fact, for many important classes of algebras  $R$ , rational ideals are identical with the locally closed points of  $\text{Spec } R$ . Specifically, we will say that the algebra  $R$  satisfies the *Nullstellensatz* if the following two conditions are satisfied:

- (i) every prime ideal of  $R$  is an intersection of primitive ideals; and
- (ii)  $\mathcal{Z}(\text{End}_R V) = \mathbb{k}$  holds for every simple  $R$ -module  $V$ .

Recall that an ideal of  $R$  is said to be (right) primitive if it is the annihilator of a simple (right)  $R$ -module. Hypothesis (i) is known as the *Jacobson property* while versions of (ii) are referred to as the *endomorphism property* [18] or the *weak Nullstellensatz* [23], [15]. The Nullstellensatz is quite common. It is guaranteed to hold, for example, if  $\mathbb{k}$  is uncountable and the algebra  $R$  is Noetherian and countably generated [18, Cor. 9.1.8], [7, II.7.16]. The Nullstellensatz also holds for any affine PI-algebra  $R$  [26, Chap. 6]. For many other classes of algebras satisfying the Nullstellensatz, see [18, Chap. 9] or [7, II.7].

If  $R$  satisfies the Nullstellensatz then the following implications hold for all primes of  $R$ :

$$\text{locally closed} \Rightarrow \text{primitive} \Rightarrow \text{rational}.$$

Here, the first implication is an immediate consequence of (i) while the second follows from (ii); see [15, Prop. 6]. The algebra  $R$  is said to satisfy the *Dixmier–Mœglin equivalence* if all three properties are equivalent for primes of  $R$ . Standard examples of algebras satisfying the Dixmier–Mœglin equivalence include affine PI-algebras, whose rational ideals are in fact maximal [24], and enveloping algebras of finite-dimensional Lie algebras; see [25, 1.9] for  $\text{char } \mathbb{k} = 0$ . (In positive characteristics, enveloping algebras are affine PI.) More recently, the Dixmier–Mœglin equivalence has been shown to hold for numerous quantum groups; see [7, II.8] for an overview.

Note that the validity of the Nullstellensatz and the Dixmier–Mœglin equivalence are intrinsic to  $R$ . However,  $G$ -actions can be useful tools in verifying the latter. Indeed, assuming the Nullstellensatz for  $R$ , Theorem 1 implies that the Dixmier–Mœglin equivalence is equivalent to  $P:G$  being locally closed in  $G\text{-Spec } R$  for every  $P \in \text{Rat } R$ . This condition is surely satisfied whenever  $G\text{-Spec } R$  is in fact finite.

**0.8.**

The final Section 5 briefly addresses the question as to when  $G\text{-Spec } R$  is a finite set. Besides being of interest in connection with the Dixmier–Mœglin equivalence (Section 0.7), this is obviously relevant for the  $G$ -stratification (5); see also [7, Prob. II.10.6]. Restricting ourselves to algebras  $R$  satisfying the Nullstellensatz, we show in Proposition 14 that finiteness of  $G\text{-Spec } R$  is equivalent to the following three conditions:

- (i) the ascending chain condition holds for  $G$ -stable semiprime ideals of  $R$ ;
- (ii)  $R$  satisfies the Dixmier–Mœglin equivalence; and
- (iii)  $G\text{-Rat } R = G\text{-Spec } R$ .

Several versions of Proposition 14 for Noetherian algebras  $R$  can be found in [7, II.8], where a profusion of algebras is exhibited for which  $G$ -Spec  $R$  is known to be finite.

Note that (i) above is no trouble for the standard classes of algebras, even in the strengthened form which ignores  $G$ -stability. Indeed, Noetherian algebras trivially satisfy the ascending chain condition for all semiprime ideals, and so do all affine PI-algebras; see [26, 6.3.36']. Moreover, as was outlined in Section 0.7, the Dixmier–Mœglin equivalence (ii) has been established for a wide variety of algebras. Therefore, in many situations of interest, Proposition 14 says in essence that finiteness of  $G$ -Spec  $R$  is tantamount to the equality  $G$ -Rat  $R = G$ -Spec  $R$ . This is also the only condition where the  $G$ -action properly enters the picture. The paper concludes with some simple examples of torus actions satisfying (iii). Further work is needed on how to assure the validity of (iii) under reasonably general circumstances.

**0.9.**

This paper owes a great deal to the ground breaking investigations of Mœglin and Rentschler and of Vonesen. The statements of our main results as well as the basic strategies employed in their proofs have roots in the aforementioned articles of these authors. We have made an effort to render our presentation reasonably self-contained while also indicating the original sources at the appropriate points in the text. The reader interested in the details of Sections 3 and 4 may wish to have a copy of [28] at hand in addition to [15].

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**Notation.** Our terminology and notation follows [15]. The notations and hypotheses introduced in the foregoing will remain in effect throughout the paper. In particular, we will work over an algebraically closed base field  $\mathbb{k}$ . Furthermore,  $G$  will be an affine algebraic  $\mathbb{k}$ -group and  $R$  will be an associative  $\mathbb{k}$ -algebra (with 1) on which  $G$  acts rationally by  $\mathbb{k}$ -algebra automorphisms. For simplicity,  $\otimes_{\mathbb{k}}$  will be written as  $\otimes$ . Finally, for any ideal  $I \trianglelefteq R$ , the largest  $G$ -stable ideal of  $R$  that is contained in  $I$  will be denoted by

$$I:G = \bigcap_{g \in G} g.I.$$

**1. Topological preliminaries**

**1.1.**

Recall that the closed sets of the *Jacobson–Zariski topology* on Spec  $R$  are exactly the subsets of the form

$$V(I) = \{P \in \text{Spec } R \mid P \supseteq I\}$$

where  $I \subseteq R$ . The topological closure of a subset  $A \subseteq \text{Spec } R$  is given by

$$\overline{A} = \mathbf{V}(\mathbf{I}(A)) \quad \text{where} \quad \mathbf{I}(A) = \bigcap_{P \in A} P. \tag{6}$$

The easily checked equalities  $\mathbf{I}(\mathbf{V}(\mathbf{I}(A))) = \mathbf{I}(A)$  and  $\mathbf{V}(\mathbf{I}(\mathbf{V}(I))) = \mathbf{V}(I)$  show that the operators  $\mathbf{V}$  and  $\mathbf{I}$  yield inverse bijections between the collection of all closed subsets of  $\text{Spec } R$  on one side and the collection of all semiprime ideals of  $R$  (i.e., ideals of  $R$  that are intersections of prime ideals) on the other. Thus, we have an inclusion reversing 1-1 correspondence

$$\left\{ \begin{array}{l} \text{closed subsets} \\ A \subseteq \text{Spec } R \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{semiprime ideals} \\ I \trianglelefteq R \end{array} \right\}. \tag{7}$$

Note that the equality  $\mathbf{I}(\mathbf{V}(\mathbf{I}(A))) = \mathbf{I}(A)$  can also be stated as

$$\mathbf{I}(\overline{A}) = \mathbf{I}(A). \tag{8}$$

**1.2.**

The action of  $G$  commutes with the operators  $\mathbf{V}$  and  $\mathbf{I}$ :  $g \cdot \mathbf{V}(I) = \mathbf{V}(g \cdot I)$  and  $g \cdot \mathbf{I}(A) = \mathbf{I}(g \cdot A)$  holds for all  $g \in G$  and all  $I \subseteq R, A \subseteq \text{Spec } R$ . In particular, the elements of  $G$  act by homeomorphisms on  $\text{Spec } R$ . Furthermore:

$$\text{If } g \cdot A \subseteq A \text{ for a closed subset } A \subseteq \text{Spec } R \text{ and } g \in G \text{ then } g \cdot A = A. \tag{9}$$

In view of the correspondence (7), this amounts to saying that  $g \cdot I \supseteq I$  forces  $g \cdot I = I$  for semiprime ideals  $I \trianglelefteq R$ . But this follows from the fact that the  $G$ -action on  $R$  is locally finite [15, 3.1]: If  $r \in I$  satisfies  $g \cdot r \notin I$  then choose a finite-dimensional  $G$ -stable subspace  $V \subseteq R$  with  $r \in V$  to get  $I \cap V \subsetneq g \cdot I \cap V \subsetneq g^2 \cdot I \cap V \subsetneq \dots$ , which is impossible.

Finally, consider the  $G$ -orbit  $G \cdot P$  of a point  $P \in \text{Spec } R$ . Since  $\mathbf{I}(G \cdot P) = P : G$ , we deduce from equation (6) that the closure of  $G \cdot P$  in  $\text{Spec } R$  is given by

$$\overline{G \cdot P} = \mathbf{V}(P : G) = \{Q \in \text{Spec } R \mid Q \supseteq P : G\}$$

and (8) gives

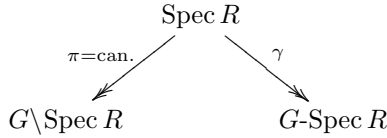
$$\mathbf{I}(\overline{G \cdot P}) = \mathbf{I}(G \cdot P) = P : G.$$

Thus, the correspondence (7) restricts to an inclusion reversing bijection

$$\begin{array}{ccc} \{G\text{-orbit closures in Spec } R\} & \xleftrightarrow{1-1} & G\text{-Spec } R, \\ \cup & & \cup \\ \overline{G \cdot P} & \longleftrightarrow & P : G. \end{array} \tag{10}$$

**1.3.**

As was mentioned in the Introduction, the spaces  $G \backslash \text{Spec } R$  and  $G\text{-Spec } R$  inherit their topology from  $\text{Spec } R$  via the surjections in (1) and (2):



with  $\pi(P) = G.P$  and  $\gamma(P) = P.G$ . Closed sets in  $G \backslash \text{Spec } R$  and in  $G\text{-Spec } R$  are exactly those subsets whose preimage in  $\text{Spec } R$  is closed. In both cases, preimages are also  $G$ -stable, and hence they have the form  $\mathbf{V}(I)$  for some  $G$ -stable semiprime ideal  $I \leq R$ .

Let  $C$  be a closed subset of  $G\text{-Spec } R$  and write  $\gamma^{-1}(C) = \mathbf{V}(I)$  as above. Since  $P \supseteq I$  is equivalent to  $P.G \supseteq I$  for  $P \in \text{Spec } R$ , we obtain

$$\begin{aligned}
 C = \gamma(\mathbf{V}(I)) &= \{P.G \mid P \in \text{Spec } R, P \supseteq I\} \\
 &= \{P.G \mid P \in \text{Spec } R, P.G \supseteq I\} \\
 &= \{J \in G\text{-Spec } R \mid J \supseteq I\}.
 \end{aligned}
 \tag{11}$$

Conversely, if  $C = \{J \in G\text{-Spec } R \mid J \supseteq I\}$  for some  $G$ -stable semiprime ideal  $I \leq R$  then  $\gamma^{-1}(C) = \mathbf{V}(I)$  is closed in  $\text{Spec } R$  and so  $C$  is closed in  $G\text{-Spec } R$ . Thus, the closed subsets of  $G\text{-Spec } R$  are exactly those of the form (11). The partial order on  $G\text{-Spec } R$  that is given by inclusion can be expressed in terms of orbit closures by (10): for  $P, Q \in \text{Spec } R$ , we have

$$Q.G \supseteq P.G \iff \overline{G.Q} \subseteq \overline{G.P} \iff Q \in \overline{G.P}.
 \tag{12}$$

The map  $\gamma$  factors through  $\pi$ ; so we have a map

$$\gamma': G \backslash \text{Spec } R \rightarrow G\text{-Spec } R$$

with  $\gamma = \gamma' \circ \pi$  as in (3). Since  $\gamma^{-1}(C) = \pi^{-1}(\gamma'^{-1}(C))$  holds for any  $C \subseteq G\text{-Spec } R$ , the map  $\gamma'$  is certainly continuous. Moreover, if  $B \subseteq G \backslash \text{Spec } R$  is closed then  $A = \pi^{-1}(B) \subseteq \text{Spec } R$  satisfies  $\gamma(A) = \gamma'(B)$  and  $A = \mathbf{V}(I)$  for some  $G$ -stable semiprime ideal  $I \leq R$ . Hence,  $\gamma'(B) = \gamma(\mathbf{V}(I))$  is closed in  $G\text{-Spec } R$  by (11). This shows that  $\gamma'$  is a closed map.

**1.4.**

Recall that a subset  $A$  of an arbitrary topological space  $X$  is *locally closed* if and only if  $\overline{A} \backslash A = \overline{A} \cap A^c$  is a closed subset of  $X$ . By (6), a prime ideal  $P$  of  $R$  is locally closed in  $\text{Spec } R$  if and only if  $\mathbf{V}(P) \backslash \{P\} = \{Q \in \text{Spec } R \mid Q \not\supseteq P\}$  is a closed subset of  $\text{Spec } R$ , which in turn is equivalent to the condition

$$P \not\supseteq \mathbf{I}(\mathbf{V}(P) \backslash \{P\}) = \bigcap_{\substack{Q \in \text{Spec } R \\ Q \not\supseteq P}} Q.
 \tag{13}$$

Similarly, (11) implies that a  $G$ -prime ideal  $I$  of  $R$  is locally closed in  $G\text{-Spec } R$  if and only if

$$I \not\supseteq \bigcap_{\substack{J \in G\text{-Spec } R \\ J \not\supseteq I}} J.
 \tag{14}$$



**Lemma 3.** *Let  $P \in \text{Spec } R$  and let  $N$  be a normal subgroup of  $G$  having finite index in  $G$ . Then  $P:G$  is locally closed in  $G\text{-Spec } R$  if and only if  $P:N$  is locally closed in  $N\text{-Spec } R$ .*

*Proof.* For brevity, put  $X = N\text{-Spec } R$ ,  $Y = G\text{-Spec } R$ , and  $H = G/N$ . Thus,  $H$  is a finite group acting by homeomorphisms on  $X$ . From (3) we obtain a continuous surjection  $X \rightarrow Y$ ,  $I \mapsto I:H$ , whose fibers are easily seen to be the  $H$ -orbits in  $X$ . Thus, we obtain a continuous bijection  $H \backslash X \rightarrow Y$ . From the description (11) of the closed sets in  $X$  and  $Y$ , we further see that this bijection is closed, and hence it is a homeomorphism. Therefore, the image  $I:H$  of a given  $I \in X$  is locally closed in  $Y$  if and only if the orbit  $H.I$  is locally closed in  $X$ ; see [5, I.5.3, Cor. of Prop. 7]. Finally, with  $\overline{\phantom{x}}$  denoting topological closure in  $X$ , one easily checks that

$$\overline{H.I} \backslash H.I = \bigcup_{h \in H} h.(\overline{\{I\}} \backslash \{I\})$$

and, consequently,  $(\overline{H.I} \backslash H.I) \cap \overline{\{I\}} = \overline{\{I\}} \backslash \{I\}$ . Thus,  $H.I$  is locally closed if and only if  $I$  is locally closed, which proves the lemma.  $\square$

**1.5.**

We now turn to the space  $\text{Rat } R$  of rational ideals of  $R$  and the associated spaces  $G \backslash \text{Rat } R$  and  $G\text{-Rat } R$  with the induced topologies from  $\text{Spec } R$ ,  $G \backslash \text{Spec } R$ , and  $G\text{-Spec } R$ , respectively; see Section 0.3. Restricting the maps  $\pi = \text{can.}$  and  $\gamma'$  in Section 1.3, we obtain a commutative triangle of continuous maps

$$\begin{array}{ccc} & \text{Rat } R & \\ \pi_{\text{rat}} = \text{can.} \swarrow & & \searrow \gamma_{\text{rat}} \\ G \backslash \text{Rat } R & \xrightarrow{\gamma'_{\text{rat}}} & G\text{-Rat } R. \end{array}$$

The map  $\gamma'_{\text{rat}}$ , identical with (4), is bijective. Furthermore, if  $B \subseteq G \backslash \text{Rat } R$  is closed then, as in Section 1.3, we have  $\gamma'_{\text{rat}}(B) = \{J \in G\text{-Rat } R \mid J \supseteq I\}$  for some  $G$ -stable semiprime ideal  $I \trianglelefteq R$ ; so  $\gamma'_{\text{rat}}(B)$  is closed in  $G\text{-Rat } R$ . This shows that  $\gamma'_{\text{rat}}$  is a homeomorphism. For general reasons, the quotient map  $\pi_{\text{rat}}$  is open and the topology on  $G \backslash \text{Rat } R$  is identical to the final topology for  $\pi_{\text{rat}}$  [5, III.2.4, Lemme 2 and Prop. 10]. By virtue of the homeomorphism  $\gamma'_{\text{rat}}$ , the same holds for  $G\text{-Rat } R$  and  $\gamma_{\text{rat}}$ .

We remark that injectivity of  $\gamma'_{\text{rat}}$  and (12) imply that

$$Q:G \not\supseteq P:G \iff Q \in \overline{G.P} \backslash G.P$$

holds for  $P, Q \in \text{Rat } R$ . Here,  $\overline{\phantom{x}}$  can be taken to be the closure in  $\text{Spec } R$  or in  $\text{Rat } R$ . Focusing on the latter interpretation, we easily conclude that

$$P:G \text{ is locally closed in } G\text{-Rat } R \iff G.P \text{ is open in its closure in } \text{Rat } R.$$

Alternatively, in view of the homeomorphism  $\gamma'_{\text{rat}}$ , local closedness of  $P:G \in G\text{-Rat } R$  is equivalent to local closedness of the point  $G.P \in G \backslash \text{Rat } R$ , which in turn is equivalent to local closedness of the preimage  $G.P = \pi_{\text{rat}}^{-1}(G.P) \subseteq \text{Rat } R$ ; see [5, I.5.3, Cor. of Prop. 7].

## 2. Ring theoretical preliminaries

### 2.1.

The extended centroid of a ring  $U$  will be denoted by

$$\mathcal{C}(U).$$

By definition,  $\mathcal{C}(U)$  is the center of the Amitsur–Martindale ring of quotients of  $U$ . We briefly recall some basic definitions and facts. For details, the reader is referred to [15].

The ring  $U$  is said to be *centrally closed* if  $\mathcal{C}(U) \subseteq U$ . If  $U$  is semiprime then

$$\tilde{U} = U\mathcal{C}(U)$$

is a centrally closed semiprime subring of the Amitsur–Martindale ring of quotients of  $U$ ; it is called the *central closure* of  $U$ . Furthermore, if  $U$  is prime then  $\tilde{U}$  is prime as well and  $\mathcal{C}(U)$  is a field. Consequently, for any  $P \in \text{Spec } U$ , we have a commutative field  $\mathcal{C}(U/P)$ .

If  $\Gamma$  is any group acting by automorphisms on  $U$  then the action of  $\Gamma$  extends uniquely to an action on the Amitsur–Martindale ring of quotients of  $U$ , and hence  $\Gamma$  acts on  $\mathcal{C}(U)$ ; see [15, 2.3]. If  $I$  is a  $\Gamma$ -prime ideal of  $U$  then the ring of  $\Gamma$ -invariants  $\mathcal{C}(U/I)^\Gamma$  is a field; see [15, Prop. 9].

### 2.2.

A ring homomorphism  $\varphi: U \rightarrow V$  is called *centralizing* if the ring  $V$  is generated by the image  $\varphi(U)$  and its centralizer,  $C_V(\varphi(U)) = \{v \in V \mid v\varphi(u) = \varphi(u)v \ \forall u \in U\}$ ; see [15, 1.5]. The lemma below is a special case of [15, Lemma 4] and it can also be found in an earlier unpublished preprint of Bergman [1, Lemma 1].

**Lemma 4.** *Let  $\varphi: U \hookrightarrow V$  be a centralizing embedding of prime rings. Then  $\varphi$  extends uniquely to a homomorphism  $\tilde{\varphi}: \tilde{U} \rightarrow \tilde{V}$  between the central closures of  $U$  and  $V$ . The extension  $\tilde{\varphi}$  is again injective and centralizing. In particular,  $\tilde{\varphi}(\mathcal{C}(U)) \subseteq \mathcal{C}(V)$ .*

*Proof.* If  $I$  is a nonzero ideal of  $U$  then  $\varphi(I)V = V\varphi(I)$  is a nonzero ideal of  $V$ . Hence  $\varphi(I)$  has zero left and right annihilator in  $V$ . The existence of the desired extension  $\tilde{\varphi}$  now follows from [15, Lemma 4], since  $\mathcal{C}_\varphi = \mathcal{C}(U)$  holds in the notation of that result. Uniqueness of  $\tilde{\varphi}$  as well as injectivity and the centralizing property are immediate from [15, Prop. 2(ii), (iii)]. For example, in order to guarantee that  $\tilde{\varphi}$  is centralizing, it suffices to check that  $C_V(\varphi(U))$  centralizes  $\tilde{\varphi}(\mathcal{C}(U))$ . To prove this, let  $q \in \mathcal{C}(U)$  and  $v \in C_V(\varphi(U))$  be given. By [15, Prop. 2(ii)] there is a nonzero ideal  $I \leq U$  so that  $qI \subseteq U$ . For  $u \in I$ , one computes

$$\tilde{\varphi}(q)v\varphi(u) = \tilde{\varphi}(q)\varphi(u)v = \varphi(qu)v = v\varphi(qu) = v\tilde{\varphi}(q)\varphi(u).$$

This shows that  $[v, \tilde{\varphi}(q)]\varphi(I) = 0$  and [15, Prop. 2(iii)] further implies that  $[v, \tilde{\varphi}(q)] = 0$ .  $\square$

The lemma implies in particular that every automorphism of a prime ring  $U$  extends uniquely to the central closure  $\tilde{U}$ . (A more general fact was already mentioned above.) We will generally use the same notation for the extended automorphism of  $\tilde{U}$ .

**2.3.**

The essence of the next result goes back to Martindale et al. [16], [9].

**Proposition 5.** *Let  $U$  be a centrally closed prime ring and let  $V$  be any  $C$ -algebra, where  $C = \mathcal{C}(U)$ . Then there are bijections*

$$\begin{array}{ccc} \{P \in \text{Spec}(U \otimes_C V) \mid P \cap U = 0\} & \xleftarrow{1-1} & \text{Spec } V, \\ P & \longmapsto & P \cap V, \\ U \otimes_C \mathfrak{p} & \longleftarrow & \mathfrak{p}. \end{array}$$

*These bijections are inverse to each other and they are equivariant with respect to all automorphisms of  $U \otimes_C V$  that stabilize both  $U$  and  $V$ . Furthermore, hearts are preserved:*

$$\mathcal{C}((U \otimes_C V)/P) \cong \mathcal{C}(V/P \cap V).$$

*Proof.* The extension  $V \hookrightarrow U \otimes_C V$  is centralizing. Therefore, contraction  $P \mapsto P \cap V$  is a well-defined map  $\text{Spec}(U \otimes_C V) \rightarrow \text{Spec } V$  which clearly has the stated equivariance property.

If  $V$  is a prime ring then so is  $U \otimes_C V$ ; this follows from the fact that every nonzero ideal of  $U \otimes_C V$  contains a nonzero element of the form  $u \otimes v$  with  $u \in U$  and  $v \in V$ ; see [15, Lemma 3(a)] or [9, Theorem 3.8(1)]. By [17, Cor. 2.5] we also know that  $\mathcal{C}(U \otimes_C V) = \mathcal{C}(V)$ . Consequently,  $\mathfrak{p} \mapsto U \otimes_C \mathfrak{p} = (U \otimes_C V)\mathfrak{p}$  gives a map  $\text{Spec } V \rightarrow \{P \in \text{Spec}(U \otimes_C V) \mid P \cap U = 0\}$  which preserves hearts.

Finally, by [15, Lemma 3(c)], the above maps are inverse to each other.  $\square$

**2.4.**

Let  $U$  be a prime ring. If  $U$  is an algebra over some commutative field  $F$  then the central closure  $\tilde{U}$  is an  $F$ -algebra as well, because  $\mathcal{Z}(U) \subseteq \mathcal{C}(U) = \mathcal{Z}(\tilde{U})$ .

**Proposition 6.** *Let  $U$  and  $V$  be algebras over some commutative field  $F$ , with  $U$  prime. Then there is a bijection*

$$\begin{array}{ccc} \{\tilde{P} \in \text{Spec}(\tilde{U} \otimes_F V) \mid \tilde{P} \cap \tilde{U} = 0\} & \longrightarrow & \{P \in \text{Spec}(U \otimes_F V) \mid P \cap U = 0\} \\ \tilde{P} & \longmapsto & \tilde{P} \cap (U \otimes_F V). \end{array}$$

*This bijection and its inverse are inclusion preserving and equivariant with respect to all automorphisms of  $\tilde{U} \otimes_F V$  that stabilize both  $\tilde{U}$  and  $U \otimes_F V$ . Furthermore, hearts are preserved under this bijection.*

*Proof.* Since the extension  $U \otimes_F V \hookrightarrow \tilde{U} \otimes_F V$  is centralizing, the contraction map  $\tilde{P} \mapsto \tilde{P} \cap (U \otimes_F V)$  sends primes to primes, and hence it yields a well-defined map between the sets in the proposition. This map is clearly inclusion preserving and equivariant as stated.

For surjectivity, let  $P$  be a prime of  $U \otimes_F V$  such that  $P \cap U = 0$ . The canonical map

$$\varphi: U \hookrightarrow U \otimes_F V \twoheadrightarrow W = (U \otimes_F V)/P$$

is a centralizing embedding of prime rings. By Lemma 4, there is a unique extension to central closures,

$$\tilde{\varphi}: \tilde{U} \longrightarrow \tilde{W}.$$

The image of the canonical map  $\psi: V \hookrightarrow U \otimes_F V \rightarrow W$  centralizes  $\varphi(U)$ , and hence it also centralizes  $\tilde{\varphi}(\tilde{U})$ ; see the proof of Lemma 4. Therefore,  $\tilde{\varphi}$  and  $\psi$  yield a ring homomorphism

$$\tilde{\varphi}_V: \tilde{U} \otimes_F V \longrightarrow \tilde{W}.$$

Put  $\tilde{P} = \text{Ker } \tilde{\varphi}_V$ . Since  $\tilde{\varphi}_V$  extends the canonical map  $U \otimes_F V \rightarrow W = (U \otimes_F V)/P$ , we have  $\tilde{P} \cap (U \otimes_F V) = P$  and

$$W \subseteq \text{Im } \tilde{\varphi}_V = (\tilde{U} \otimes_F V)/\tilde{P} \subseteq \tilde{W}. \tag{15}$$

In particular, every nonzero ideal of  $\text{Im } \tilde{\varphi}_V$  has a nonzero intersection with  $W$ , and hence  $\text{Im } \tilde{\varphi}_V$  is a prime ring and  $\tilde{P}$  is a prime ideal. Since  $\tilde{P} \cap U = P \cap U = 0$ , it follows that  $\tilde{P} \cap \tilde{U} = 0$ . This proves surjectivity. Furthermore, since the inclusions in (15) are centralizing inclusions of prime rings, they yield inclusions of extended centroids,  $\mathcal{C}(W) \subseteq \mathcal{C}((\tilde{U} \otimes_F V)/\tilde{P}) \subseteq \mathcal{C}(\tilde{W}) = \mathcal{C}(W)$  by Lemma 4. Therefore,  $\tilde{P}$  has the same heart as  $P$ .

To prove injectivity, let  $\tilde{P}$  and  $\tilde{P}'$  be primes of  $\tilde{U} \otimes_F V$ , both disjoint from  $\tilde{U} \setminus \{0\}$ , such that  $\tilde{P} \cap (U \otimes_F V) \subseteq \tilde{P}' \cap (U \otimes_F V)$ . We claim that  $\tilde{P} \subseteq \tilde{P}'$ . Indeed, it follows from [15, Prop. 2(ii)] that, for any  $q \in \tilde{U} \otimes_F V$ , there is a nonzero ideal  $I$  of  $U$  such that  $qI \subseteq U \otimes_F V$ . For  $q \in \tilde{P}$ , we conclude that  $qI \subseteq \tilde{P}'$ . Since  $I(\tilde{U} \otimes_F V)$  is an ideal of  $\tilde{U} \otimes_F V$  that is not contained in  $\tilde{P}'$ , we must have  $q \in \tilde{P}'$ . Therefore,  $\tilde{P} \subseteq \tilde{P}'$  as claimed. Injectivity follows and we also obtain that the inverse bijection preserves inclusions. This completes the proof.  $\square$

We will apply the equivariance property of the above bijection to automorphisms of the form  $\alpha \otimes_F \beta$  with  $\alpha \in \text{Aut}_{F\text{-alg}}(U)$ , extended uniquely to  $\tilde{U}$  as in Section 2.2, and  $\beta \in \text{Aut}_{F\text{-alg}}(V)$ .

**2.5.**

The following two technical results have been extracted from Mœglin and Rentschler [19, 3.4–3.6]; see also Vonessen [28, Proof of Prop. 8.12]. As above,  $F$  denotes a commutative field. If a group  $\Gamma$  acts on a ring  $U$  then  $U$  is called a  $\Gamma$ -ring; similarly for fields. A  $\Gamma$ -ring  $U$  is called  $\Gamma$ -simple if  $U$  has no  $\Gamma$ -stable ideals other than 0 and  $U$ .

**Lemma 7.** *Let  $F \subseteq L \subseteq K$  be  $\Gamma$ -fields. Assume that  $K = \text{Fract } A$  for some  $\Gamma$ -stable  $F$ -subalgebra  $A$  which is  $\Gamma$ -simple and affine (finitely generated). Then  $L = \text{Fract } B$  for some  $\Gamma$ -stable  $F$ -subalgebra  $B$  which is  $\Gamma$ -simple and generated by finitely many  $\Gamma$ -orbits.*

*Proof.* We need to construct a  $\Gamma$ -stable  $F$ -subalgebra  $B \subseteq L$  satisfying:

- (i)  $L = \text{Fract } B$ ;
- (ii)  $B$  is generated as  $F$ -algebra by finitely many  $\Gamma$ -orbits; and
- (iii)  $B$  is  $\Gamma$ -simple.

Note that  $L$  is a finitely generated field extension of  $F$ , because  $K$  is. Fix a finite set  $X_0 \subseteq L$  of field generators and let  $B_0 \subseteq L$  denote the  $F$ -subalgebra that is generated by  $\bigcup_{x \in X_0} \Gamma.x$ . Then  $B_0$  certainly satisfies (i) and (ii). We will show that the intersection of all nonzero  $\Gamma$ -stable semiprime ideals of  $B_0$  is nonzero. Consider the algebra  $B' = B_0A \subseteq K$ ; this algebra is  $\Gamma$ -stable and affine over  $B_0$ . By generic flatness [8, 2.6.3], there exists some nonzero  $t \in B_0$  so that  $B'[t^{-1}]$  is free over  $B_0[t^{-1}]$ . We claim that if  $\mathfrak{b}_0$  is any  $\Gamma$ -stable semiprime ideal of  $B_0$  not containing  $t$  then  $\mathfrak{b}_0$  must be zero. Indeed,  $\mathfrak{b}_0B_0[t^{-1}]$  is a proper ideal of  $B_0[t^{-1}]$ , and hence  $\mathfrak{b}_0B'[t^{-1}]$  is a proper ideal of  $B'[t^{-1}]$ . Therefore,  $\mathfrak{b}_0A \cap A$  is a proper ideal of  $A$  which is clearly  $\Gamma$ -stable. Since  $A$  is  $\Gamma$ -simple, we must have  $\mathfrak{b}_0A \cap A = 0$ . Finally, since  $\mathfrak{b}_0 \subseteq K = \text{Fract } A$ , we conclude that  $\mathfrak{b}_0 = 0$  as desired. Thus,  $t$  belongs to every nonzero  $\Gamma$ -stable semiprime ideal of  $B_0$ . Now let  $B$  be the  $F$ -subalgebra of  $L$  that is generated by  $B_0$  and the  $\Gamma$ -orbit of  $t^{-1}$ . Clearly,  $B$  still satisfies (i) and (ii). Moreover, if  $\mathfrak{b}$  is any nonzero  $\Gamma$ -stable semiprime ideal of  $B$  then  $\mathfrak{b} \cap B_0$  is a  $\Gamma$ -stable semiprime ideal of  $B_0$  which is nonzero, because  $L = \text{Fract } B_0$ . Hence  $t \in \mathfrak{b}$  and so  $\mathfrak{b} = B$ . This implies that  $B$  is  $\Gamma$ -simple which completes the construction of  $B$ .  $\square$

The lemma above will only be used in the proof of the following “lying over” result which will be crucial later on. For a given  $\Gamma$ -ring  $U$ , we let

$$\text{Spec}^\Gamma U$$

denote the collection of all  $\Gamma$ -stable primes of  $U$ . These primes are certainly  $\Gamma$ -prime, but the converse need not hold in general.

**Proposition 8.** *Let  $U$  be a prime  $F$ -algebra and let  $\Gamma$  be a group acting by  $F$ -algebra automorphisms on  $U$ . Suppose that there is a  $\Gamma$ -equivariant embedding  $\mathcal{C}(U) \hookrightarrow K$ , where  $K$  is a  $\Gamma$ -field such that  $K = \text{Fract } A$  for some  $\Gamma$ -stable affine  $F$ -subalgebra  $A$  which is  $\Gamma$ -simple.*

*Then there exists a nonzero ideal  $D \trianglelefteq U$  such that, for every  $P \in \text{Spec}^\Gamma U$  not containing  $D$ , there is a  $\tilde{P} \in \text{Spec}^\Gamma \tilde{U}$  satisfying  $\tilde{P} \cap U = P$ .*

*Proof.* Applying Lemma 7 to the given embedding  $F \subseteq C = \mathcal{C}(U) \hookrightarrow K$  we obtain a  $\Gamma$ -stable  $F$ -subalgebra  $B \subseteq C$  such that  $C = \text{Fract } B$ ,  $B$  is  $\Gamma$ -simple, and  $B$  is generated as an  $F$ -algebra by finitely many  $\Gamma$ -orbits. Fix a finite subset  $X \subseteq B$  such that  $B$  is generated by  $\bigcup_{x \in X} \Gamma.x$  and let

$$D = \{u \in U \mid xu \in U \text{ for all } x \in X\};$$

this is a nonzero ideal of  $U$  by [15, Prop. 2(ii)]. In order to show that  $D$  has the desired property, we first make the following

**Claim.** *Suppose  $P \in \text{Spec}^\Gamma U$  does not contain  $D$ . Then, given  $w_1, \dots, w_n \in UB \subseteq \tilde{U}$ , there exists an ideal  $I \trianglelefteq U$  with  $I \not\subseteq P$  and such that  $w_i I \subseteq U$  for all  $i$ .*

To see this, note that every element of  $UB$  is a finite sum of terms of the form

$$w = u(g_1.x_1) \cdots (g_r.x_r)$$

with  $u \in U$ ,  $g_j \in G$  and  $x_j \in X$ . The ideal  $J = \left(\bigcap_{j=1}^r g_j \cdot D\right)^r$  of  $U$  satisfies

$$wJ \subseteq ug_1 \cdot (x_1 D) \cdots g_r \cdot (x_r D) \subseteq U.$$

Moreover,  $J \not\subseteq P$ , because otherwise  $g_j \cdot D \subseteq P$  for some  $j$  and so  $D \subseteq P$  contrary to our hypothesis. Now write the given  $w_i$  as above, collect all occurring  $g_j$  in the (finite) subset  $E \subseteq G$  and let  $s$  be the largest occurring  $r$ . Then the ideal  $I = \left(\bigcap_{g \in E} g \cdot D\right)^s$  does what is required.

Next, we show that

$$PB \cap U = P.$$

Indeed, for any  $u \in PB \cap U$ , the above claim yields an ideal  $I \trianglelefteq U$  with  $I \not\subseteq P$  and such that  $uI \subseteq P$ . Since  $P$  is prime, we must have  $u \in P$  which proves the above equality. Now choose an ideal  $Q \trianglelefteq UB$  which contains the ideal  $PB$  and is maximal with respect to the condition  $Q \cap U = P$  (Zorn’s lemma). It is routine to check that  $Q$  is prime. Therefore,  $\tilde{Q} = Q : \Gamma$  is at least a  $\Gamma$ -prime ideal of  $UB$  satisfying  $\tilde{Q} \cap U = P$ . We show that  $\tilde{Q}$  is in fact prime. Let  $w_1, w_2 \in UB$  be given such that  $w_1 UB w_2 \subseteq \tilde{Q}$ . Choosing  $I$  as in the claim for  $w_1, w_2$ , we have  $w_1 I U w_2 I \subseteq U \cap \tilde{Q} = P$ . Since  $P$  is prime, we conclude that  $w_j I \subseteq P$  for  $j = 1$  or  $2$ . Hence,  $w_j IB \subseteq \tilde{Q} = \bigcap_{g \in \Gamma} g \cdot Q$ . Since each  $g \cdot Q$  is prime in  $UB$  and  $IB$  is an ideal of  $UB$  not contained in  $g \cdot Q$ , we obtain that  $w_j \in \bigcap_{g \in \Gamma} g \cdot Q = \tilde{Q}$ . This shows that  $\tilde{Q}$  is indeed prime.

Finally,  $\tilde{U}$  is the (central) localization of  $UB$  at the nonzero elements of  $B$ . Furthermore,  $\tilde{Q} \cap B = 0$ , since  $\tilde{Q} \cap B$  is a proper  $\Gamma$ -stable ideal of  $B$ . It follows that  $\tilde{P} = \tilde{Q}C$  is a prime ideal of  $\tilde{U}$  which is clearly  $\Gamma$ -stable and satisfies  $\tilde{P} \cap UB = \tilde{Q}$ . Consequently,  $\tilde{P} \cap U = \tilde{Q} \cap U = P$ , thereby completing the proof.  $\square$

### 3. Description of $G$ -strata

#### 3.1.

We now return to the setting of Section 0.1. The  $G$ -action on  $R$  will be denoted by

$$\rho = \rho_R : G \longrightarrow \text{Aut}_{\mathbb{k}\text{-alg}}(R) \tag{16}$$

when it needs to be explicitly referred to; so

$$g \cdot r = \rho(g)(r).$$

Recall from [15, 3.1, 3.4] that rationality of the action of  $G$  on  $R$  is equivalent to the existence of a  $\mathbb{k}$ -algebra map

$$\Delta_R : R \longrightarrow R \otimes \mathbb{k}[G], \quad r \mapsto \sum_i r_i \otimes f_i,$$

such that

$$g \cdot r = \sum_i r_i f_i(g)$$

holds for all  $g \in G$  and  $r \in R$ . Here,  $\mathbb{k}[G]$  denotes the Hopf algebra of regular functions on  $G$ , as usual. The  $\mathbb{k}[G]$ -linear extension of the map  $\Delta_R$  is an automorphism of  $\mathbb{k}[G]$ -algebras which will also be denoted by  $\Delta_R$ :

$$\Delta_R: R \otimes \mathbb{k}[G] \xrightarrow{\sim} R \otimes \mathbb{k}[G]; \tag{17}$$

see [15, 3.4].

**3.2.**

As in [15], the right and left *regular representations* of  $G$  will be denoted by

$$\rho_r, \rho_\ell: G \rightarrow \text{Aut}_{\mathbb{k}\text{-alg}}(\mathbb{k}[G]);$$

they are defined by  $(\rho_r(x)f)(y) = f(yx)$  and  $(\rho_\ell(x)f)(y) = f(x^{-1}y)$  for  $x, y \in G$ .

The group  $G$  acts (rationally) on the  $\mathbb{k}$ -algebra  $R \otimes \mathbb{k}[G]$  by means of the maps  $\text{Id}_R \otimes \rho_{r,\ell}$  and  $\rho \otimes \rho_{r,\ell}$ . The following intertwining formulas hold for all  $g \in G$ :

$$\Delta_R \circ (\rho \otimes \rho_r)(g) = (\text{Id}_R \otimes \rho_r)(g) \circ \Delta_R \tag{18}$$

and

$$\Delta_R \circ (\text{Id}_R \otimes \rho_\ell)(g) = (\rho \otimes \rho_\ell)(g) \circ \Delta_R, \tag{19}$$

where  $\Delta_R$  is the automorphism (17); see [15, 3.3].

In the following,

$$\mathbb{k}(G) = \text{Fract } \mathbb{k}[G]$$

will denote the algebra  $\mathbb{k}(G)$  of rational functions on  $G$ ; this is the full ring of fractions of  $\mathbb{k}[G]$ , and  $\mathbb{k}(G)$  is also equal to the direct product of the rational function fields of the irreducible components of  $G$  [2, AG 8.1]. The above  $G$ -actions extend uniquely to  $\mathbb{k}(G)$  and to  $R \otimes \mathbb{k}(G)$ . We will use the same notations as above for the extended actions. The intertwining formulas (18) and (19) remain valid in this setting, with  $\Delta_R$  replaced by its unique extension to a  $\mathbb{k}(G)$ -algebra automorphism

$$\Delta_R: R \otimes \mathbb{k}(G) \xrightarrow{\sim} R \otimes \mathbb{k}(G). \tag{20}$$

**3.3.**

We are now ready to describe the  $G$ -stratum

$$\text{Spec}_I R = \{P \in \text{Spec } R \mid P:G = I\}$$

over a given  $I \in G\text{-Spec } R$ . For simplicity, we will assume that  $G$  is connected; so  $\mathbb{k}(G)$  is a field which is unirational over  $\mathbb{k}$  [2, 18.2]. Furthermore,  $I$  is a prime ideal of  $R$  by [15, Prop. 19(a)] and so  $\mathcal{C}(R/I)$  is a field as well. The group  $G$  acts on  $R/I$  by means of the map  $\rho$  in (16). This action extends uniquely to an action on the central closure  $\widetilde{R/I}$ , and hence we also have a  $G$ -action on  $\mathcal{C}(R/I) = \widetilde{\mathcal{Z}(R/I)}$ . Denoting the latter two actions by  $\rho$  again, we obtain  $G$ -actions  $\rho \otimes \rho_r$  on  $\widetilde{R/I} \otimes \mathbb{k}(G)$  and on  $\mathcal{C}(R/I) \otimes \mathbb{k}(G)$ . As in Section 2.5, we will write

$$\text{Spec}^G(T) = \{\mathfrak{p} \in \text{Spec } T \mid \mathfrak{p} \text{ is } (\rho \otimes \rho_r)(G)\text{-stable}\}$$

for  $T = \mathcal{C}(R/I) \otimes \mathbb{k}(G)$  or  $T = \widetilde{R/I} \otimes \mathbb{k}(G)$ . The group  $G$  also acts on both algebras  $T$  via  $\text{Id} \otimes \rho_\ell$  and the latter action commutes with  $\rho \otimes \rho_r$ . Hence,  $G$  acts on  $\text{Spec}^G(T)$  through  $\text{Id} \otimes \rho_\ell$ . We are primarily interested in the first of these algebras,

$$T_I = \mathcal{C}(R/I) \otimes \mathbb{k}(G),$$

a commutative domain and a tensor product of two fields.

**Theorem 9.** *For a given  $I \in G\text{-Spec } R$ , let  $T_I = \mathcal{C}(R/I) \otimes \mathbb{k}(G)$ . There is a bijection*

$$c: \text{Spec}_I R \xrightarrow{\text{bij.}} \text{Spec}^G(T_I)$$

having the following properties, for  $P, P' \in \text{Spec}_I R$  and  $g \in G$ :

- (a)  $G$ -equivariance:  $c(g.P) = (\text{Id} \otimes \rho_\ell(g))(c(P))$ ;
- (b) inclusions:  $P \subseteq P' \iff c(P) \subseteq c(P')$ ;
- (c) hearts: *there is an isomorphism of  $\mathbb{k}(G)$ -fields*

$$\Psi_P: \mathcal{C}(T_I/c(P)) \xrightarrow{\sim} \mathcal{C}((R/P) \otimes \mathbb{k}(G))$$

satisfying

$$\begin{aligned} \Psi_P \circ (\rho \otimes \rho_r)(g) &= (\text{Id}_{R/P} \otimes \rho_r)(g) \circ \Psi_P, \\ \Psi_{g.P} \circ (\text{Id} \otimes \rho_\ell)(g) &= (\rho \otimes \rho_\ell)(g) \circ \Psi_P; \end{aligned}$$

- (d) rationality:  *$P$  is rational if and only if  $T_I/c(P) \cong \mathbb{k}(G)$ .*

Note that  $\mathcal{C}(T_I/c(P))$  in (c) is just the classical field of fractions of the commutative domain  $T_I/c(P)$ . Furthermore, in the second identity in (c), we have

$$\begin{aligned} (\text{Id} \otimes \rho_\ell)(g): \mathcal{C}(T_I/c(P)) &\xrightarrow{\sim} \mathcal{C}(T_I/c(g.P)), \\ (\rho \otimes \rho_\ell)(g): \mathcal{C}((R/P) \otimes \mathbb{k}(G)) &\xrightarrow{\sim} \mathcal{C}((R/g.P) \otimes \mathbb{k}(G)), \end{aligned}$$

in the obvious way. For (d), recall from (4) that there exists a rational  $P \in \text{Spec}_I R$  if and only if  $I$  is  $G$ -rational.

*Proof.* Replacing  $R$  by  $R/I$ , we may assume that  $I = 0$ . Our goal is to establish a bijection between  $\text{Spec}_0 R = \{P \in \text{Spec } R \mid P:G = 0\}$  and  $\text{Spec}^G(T_0)$ . For (a), this bijection needs to be equivariant for the  $G$ -action by  $\rho$  on  $R$  and by  $\text{Id} \otimes \rho_\ell$  on  $T_0$ .

As was pointed out above,  $R$  is a prime ring. For brevity, we will put

$$C = \mathcal{C}(R) \quad \text{and} \quad K = \mathbb{k}(G).$$

Thus,  $C$  and  $K$  are fields and  $T_0 = C \otimes K$ . Let  $\widetilde{R} = RC$  denote the central closure of  $R$ ; so  $\widetilde{R}$  is a centrally closed prime ring and

$$\widetilde{R} \otimes K = \widetilde{R} \otimes_C T_0.$$



By Proposition 5,  $\text{Spec } T_0$  is in bijection with the set of all primes  $\tilde{Q} \in \text{Spec}(\tilde{R} \otimes K)$  such that  $\tilde{Q} \cap \tilde{R} = 0$ . This bijection is equivariant with respect to the subgroups  $(\rho \otimes \rho_r)(G)$  and  $(\text{Id} \otimes \rho_\ell)(G)$  of  $\text{Aut}_{\mathbb{k}\text{-alg}}(\tilde{R} \otimes K)$ , because these subgroups stabilize both  $\tilde{R}$  and  $T_0$ . Therefore, the bijection in Proposition 5 restricts to a bijection

$$\text{Spec}^G(T_0) \xrightarrow{1-1} \{\tilde{Q} \in \text{Spec}^G(\tilde{R} \otimes K) \mid \tilde{Q} \cap \tilde{R} = 0\},$$

which is equivariant for the  $G$ -action by  $\text{Id} \otimes \rho_\ell$  on  $T_0$  and on  $\tilde{R} \otimes K$ . Furthermore, Proposition 6 gives a bijection  $\{\tilde{Q} \in \text{Spec}(\tilde{R} \otimes K) \mid \tilde{Q} \cap \tilde{R} = 0\} \xrightarrow{1-1} \{Q \in \text{Spec}(R \otimes K) \mid Q \cap R = 0\}$  and this bijection is equivariant with respect to both  $(\rho \otimes \rho_r)(G)$  and  $(\text{Id} \otimes \rho_\ell)(G)$ . As above, we obtain a bijection

$$\{\tilde{Q} \in \text{Spec}^G(\tilde{R} \otimes K) \mid \tilde{Q} \cap \tilde{R} = 0\} \xrightarrow{1-1} \{Q \in \text{Spec}^G(R \otimes K) \mid Q \cap R = 0\},$$

which is equivariant for the  $G$ -action by  $\text{Id} \otimes \rho_\ell$  on  $R \otimes K$  and on  $\tilde{R} \otimes K$ . Hence it suffices to construct a bijection

$$d: \text{Spec}_0 R \longrightarrow \{Q \in \text{Spec}^G(R \otimes K) \mid Q \cap R = 0\} \tag{21}$$

which is equivariant for the  $G$ -action by  $\rho$  on  $R$  and by  $\text{Id} \otimes \rho_\ell$  on  $R \otimes K$ .

For a given  $P \in \text{Spec}_0 R$ , consider the homomorphism of  $K$ -algebras

$$\varphi_P: R \otimes K \xrightarrow{\Delta_R} R \otimes K \xrightarrow{\text{can.}} S_P = (R/P) \otimes K, \tag{22}$$

where  $\Delta_R$  is the automorphism (20) and the second map is the  $K$ -linear extension of the canonical epimorphism  $R \twoheadrightarrow R/P$ . The algebra  $S_P$  is prime, since  $K$  is unirational over  $\mathbb{k}$ . Therefore,

$$d(P) = \text{Ker } \varphi_P = \Delta_R^{-1}(P \otimes K) \tag{23}$$

is a prime ideal of  $R \otimes K$ . From [15, Lemma 17], we infer that  $P:G = d(P) \cap R$ , and so  $d(P) \cap R = 0$ . Furthermore,  $d(P)$  clearly determines  $P$ . Hence, the map  $P \mapsto d(P)$  yields an injection of  $\text{Spec}_0 R$  into  $\{Q \in \text{Spec}(R \otimes K) \mid Q \cap R = 0\}$ .

We now check that this injection is  $G$ -equivariant and has image in  $\text{Spec}^G(R \otimes K)$ . Note that (18) and (19) imply the following equalities for all  $g \in G$ :

$$\varphi_P \circ (\rho \otimes \rho_r)(g) = (\text{Id}_{R/P} \otimes \rho_r)(g) \circ \varphi_P, \tag{24}$$

$$\varphi_{g.P} \circ (\text{Id}_R \otimes \rho_\ell)(g) = (\rho \otimes \rho_\ell)(g) \circ \varphi_P. \tag{25}$$

In (25), we view  $(\rho \otimes \rho_\ell)(g)$  as an isomorphism  $S_P \xrightarrow{\sim} S_{g.P}$  in the obvious way. In particular, (24) shows that  $d(P)$  is stable under  $(\rho \otimes \rho_r)(G)$  while (25) gives

$$d(g.P) = (\text{Id}_R \otimes \rho_\ell)(g)(P);$$

so the map  $P \mapsto d(P)$  is  $G$ -equivariant.

For surjectivity of  $d$  and the inverse map, let  $Q \in \text{Spec}^G(R \otimes K)$  be given such that  $Q \cap R = 0$ . Put

$$P = R \cap \Delta_R(Q).$$

Then  $P$  is prime in  $R$ , because the extension  $R \hookrightarrow R \otimes K$  is centralizing. Furthermore, [15, Lemma 17] gives  $P:G = \Delta_R^{-1}(P \otimes K) \cap R = Q \cap R = 0$ ; so  $P \in \text{Spec}_0 R$ . We claim that  $Q = d(P)$  or, equivalently,  $\Delta_R(Q) = P \otimes K$ . To see this, note that  $\Delta_R(Q)$  is stable under  $(\text{Id} \otimes \rho_r)(G)$  by (18). Thus, the desired equality  $\Delta_R(Q) = P \otimes K$  follows from [6, Cor. to Prop. V.10.6], because the field of  $\rho_r(G)$ -invariants in  $K$  is  $\mathbb{k}$ . Thus,  $d$  is surjective and the inverse of  $d$  is given by

$$d^{-1}(Q) = R \cap \Delta_R(Q). \tag{26}$$

This finishes the construction of the desired  $G$ -equivariant bijection (21).

To summarize, we have constructed a bijection

$$c: \text{Spec}_0 R \longrightarrow \text{Spec}^G(T_0)$$

with property (a); it arises as the composite of the following bijections:

$$\begin{array}{ccc} \text{Spec}_0 R & \xrightarrow{d} & \{Q \in \text{Spec}^G(R \otimes K) \mid Q \cap R = 0\} \\ \downarrow c = f \circ e^{-1} \circ d & & \uparrow e \\ \text{Spec}^G(T_0) & \xleftarrow{f} & \{\tilde{Q} \in \text{Spec}^G(\tilde{R} \otimes K) \mid \tilde{Q} \cap \tilde{R} = 0\} \end{array} \tag{27}$$

Formulas for  $d$  and its inverse are given in (23) and (26), respectively. The other maps are as follows:

$$e(\tilde{Q}) = \tilde{Q} \cap (R \otimes K), \tag{28}$$

$$f(\tilde{Q}) = \tilde{Q} \cap T_0, \tag{29}$$

$$f^{-1}(\mathfrak{p}) = \tilde{R} \otimes_C \mathfrak{p} = (\tilde{R} \otimes K)\mathfrak{p}; \tag{30}$$

see Propositions 5 and 6. The maps  $d^{\pm 1}$ ,  $f^{\pm 1}$ , and  $e$  visibly preserve inclusions, and Proposition 6 tells us that this also holds for  $e^{-1}$ . Property (b) follows. By Propositions 5 and 6, both  $e$  and  $f$  also preserve hearts. Thus, we have an isomorphism of  $K$ -fields

$$\Psi_P: \mathcal{C}(T_0/c(P)) \xrightarrow{\sim} \mathcal{C}((R \otimes K)/d(P)) \xrightarrow{\sim} \mathcal{C}(S_P).$$

The desired identities for  $\Psi$  are consequences of (24) and (25). This proves property (c).

For (d), note that

$$K \subseteq T_0/c(P) \subseteq \mathcal{C}(T_0/c(P)) = \text{Fract}(T_0/c(P)) \cong \mathcal{C}(S_P) \tag{31}$$

holds for any  $P \in \text{Spec}_0 R$ . If  $P$  is rational then  $\mathcal{C}(S_P) = K$  by [15, Lemma 7], and so  $T_0/c(P) = K$ . Conversely, if  $T_0/c(P) = K$  then (31) gives  $\mathcal{C}(S_P) = K$ . Since there always is a  $K$ -embedding  $\mathcal{C}(R/P) \otimes K \hookrightarrow \mathcal{C}(S_P)$  by [15, Eq. (1-2)], we conclude that  $\mathcal{C}(R/P) = \mathbb{k}$ ; so  $P$  is rational. This proves (d), and hence the proof of the theorem is complete.  $\square$

3.4.

Note that Theorem 9(b) and (d) together imply that rational ideals are maximal in their  $G$ -strata. The following result, which generalizes Vonessen [28, Theorem 2.3], is a marginal strengthening of this fact. In particular, the group  $G$  is no longer assumed to be connected.

**Proposition 10.** *Let  $P \in \text{Rat } R$  and let  $I \trianglelefteq R$  be any ideal of  $R$  such that  $I \supseteq P$ . If  $I:G = P:G$  then  $I = P$ .*

*Proof.* Let  $G^0$  denote the connected component of the identity in  $G$ ; this is a normal subgroup of finite index in  $G$  [2, 1.2]. Putting  $I^0 = I:G^0$  and  $P^0 = P:G^0$ , we have

$$I^0 \supseteq P^0 \supseteq P:G = I:G = \bigcap_{x \in G/G^0} x.I^0.$$

Since  $P^0$  is  $G^0$ -prime and all  $x.I^0$  are  $G^0$ -stable ideals of  $R$ , we conclude that  $I^0 \supseteq P^0 \supseteq x.I^0$  for some  $x$  and (9) further implies that  $I^0 = P^0$ . Therefore, we may replace  $G$  by  $G^0$ , thereby reducing to the case where  $G$  is connected. Furthermore, replacing  $I$  by an ideal that is maximal subject to the condition  $I:G = P:G$ , we may also assume that  $I$  is prime; see [15, Proof of Prop. 8]. Thus,  $P$  and  $I$  both belong to  $\text{Spec}_{P:G} R$  and Theorem 9(b), (d) yields the result.  $\square$

**Corollary 11.** *Let  $I \in G\text{-Spec } R$  be locally closed in  $G\text{-Spec } R$ . Then the maximal members of the  $G$ -stratum  $\text{Spec}_I R$  are locally closed in  $\text{Spec } R$ . In particular, if  $R$  satisfies the Nullstellensatz (see Section 0.7) then the maximal members of the  $G$ -stratum  $\text{Spec}_I R$  are exactly the rational ideals in  $\text{Spec}_I R$ .*

*Proof.* Let  $P \in \text{Spec } R$  be maximal in its  $G$ -stratum  $\text{Spec}_I R$ , where  $I = P:G$ . Then, for any  $Q \in \text{Spec } R$  with  $Q \not\supseteq P$ , we have  $Q:G \not\supseteq I$ . Since  $I$  is locally closed, it follows that  $I \neq \bigcap_{Q \not\supseteq P} Q:G$ . Hence  $P \neq \bigcap_{Q \not\supseteq P} Q$  which proves that  $P$  is locally closed in  $\text{Spec } R$ .

Finally, rational ideals are always maximal in their  $G$ -strata by Proposition 10. In the presence of the Nullstellensatz, the converse follows from the preceding paragraph.  $\square$

3.5.

We review some general results of Mœglin and Rentschler [22] and of Vonessen [28]. Some of the constructions below were already used, in a more specialized form, in the proof of Theorem 9. The affine algebraic group  $G$  need not be connected here, but we will only use this material in the connected case later on.

Fix a closed subgroup  $H \leq G$  and let

$$\mathbb{k}(G)^H = \mathbb{k}(H \backslash G) \subseteq \mathbb{k}(G)$$

denote the subalgebra of invariants for the left regular action  $\rho_\ell|_H$  on  $\mathbb{k}(G)$ . Following Mœglin and Rentschler [22] we define, for an arbitrary ideal  $I \trianglelefteq R$ ,

$$I^\natural = \Delta_R^{-1}(I \otimes \mathbb{k}(G)) \cap (R \otimes \mathbb{k}(G)^H). \tag{32}$$

Here  $\Delta_R$  is the automorphism (20) of  $R \otimes \mathbb{k}(G)$ . Thus,  $I^\natural$  is certainly an ideal of  $R \otimes \mathbb{k}(G)^H$ . Furthermore:

- If  $I$  is semiprime then  $I \otimes \mathbb{k}(G)$  is a semiprime ideal of  $R \otimes \mathbb{k}(G)$ , because  $\mathbb{k}(G)$  is a direct product of fields that are unirational over  $\mathbb{k}$ . Therefore,  $I^\natural$  is semiprime in this case. For connected  $G$ , we also see that if  $I$  is prime then  $I^\natural$  is likewise, as in (23).
- The group  $G$  acts on  $R \otimes \mathbb{k}(G)^H$  by means of  $\rho \otimes \rho_r$ . Formula (18) implies that  $I^\natural$  is always stable under this action. Moreover, if the ideal  $I$  is  $H$ -stable then formula (19) implies that the ideal  $\Delta_R^{-1}(I \otimes \mathbb{k}(G))$  of  $R \otimes \mathbb{k}(G)$  is stable under the automorphism group  $\text{Id}_R \otimes \rho_\ell(H)$ . Therefore, [28, Lemma 6.3] (or [6, Cor. to Prop. V.10.6] for connected  $G$ ) implies that  $\Delta_R^{-1}(I \otimes \mathbb{k}(G)) = I^\natural \otimes_{\mathbb{k}(G)^H} \mathbb{k}(G)$  holds in this case, and hence

$$I = \Delta_R (I^\natural \otimes_{\mathbb{k}(G)^H} \mathbb{k}(G)) \cap R. \tag{33}$$

To summarize, the map  $I \mapsto I^\natural$  gives an injection of the set of all  $H$ -stable semiprime ideals of  $R$  into the set of all  $(\rho \otimes \rho_r)(G)$ -stable semiprime ideals of  $R \otimes \mathbb{k}(G)^H$ . In fact:

**Proposition 12** (Moeglin and Rentschler, Vonessen). *Let  $H$  be a closed subgroup of  $G$ . The map  $I \mapsto I^\natural$  in (32) gives a bijection from the set of all  $H$ -stable semiprime ideals of  $R$  to the set of all  $(\rho \otimes \rho_r)(G)$ -stable semiprime ideals of  $R \otimes \mathbb{k}(G)^H$ . This bijection and its inverse, given by (33), preserve inclusions.*

For the complete proof, see [28, Theorem 6.6(a)].

### 4. Proof of Theorem 1

#### 4.1. Proof of Theorem 1 (b) $\Rightarrow$ (a)

Let  $P \in \text{Rat } R$  be given such that  $I = P:G$  is a locally closed point of  $G\text{-Spec } R$ . Then the preimage  $\gamma^{-1}(I) = \text{Spec}_I R$  under the continuous map (2) is a locally closed subset of  $\text{Spec } R$ , and hence so is  $\text{Spec}_I R \cap \{P\}$ . By Proposition 10,  $\text{Spec}_I R \cap \overline{\{P\}} = \{P\}$ , which proves (a).

#### 4.2. Proof of Theorem 1 (a) $\Rightarrow$ (b)

Besides making crucial use of Theorem 9, our proof closely follows Vonessen [28, Sect. 8] which in turn is based on Moeglin and Rentschler [19, Sect. 3].

*Some reductions.* First, recall that the connected component of the identity in  $G$  is always a normal subgroup of finite index in  $G$ . Therefore, Lemma 3 allows us to assume that  $G$  is connected and we will do so for the remainder of this section.

We are given an ideal  $P \in \text{Rat } R$  satisfying (13) and our goal is to show that  $P:G$  is distinct from the intersection of all  $G$ -primes of  $R$  which properly contain  $P:G$ ; see (14). For this, we may clearly replace  $R$  by  $R/P:G$ . Hence we may assume that

$$P:G = 0.$$

Thus, the algebra  $R$  is prime by [15, Prop. 19(a)], and so  $\mathcal{C}(R)$  is a field. Our goal now is to show that the intersection of all nonzero  $G$ -primes of  $R$  is nonzero again.

Note that this intersection is identical to the intersection of all nonzero  $G$ -stable semiprime ideals of  $R$ , because  $G$ -stable semiprimes are exactly the intersections of  $G$ -primes. The intersection in question is also identical to the intersection of all nonzero  $G$ -stable prime ideals of  $R$ , because  $G$ -primes are the same as  $G$ -stable primes of  $R$  by [15, Prop. 19(a)].

*The main lemma.* Let  $\tilde{R} = RC(R)$  denote the central closure of  $R$ . In the lemma below, which corresponds to Vonessen [28, Prop. 8.7], we achieve our goal for  $\tilde{R}$  in place of  $R$ .

**Lemma 13.** *The intersection of all nonzero  $G$ -stable semiprime ideals of  $\tilde{R}$  is nonzero.*

*Proof.* Let  $G_P$  denote the stabilizer of  $P$  in  $G$  and recall that  $G_P$  is a closed subgroup of  $G$  [13, I.2.12(5)]. Since  $P$  is locally closed,  $P$  is distinct from the intersection of all  $G_P$ -stable semiprime ideals of  $R$  that properly contain  $P$ . By Proposition 12 with  $H = G_P$ , we conclude that the ideal  $P^\natural \in \text{Spec}(R \otimes K^{G_P})$  is distinct from the intersection of all  $(\rho \otimes \rho_r)(G)$ -stable semiprime ideals of  $R \otimes K^{G_P}$  that properly contain  $P^\natural$ . Here, we have put  $K = \mathbb{k}(G)$  and  $K^{G_P} = \mathbb{k}(G_P \backslash G)$  denotes the invariant subfield of  $K$  for the left regular action  $\rho_\ell|_{G_P}$  as in Section 3.5. In other words, the intersection of all nonzero  $(\rho \otimes \rho_r)(G)$ -stable semiprime ideals of  $(R \otimes K^{G_P})/P^\natural$  is nonzero. By Vonessen [28, Lemma 8.6], the lemma will follow if we can show that there is a finite centralizing embedding

$$\tilde{R} \hookrightarrow (R \otimes K^{G_P})/P^\natural \tag{34}$$

such that the  $G$ -action via  $\rho \otimes \rho_r$  on  $(R \otimes K^{G_P})/P^\natural$  restricts to the  $G$ -action on  $\tilde{R}$  via the unique extension of  $\rho$ .

To construct the desired embedding, write  $C = \mathcal{C}(R)$  and consider the  $\mathbb{k}$ -algebra map

$$\psi_P : C \hookrightarrow T_0 = C \otimes K \rightarrow T_0/c(P) \xrightarrow{\sim} K, \tag{35}$$

where the first two maps are canonical and the last map is the  $K$ -isomorphism in Theorem 9(d); it is given by the isomorphism  $\Psi_P$  in Theorem 9(c). (We remark that the map  $\psi_P$  is identical to the one constructed in [15, Theorem 22].) The identities for  $\Psi_P$  in Theorem 9(c) yield the following formulas:

$$\psi_P \circ \rho(g)|_C = \rho_r(g) \circ \psi_P, \tag{36}$$

$$\psi_{g.P} = \rho_\ell(g) \circ \psi_P. \tag{37}$$

(See also (a) in the proof of [15, Theorem 22].) Consider the subfield

$$K_P = \text{Im } \psi_P \subseteq K.$$

Equation (36) implies that  $K_P$  is a  $\rho_r(G)$ -stable subfield of  $K$ . More precisely, identity (37) shows that

$$K_P \subseteq K^{G_P}.$$

Moreover, if  $g \notin G_P$  then  $c(g.P) \neq c(P)$  and hence  $\psi_{g.P} \neq \psi_P$ . Therefore, we deduce from (37) that  $G_P = \{g \in G \mid \rho_\ell(g)(x) = x \text{ for all } x \in K_P\}$ . Now Vonessen [28, Prop. 4.5] gives that the field extension  $K^{G_P}/K_P$  is finite (and purely inseparable). Thus, the desired embedding (34) will follow if we can show that there is a  $(\rho \otimes \rho_r)(G)$ -equivariant isomorphism

$$(R \otimes K^{G_P})/P^\natural \cong \tilde{R} \otimes_C K^{G_P},$$

where  $c \otimes 1 = 1 \otimes \psi_P(c)$  holds for all  $c \in C$  in the ring on the right. But the map  $\tilde{R} \otimes K \rightarrow \tilde{R} \otimes_C K \cong \tilde{R} \otimes_C (T_0/c(P))$  has kernel  $(e^{-1} \circ d)(P)$  in the notation of (27), and it is  $(\rho \otimes \rho_r)(G)$ -equivariant. The restriction of this map to  $R \otimes K^{G_P}$  has image  $\tilde{R} \otimes_C K^{G_P}$  and kernel  $(e^{-1} \circ d)(P) \cap (R \otimes K^{G_P}) = P^\natural$ . This finishes the proof of the lemma.  $\square$

*End of proof.* We now complete the proof of Theorem 1 by showing that the intersection of all nonzero  $G$ -stable prime ideals of  $R$  is nonzero.

We use the  $G$ -equivariant embedding  $\psi_P: C \hookrightarrow K = \mathbb{k}(G)$ ; see (35) and (36). Note that  $K = \text{Fract } A$  where  $A = \mathbb{k}[G]$  is a  $G$ -stable affine domain over  $\mathbb{k}$  whose maximal ideals form one  $G$ -orbit. Therefore,  $A$  is  $G$ -simple. By Proposition 8, there exists an ideal  $0 \neq D \trianglelefteq R$  such that, for every  $G$ -stable prime  $P$  of  $R$  not containing  $D$ , there is a  $G$ -stable prime of  $\tilde{R}$  lying over  $P$ .

Now let  $N$  denote the intersection of all nonzero  $G$ -stable semiprime ideals of  $\tilde{R}$ ; so  $N \neq 0$  by Lemma 13 and hence  $N \cap R \neq 0$  by [15, Prop. 2(ii), (iii)]. We conclude from the preceding paragraph that every nonzero  $G$ -stable prime ideal of  $R$  either contains  $D$  or else it contains  $N \cap R$ . Therefore, the intersection of all nonzero  $G$ -stable prime ideals of  $R$  contains the ideal  $D \cap N \cap R$  which is nonzero, because  $R$  is prime. This completes the proof of Theorem 1.  $\square$

### 5. Finiteness of $G\text{-Spec } R$

#### 5.1.

The following finiteness result is an application of Theorem 1 and [15, Theorem 1].

**Proposition 14.** *Assume that  $R$  satisfies the Nullstellensatz. Then the following conditions are equivalent:*

- (a)  $R$  has finitely many  $G$ -stable semiprime ideals;
- (b)  $G\text{-Spec } R$  is finite;
- (c)  $G\text{-Rat } R$  is finite;
- (d)  $G$  has finitely many orbits in  $\text{Rat } R$ ;
- (e)  $R$  satisfies:
  - (i) the ascending chain condition for  $G$ -stable semiprime ideals;
  - (ii) the Dixmier–Mœglin equivalence; and
  - (iii)  $G\text{-Rat } R = G\text{-Spec } R$ .

*If these conditions are satisfied then rational ideals of  $R$  are exactly the primes that are maximal in their  $G$ -strata.*

*Proof.* The implications (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c) are trivial and (c)  $\Leftrightarrow$  (d) is clear from (4). Moreover, the Nullstellensatz implies that the  $G$ -stable semiprime ideals of  $R$  are exactly the intersections of  $G$ -rational ideals. Thus, (c) implies (a), and hence conditions (a)–(d) are all equivalent.

We now show that (a)–(d) imply (e). First, (i) is trivial from (a). For (ii), note that (b) implies that all points of  $G\text{-Spec } R$  are locally closed. Hence, all rational ideals of  $R$  are locally closed in  $\text{Spec } R$  by Theorem 1, proving (ii). Finally, in order to prove (iii), write a given  $I \in G\text{-Spec } R$  as an intersection of  $G$ -rational ideals. The intersection involves only finitely many members by (c), and so one of them must be equal to  $I$  by  $G$ -primeness. Thus,  $I \in G\text{-Rat } R$  which takes care of (iii).

To complete the proof of the equivalence of (a)–(e), we will show that (e) implies (b). By a familiar argument, hypothesis (i) allows us to assume that the algebra  $R$  is  $G$ -prime and  $G\text{-Spec } R/I$  is finite for all nonzero  $G$ -stable semiprime ideals  $I$  of  $R$ . By (iii) and (4), we know that  $P:G = 0$  holds for some  $P \in \text{Rat } R$ . Since  $P$  is locally closed in  $\text{Spec } R$  by (ii), Theorem 1 implies that  $0$  is locally closed in  $G\text{-Spec } R$ , that is,  $I = \bigcap_{0 \neq J \in G\text{-Spec } R} J$  is nonzero; see (14). Therefore,  $G\text{-Spec } R = \{0\} \cup G\text{-Spec } R/I$  is finite.

Finally, the last assertion is clear from Corollary 11, because all points of  $G\text{-Spec } R$  are locally closed by (b).  $\square$

**5.2.**

We now concentrate on the case where  $G$  is an algebraic torus:  $G \cong \mathbb{G}_m^n$  with  $\mathbb{G}_m = \mathbb{k}^*$ . In particular,  $G$  is connected and so  $G\text{-Spec } R$  is simply the set of all  $G$ -stable primes of  $R$  by [15, Prop. 19(a)]. Moreover, every  $G$ -module  $M$  has the form

$$M = \bigoplus_{\lambda \in X(G)} M_\lambda,$$

where  $X(G)$  is the collection of all morphisms of algebraic groups  $\lambda: G \rightarrow \mathbb{G}_m$  and

$$M_\lambda = \{m \in M \mid g.m = \lambda(g)m \text{ for all } g \in G\}$$

is the set of  $G$ -eigenvectors of weight  $\lambda$  in  $M$ .

**Lemma 15.** *If  $\dim_{\mathbb{k}} R_\lambda \leq 1$  holds for all  $\lambda \in X(G)$  then  $G\text{-Spec } R = G\text{-Rat } R$ .*

*Proof.* Let  $P \in G\text{-Spec } R$  be given. The condition  $\dim_{\mathbb{k}} R_\lambda \leq 1$  for all  $\lambda \in X(G)$  passes to  $R/P$ . Therefore, replacing  $R$  by  $R/P$ , we may assume that  $R$  is prime and we must show that  $\mathcal{C}(R)^G = \mathbb{k}$ . For a given  $q \in \mathcal{C}(R)^G$  put  $I = \{r \in R \mid qr \in R\}$ ; this is a nonzero  $G$ -invariant ideal of  $R$ . Hence  $I = \bigoplus_{\lambda \in X(G)} I_\lambda$  and so there is a nonzero element  $r \in I_\lambda$  for some  $\lambda$ . Since  $q$  is  $G$ -invariant, we have  $qr \in R_\lambda = \mathbb{k}r$ . Therefore,  $(q - k)r = 0$  holds in the central closure  $\tilde{R}$  for some  $k \in \mathbb{k}$ . Inasmuch as nonzero elements of  $\mathcal{C}(R)$  are units in  $\tilde{R}$ , we conclude that  $q = k \in \mathbb{k}$ , which proves the lemma.  $\square$

*Example: Affine commutative algebras.* The following proposition is a standard result on  $G$ -varieties [14, II.3.3 Satz 5]. It is also immediate from the foregoing:

**Proposition 16.** *Let  $R$  be an affine commutative domain over  $\mathbb{k}$  and let  $G$  be an algebraic  $\mathbb{k}$ -torus acting rationally on  $R$ . Then the following are equivalent:*

- (i)  $G\text{-Spec } R$  is finite;
- (ii)  $(\text{Fract } R)^G = \mathbb{k}$ ;
- (iii)  $\dim_{\mathbb{k}} R_{\lambda} \leq 1$  for all  $\lambda \in X(G)$ .

*Proof.* Since affine commutative algebras satisfy the Nullstellensatz, the Dixmier–Mœglin equivalence and the ascending chain condition for ideals, Proposition 14 tells us that (i) amounts to the equality  $G\text{-Spec } R = G\text{-Rat } R$ . The implication (iii)  $\Rightarrow$  (i) therefore follows from Lemma 15. Furthermore, (i) implies that  $0 \in G\text{-Rat } R$ ; so  $\mathcal{C}(R)^G = \mathbb{k}$ . Since  $\mathcal{C}(R) = \text{Fract } R$ , (ii) follows. Finally, if  $a, b \in R_{\lambda}$  are linearly independent then  $a/b \in \text{Fract } R$  is not a scalar. Hence (ii) implies (iii).  $\square$

*Example: Quantum affine toric varieties.* Affine domains  $R$  with a rational action by an algebraic torus  $G$  satisfying the condition  $\dim_{\mathbb{k}} R_{\lambda} \leq 1$  for all  $\lambda \in X(G)$  as in Lemma 15 are called *quantum affine toric varieties* in Ingalls [12].

A particular example is *quantum affine space*  $R = \mathcal{O}_{\mathbf{q}}(\mathbb{k}^n) = \mathbb{k}[x_1, \dots, x_n]$ . Here,  $\mathbf{q}$  denotes a family of parameters  $q_{ij} \in \mathbb{k}^*$  ( $1 \leq i < j \leq n$ ) and the defining relations of  $R$  are

$$x_j x_i = q_{ij} x_i x_j \quad (i < j).$$

The torus  $G = \mathbb{G}_m^n$  acts on  $R$ , with  $\alpha = (\alpha_1, \dots, \alpha_n) \in G$  acting by

$$\alpha \cdot x_i = \alpha_i x_i$$

for all  $i$ . The standard PBW-basis of  $R$ ,

$$\{x^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n} \mid \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n\},$$

consists of  $G$ -eigenvectors:  $x^{\mathbf{m}} \in R_{\lambda_{\mathbf{m}}}$  with

$$\lambda_{\mathbf{m}}(\alpha) = \alpha^{\mathbf{m}} = \alpha_1^{m_1} \cdots \alpha_n^{m_n}.$$

Therefore, the condition  $\dim_{\mathbb{k}} R_{\lambda} \leq 1$  for all  $\lambda$  is satisfied. Moreover,  $R = \mathcal{O}_{\mathbf{q}}(\mathbb{k}^n)$  is Noetherian and satisfies the Dixmier–Mœglin equivalence as long as  $\mathbb{k}$  contains a non-root of unity [7, II.8.4].

Any quantum affine toric variety  $R$  is a quotient of some  $\mathcal{O}_{\mathbf{q}}(\mathbb{k}^n)$  [12, p. 6]. Hence,  $R$  is Noetherian and satisfies the Dixmier–Mœglin equivalence (in the presence of non-roots of unity). Therefore,  $G\text{-Spec } R$  is finite by Proposition 14 and Lemma 15.

*Example: Quantum  $2 \times 2$  matrices.* Let  $R = \mathcal{O}_{\mathbf{q}}(M_2)$ ; this is the algebra with generators  $a, b, c, d$  and defining relations

$$\begin{aligned} ab &= qba, & ac &= qca, & bc &= cb, \\ bd &= qdb, & cd &= qdc, & ad - da &= (q - q^{-1})bd. \end{aligned}$$



The torus  $\mathbb{G}_m^4$  acts on  $R$  as in [7, II.1.6(c)], with  $D = \{(\alpha, \alpha, \alpha^{-1}, \alpha^{-1}) \mid \alpha \in \mathbb{k}^*\}$  acting trivially. Thus,  $G = \mathbb{G}_m^4/D \cong \mathbb{G}_m^3$  acts on  $R$ :

$$\begin{aligned} (\alpha, \beta, \gamma).a &= \beta a, & (\alpha, \beta, \gamma).b &= \gamma b, \\ (\alpha, \beta, \gamma).c &= \alpha \beta c, & (\alpha, \beta, \gamma).d &= \alpha \gamma d. \end{aligned}$$

This action does not satisfy condition  $\dim_{\mathbb{k}} R_\lambda \leq 1$  for all  $\lambda \in X(G)$ . Indeed, the PBW-basis  $\{a^i b^j c^l d^m \mid i, j, l, m \in \mathbb{Z}_{\geq 0}\}$  of  $R$  consists of  $G$ -eigenvectors:  $a^i b^j c^l d^m$  corresponds to the character  $(\alpha, \beta, \gamma) \mapsto \alpha^{i+m} \beta^{i+l} \gamma^{j+m}$ . Defining

$$f: \mathbb{Z}^4 \longrightarrow \mathbb{Z}^3, \quad (i, j, l, m) \mapsto (l + m, i + l, j + m),$$

we have, for any given  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ ,

$$\dim_{\mathbb{k}} R_\lambda = \#(f^{-1}(\lambda) \cap \mathbb{Z}_{\geq 0}^4).$$

In order to determine this number, note that  $\text{Ker } f = \mathbb{Z}(1, -1, -1, 1)$ . Hence, we must count the possible  $t \in \mathbb{Z}$  so that  $z_\lambda + t(1, -1, -1, 1) \in \mathbb{Z}_{\geq 0}^4$  where we have put  $z_\lambda = (\lambda_2 - \lambda_1, \lambda_3, \lambda_1, 0)$ . We obtain the following conditions on  $t$ :  $\lambda_2 - \lambda_1 + t \geq 0$ ,  $\lambda_3 - t \geq 0$ ,  $\lambda_1 - t \geq 0$ , and  $t \geq 0$ . Therefore,

$$\dim_{\mathbb{k}} R_\lambda = \max\{0, \min\{\lambda_1, \lambda_3\} - \max\{\lambda_1 - \lambda_2, 0\} + 1\} \tag{38}$$

which can be arbitrarily large.

Now consider the algebra  $\overline{R} = R/(D_q)$  where  $D_q = ad - qbc \in \mathcal{Z}R$  is the quantum determinant; see [7, I.1.9]. Since  $D_q \in R_\pi$  with  $\pi = (1, 1, 1)$ , we have

$$\dim_{\mathbb{k}} \overline{R}_\lambda = \dim_{\mathbb{k}} R_\lambda - \dim_{\mathbb{k}} R_{\lambda-\pi} \leq 1$$

by (38). Therefore  $\dim_{\mathbb{k}} \overline{R}_\lambda \leq 1$  for all  $\lambda \in X(G)$ . Moreover, assuming  $q$  to be a non-root of unity,  $\overline{R}$  satisfies the Nullstellensatz and the Dixmier–Mœglin equivalence. Indeed, the algebra  $\overline{R}$  is an image of quantum 4-space, since  $ad \equiv q^2 da \pmod{D_q}$ . Therefore, we know from Proposition 14 and Lemma 15 that there are finitely many  $G$ -primes of  $R$  that contain  $D_q$ . The remaining  $G$ -primes correspond to  $G\text{-Spec } \mathcal{O}_q(\text{GL}_2)$ , and by [7, Exer. II.2.M], there are only four of these.

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