

# KOSZUL ALGEBRAS AND THE QUANTUM MACMAHON MASTER THEOREM

PHÙNG HÔ HAI AND MARTIN LORENZ

## ABSTRACT

We give a new proof of the quantum version of MacMahon’s master theorem due to Garoufalidis, Lê and Zeilberger (in the one-parameter case) and to Konvalinka and Pak (in the multiparameter case) by deriving it from known facts about Koszul algebras.

## 1. Introduction

In [9], Garoufalidis, Lê and Zeilberger prove a quantum version of MacMahon’s celebrated ‘master theorem’ [17, pp. 97–98]. As stated in [9], the generalization was motivated in part by considerations in quantum topology and knot theory, and it also answers a long-standing open question by G. Andrews [1, Problem 5]. An abundance of different proofs of the original master theorem can be found in the literature; particularly noteworthy are Good’s short argument [10] using tools from analysis, and a more recent homological approach due to Huang [13] which relies on Grothendieck duality. The quantum generalization of the master theorem in [9] is proved by an application of the calculus of difference operators developed by Zeilberger in [20].

Our goal here is to derive the quantum MacMahon master theorem of Garoufalidis, Lê, and Zeilberger, along with its multiparameter extension, proved subsequently by Konvalinka and Pak [15], fairly effortlessly from basic properties of Koszul algebras. Indeed, the Koszul complex immediately leads to a generalized MacMahon identity that is valid for any (quadratic) Koszul algebra; it is stated as Corollary 4 below. The quantum MacMahon master theorem is the special case of Corollary 4, where the Koszul algebra in question is the so-called ‘quantum affine space’. Thus, neither the main result of this note nor the methods employed are ours, but we believe that the connection between Koszul algebras and the master theorem in its various incarnations deserves to be explicitly stated and fully exploited. The link to quantum affine space and quantum matrices has in fact already been briefly mentioned in the last section of [9], but the algebraic proof given here appears to be new.

After posting the original preprint of this paper on the web, we learned of independent work by Foata and Han [6–8] giving an alternative new proof of the quantum MacMahon master theorem using a combinatorial approach. In addition, these papers offer a detailed analysis of the algebra of right quantum matrices and other noncommutative versions of the master theorem. Further noncommutative extensions, notably a multiparameter formulation, were subsequently obtained via a different combinatorial method by Konvalinka and Pak [15]. Most recently, following the outline of the algebraic approach taken in the present article, Etingof and Pak [5] have proved a master theorem that is based on a certain generalized (nonquadratic) Koszul algebra defined by Berger [3]. An extension of the algebraic apparatus underlying the

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master theorem to generalized Koszul algebras in a  $\mathbb{Z}/2$ -graded setting is given in the preprint [12] by P. H. Hai, B. Kriegk and M. Lorenz.

The current version of this paper incorporates a number of additions to the bibliography and some expository improvements suggested by the referee. Furthermore, following a suggestion by Roland Berger and a comment in [15], we have made the small modifications required to cover the multiparameter case of the master theorem. In view of the interdisciplinary nature of its topic, we have tried to keep this note reasonably self-contained and accessible to readers unfamiliar with Koszul algebras. Sections 2 and 3 serve to deploy the pertinent background material concerning Koszul algebras and characters in some detail. The operative technicalities for our proof are collected in Lemmas 1 and 2 below; they are presented here with full proofs for lack of a suitable reference. The quantum MacMahon master theorem (see [9, Theorem 1] and [15, Theorem 1.2]) is then stated and proved in Section 4. The short final Section 5 discusses certain modifications of the MacMahon identity.

Our basic reference for Koszul algebras are Manin’s notes [18]; for bialgebras, our terminology follows Kassel [14]. We work over a commutative base field  $\mathbb{k}$  except in Sections 3.1 and 3.2, where  $\mathbb{k}$  can be any commutative ring at no extra cost. Throughout, ‘ $\otimes$ ’ will stand for ‘ $\otimes_{\mathbb{k}}$ ’.

## 2. Koszul algebras

### 2.1. Quadratic algebras

A *quadratic algebra* is a factor of the tensor algebra  $\mathbb{T}(V)$  of some finite-dimensional  $\mathbb{k}$ -vector space  $V$  modulo the ideal generated by some subspace  $R(A) \subseteq \mathbb{T}(V)_2 = V^{\otimes 2}$ . Thus,  $A \cong \mathbb{T}(V)/(R(A))$ . The natural grading of  $\mathbb{T}(V)$  descends to a grading  $A = \bigoplus_{d \geq 0} A_d$  of  $A$  with  $A_0 = \mathbb{k}$  and  $A_1 \cong V$ . In practice, one often fixes a  $\mathbb{k}$ -basis  $\tilde{x}_1, \dots, \tilde{x}_n$  of  $V$ . Then  $\mathbb{T}(V)$  can be viewed as the free algebra  $\mathbb{k}\langle \tilde{x}_1, \dots, \tilde{x}_n \rangle$ . The images  $x_i = \tilde{x}_i \pmod{R(A)}$  of the elements  $\tilde{x}_i$  in  $A$  form a set of algebra generators for  $A$ .

EXAMPLE (Quantum affine  $n$ -space). This is the quadratic algebra  $A = A_{\mathbf{q}}^{n|0}$  that is defined, for fixed scalars  $0 \neq q_{ij} \in \mathbb{k}$  ( $1 \leq i < j \leq n$ ), as the factor of  $\mathbb{k}\langle \tilde{x}_1, \dots, \tilde{x}_n \rangle$  modulo the ideal generated by the 2-homogeneous elements  $\tilde{x}_j \tilde{x}_i - q_{ij} \tilde{x}_i \tilde{x}_j$  for  $i < j$ . Thus, the algebra  $A_{\mathbf{q}}^{n|0}$  is generated by elements  $x_1, \dots, x_n$  satisfying the relations

$$x_j x_i = q_{ij} x_i x_j \quad (i < j). \tag{2.1}$$

### 2.2. Quadratic dual

Given a quadratic algebra

$$A \cong \mathbb{T}(V)/(R(A)),$$

consider the subspace  $R(A)^\perp$  of the linear dual  $(V^{\otimes 2})^*$  consisting of all linear forms on  $V^{\otimes 2}$  that vanish on  $R(A)$ . Identifying  $(V^{\otimes 2})^*$  with  $(V^*)^{\otimes 2}$  via  $\langle f \otimes f', v \otimes v' \rangle = \langle f, v \rangle \langle f', v' \rangle$  as in [4, II.4.4], we may view  $R(A)^\perp \subseteq (V^*)^{\otimes 2}$  and hence define a new quadratic algebra by

$$A^! = \mathbb{T}(V^*) / (R(A)^\perp).$$

The algebra  $A^!$  is called the *quadratic dual* of  $A$ . If  $\tilde{x}_1, \dots, \tilde{x}_n$  is a fixed  $\mathbb{k}$ -basis of  $V$ , as above, then one usually chooses the dual basis  $\tilde{x}^1, \dots, \tilde{x}^n$  for  $V^*$ :  $\langle \tilde{x}^i, \tilde{x}_j \rangle = \delta_{i,j}$  (Kronecker delta). This yields algebra generators  $x^i = \tilde{x}^i \pmod{R(A)^\perp}$  for  $A^!$ .

EXAMPLE (Quantum exterior algebra). Consider the algebra  $A = A_{\mathbf{q}}^{n|0}$  as above. Following [18], the quadratic dual  $A^!$  will be denoted by  $A_{\mathbf{q}}^{0|n}$ . The above procedure yields algebra

generators  $x^1, \dots, x^n$  for  $A_{\mathbf{q}}^{0|n}$  satisfying the defining relations

$$x^\ell x^\ell = 0 \quad \text{for all } \ell \tag{2.2}$$

and

$$x^k x^\ell + q_{k\ell} x^\ell x^k = 0 \quad (k < \ell). \tag{2.3}$$

### 2.3. The bialgebra $\underline{\text{end}} A$

Given quadratic algebras  $A \cong \mathbb{T}(A_1)/(R(A))$  and  $B \cong \mathbb{T}(B_1)/(R(B))$ , one defines the quadratic algebra

$$A \bullet B = \mathbb{T}(A_1 \otimes B_1)/(S_{23}(R(A) \otimes R(B))).$$

Here,  $S_{23}: A_1^{\otimes 2} \otimes B_1^{\otimes 2} \rightarrow (A_1 \otimes B_1)^{\otimes 2}$  switches the second and third factors.

The algebra  $\underline{\text{end}} A = A^1 \bullet A$  is particularly important. Identifying  $A_1$  with  $V$  as above, we have

$$\underline{\text{end}} A = \mathbb{T}(V^* \otimes V) / (S_{23}(R(A)^\perp \otimes R(A))).$$

Fix generators  $x_i = \tilde{x}_i \bmod R(A)$  for  $A$  and  $x^i = \tilde{x}^i \bmod R(A)^\perp$  for  $A^1$  as in Sections 2.1 and 2.2. Then the elements  $\tilde{z}_i^j = \tilde{x}^j \otimes \tilde{x}_i$  form a basis of  $V^* \otimes V$ , and their images  $z_i^j = \tilde{z}_i^j \bmod R(\underline{\text{end}} A)$  form algebra generators for  $\underline{\text{end}} A$ . The algebra  $\underline{\text{end}} A$  is endowed with a comultiplication

$$\Delta: \underline{\text{end}} A \longrightarrow \underline{\text{end}} A \otimes \underline{\text{end}} A, \quad \Delta(z_i^j) = \sum_\ell z_i^\ell \otimes z_\ell^j \tag{2.4}$$

and a counit

$$\varepsilon: \underline{\text{end}} A \longrightarrow \mathbb{k}, \quad \varepsilon(z_i^j) = \delta_{i,j},$$

which make  $\underline{\text{end}} A$  a bialgebra over  $\mathbb{k}$ ; see [18, 5.7 and 5.8]. Furthermore, if we define a coaction

$$\delta_A: A \longrightarrow \underline{\text{end}} A \otimes A, \quad \delta_A(x_i) = \sum_j z_i^j \otimes x_j, \tag{2.5}$$

the algebra  $A$  becomes a left  $\underline{\text{end}} A$ -comodule algebra:  $\delta_A$  is a  $\mathbb{k}$ -algebra map that makes  $A$  a comodule for  $\underline{\text{end}} A$ ; see [18, 5.4].

EXAMPLE (Right-quantum matrices). Returning to quantum affine space  $A = A_{\mathbf{q}}^{n|0}$ , we now describe the defining relations between the generators  $z_i^j$  ( $i, j = 1, \dots, n$ ) of the algebra  $\underline{\text{end}} A_{\mathbf{q}}^{n|0}$ . Using the notation and the relations of the examples in Sections 2.1 and 2.2, the foregoing leads to the following set of generators for  $R(\underline{\text{end}} A_{\mathbf{q}}^{n|0}) \subseteq (V^* \otimes V)^{\otimes 2}$ : for all  $\ell$  and  $i < j$  we have a generator  $\tilde{z}_j^\ell \otimes \tilde{z}_i^\ell - q_{ij} \tilde{z}_i^\ell \otimes \tilde{z}_j^\ell$ , and for all  $i < j$  and  $k < \ell$  there is

$$\tilde{z}_j^k \otimes \tilde{z}_i^\ell - q_{ij} \tilde{z}_i^k \otimes \tilde{z}_j^\ell + q_{k\ell} \tilde{z}_j^\ell \otimes \tilde{z}_i^k - q_{k\ell} q_{ij} \tilde{z}_i^\ell \otimes \tilde{z}_j^k.$$

Thus we obtain the following relations between the generators  $z_i^j$  of  $\underline{\text{end}} A_{\mathbf{q}}^{n|0}$ :

$$z_j^\ell z_i^\ell = q_{ij} z_i^\ell z_j^\ell \quad \text{for all } \ell \text{ and } i < j \tag{2.6}$$

and

$$q_{ij} z_i^k z_j^\ell - q_{k\ell} z_j^\ell z_i^k = z_j^k z_i^\ell - q_{k\ell} q_{ij} z_i^\ell z_j^k \quad \text{for } i < j \text{ and } k < \ell. \tag{2.7}$$

The relations (2.6) and (2.7) are called *column relations* and *cross relations*, respectively; they are identical to the relations considered in [9, Section 1.2] (for the one-parameter case) and in [15, Section 1.3]. Therefore, following the terminology of [9] and [15], we will call the  $n \times n$ -matrix  $Z = (z_i^j)$  a generic *right-quantum  $\mathbf{q}$ -matrix*. Right-quantum  $\mathbf{q}$ -matrices over an arbitrary  $\mathbb{k}$ -algebra  $R$  are exactly the matrices of the form  $\varphi Z$  for an algebra map  $\varphi: \underline{\text{end}} A_{\mathbf{q}}^{n|0} \rightarrow R$  (an ‘ $R$ -point’ of the space defined by  $\underline{\text{end}} A_{\mathbf{q}}^{n|0}$ ).

2.4. *The Koszul complex*

For any quadratic algebra  $A = \mathbb{T}(V)/(R(A))$ , one can define Koszul complexes

$$K^{\ell, \bullet}(A): 0 \longrightarrow A_{\ell}^! \longrightarrow A_{\ell-1}^! \otimes A_1 \longrightarrow \cdots \longrightarrow A_1^! \otimes A_{\ell-1} \longrightarrow A_{\ell} \longrightarrow 0$$

as in [18, 9.6]. The differentials of these complexes have the following description. Put

$$R_m(A) = (R(A)) \cap \mathbb{T}(V)_m = \sum_{i=0}^{m-2} V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-2-i};$$

so  $A_m = V^{\otimes m}/R_m(A)$ . Since  $R_m(A^!) = \sum_{i=0}^{m-2} V^* \otimes^i \otimes R(A)^{\perp} \otimes V^* \otimes^{m-2-i}$ , the linear dual of  $A_m^! = V^* \otimes^m / R_m(A^!)$  is

$$A_m^{!*} = \begin{cases} V^{\otimes m} & \text{if } m < 2, \\ \bigcap_{i=0}^{m-2} V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-2-i} & \text{if } m \geq 2. \end{cases} \tag{2.8}$$

According to [18, 9.6], the diagram

$$\begin{array}{ccc} A_{m+1}^{!*} \otimes V^{\otimes n-1} & \hookrightarrow & A_m^{!*} \otimes V^{\otimes n} \\ \cup & & \cup \\ A_{m+1}^{!*} \otimes R_{n-1}(A) & \hookrightarrow & A_m^{!*} \otimes R_n(A) \end{array}$$

induces a map

$$d: A_{m+1}^{!*} \otimes A_{n-1} = A_{m+1}^{!*} \otimes \frac{V^{\otimes n-1}}{R_{n-1}(A)} \longrightarrow A_m^{!*} \otimes \frac{V^{\otimes n}}{R_n(A)} = A_m^{!*} \otimes A_n.$$

One can easily check that  $d^2: A_{m+1}^{!*} \otimes A_{n-1} \rightarrow A_{m-1}^{!*} \otimes A_{n+1}$  is the zero map by noticing the following chain of inclusions:

$$\begin{aligned} A_{m+1}^{!*} \otimes V^{\otimes n-1} &= \bigcap_{i=0}^{m-1} V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-1-i+n-1} \\ &\quad \cap \\ &\left( \bigcap_{i=0}^{m-3} V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-3-i+n+1} \right) \cap V^{\otimes m-1} \otimes R(A) \otimes V^{\otimes n-1} \\ &\quad \cap \\ A_{m-1}^{!*} \otimes V^{\otimes n+1} \cap V^{\otimes m-1} \otimes R_{n+1}(A) &= A_{m-1}^{!*} \otimes R_{n+1}(A). \end{aligned}$$

In [16], the reader can find an alternative description of the Koszul differential which uses the coalgebra structure of the graded dual  $A^{!*} = \bigoplus_n A_n^{!*}$  obtained by dualizing the multiplication of  $A^!$ .

LEMMA 1. *Let  $A$  be a quadratic algebra. Then all  $A_i$  and all  $A_j^{!*}$  are (left) comodules over  $\underline{\text{end}} A$ , and hence so are the components  $K^{\ell, i}(A) = A_{\ell-i}^{!*} \otimes A_i$  of the Koszul complex. Moreover, the Koszul differential is an  $\underline{\text{end}} A$ -comodule map.*

*Proof.* We will write  $B = \underline{\text{end}} A$  for brevity; so  $B_1 = V^* \otimes V$ . Equation (2.5) shows that  $\delta_A$  sends  $V = A_1$  to  $B_1 \otimes V$ . Thus,  $A_i$  is mapped to  $B_i \otimes A_i \subseteq B \otimes A_i$ , and so each  $A_i$  is a  $B$ -comodule. Moreover, as is shown in [18, 5.5], the map  $\delta_A$  comes from a map

$$\tilde{\delta}: \mathbb{T}(V) \longrightarrow \mathbb{T}(B_1) \otimes \mathbb{T}(V), \quad \tilde{\delta}(\tilde{x}_i) = \sum_j \tilde{z}_i^j \otimes \tilde{x}_j,$$

which satisfies  $\tilde{\delta}(R(A)) \subseteq R(B) \otimes \mathbb{T}(V) + \mathbb{T}(B_1) \otimes R(A)$ . Following  $\tilde{\delta}$  by the canonical map  $\mathbb{T}(B_1) \twoheadrightarrow B = \mathbb{T}(B_1)/R(B)$  tensored with  $\text{Id}_{\mathbb{T}(V)}$ , we obtain a map

$$\delta' : \mathbb{T}(V) \longrightarrow B \otimes \mathbb{T}(V), \quad \delta'(\tilde{x}_i) = \sum_j z_i^j \otimes \tilde{x}_j \tag{2.9}$$

satisfying

$$\delta'(V^{\otimes i}) \subseteq B_i \otimes V^{\otimes i} \quad \text{and} \quad \delta'(R(A)) \subseteq B \otimes R(A).$$

Therefore, all  $V^{\otimes i}$  are  $B$ -comodules and  $R(A)$  is a  $B$ -subcomodule of  $V^{\otimes 2}$ . More generally, the subspaces  $V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-2-i}$  for  $0 \leq i \leq m-2$  are  $B$ -subcomodules of  $V^{\otimes m}$ , and hence so are the subspaces

$$R_m(A) \quad \text{and} \quad \bigcap_{i=0}^{m-2} V^{\otimes i} \otimes R(A) \otimes V^{\otimes m-2-i}.$$

Consequently,  $A_m$  and  $A_m^!*$  are  $B$ -comodules, and the differential  $d$  is a  $B$ -comodule map.  $\square$

### 2.5. Koszul algebras

The quadratic algebra  $A$  is called a *Koszul algebra* if all complexes  $K^{\ell,\bullet}(A)$  for  $\ell > 0$  are exact. It is known that if  $A$  and  $B$  are Koszul, then so are  $A^!$  and  $A \bullet B$ . Moreover, if  $A$  has a so-called PBW-basis consisting of certain standard monomials, then  $A$  is Koszul; see, for example, [19, Theorem 4.3.1]. This applies in particular to the quantum affine  $n$ -space  $A = A_{\mathbf{q}}^{n|0}$ , which is therefore Koszul [19, 4.2 Example 1].

## 3. Bialgebras and characters

### 3.1. The Grothendieck ring

For now, let  $B$  denote an arbitrary bialgebra over some commutative base ring  $\mathbb{k}$ . We let  $\mathcal{R}_B$  denote the Grothendieck ring of all (left)  $B$ -comodules that are finitely generated (f.g.) projective over  $\mathbb{k}$ . Thus, for each such  $B$ -comodule  $V$ , there is an element  $[V] \in \mathcal{R}_B$  and any short exact sequence  $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$  of  $B$ -comodules (f.g. projective over  $\mathbb{k}$ ) gives rise to an equation  $[V] = [U] + [W]$  in  $\mathcal{R}_B$ . Multiplication in  $\mathcal{R}_B$  is given by the tensor product of  $B$ -comodules; see [14, III.6].

### 3.2. Characters

Continuing with the notation of Section 3.1, let  $V$  be a  $B$ -comodule that is f.g. projective over  $\mathbb{k}$ . The structure map  $\delta_V : V \rightarrow B \otimes V$  is an element of  $\text{Hom}_{\mathbb{k}}(V, B \otimes V)$ . Using the standard isomorphisms  $\text{Hom}_{\mathbb{k}}(V, B \otimes V) \cong B \otimes V \otimes V^*$  (see, for example, [4, II.4.2]) and letting  $\langle \cdot, \cdot \rangle : V \otimes V^* \rightarrow \mathbb{k}$  denote the evaluation map, we have a homomorphism

$$\text{Hom}_{\mathbb{k}}(V, B \otimes V) \cong B \otimes V \otimes V^* \xrightarrow{\text{Id}_B \otimes \langle \cdot, \cdot \rangle} B \otimes \mathbb{k} \cong B.$$

The image of  $\delta_V$  under this map will be denoted by  $\chi_V$ . If  $V$  is free over  $\mathbb{k}$ , with basis  $\{v_i\}$ , and  $\delta_V(v_j) = \sum_i b_{i,j} \otimes v_i$ , then

$$\chi_V = \sum_i b_{i,i}. \tag{3.1}$$

LEMMA 2. *The map  $[V] \mapsto \chi_V$  yields a well-defined ring homomorphism  $\chi : \mathcal{R}_B \rightarrow B$ .*

*Proof.* The assertion is equivalent to the identities  $\chi_{V \otimes W} = \chi_V \chi_W$  for any two  $B$ -comodules  $V$  and  $W$  (f.g. projective over  $\mathbb{k}$ ) and  $\chi_V = \chi_U + \chi_W$  for any short exact sequence

$$0 \longrightarrow U \xrightarrow{\mu} V \xrightarrow{\pi} W \longrightarrow 0.$$

Both are easy to check; we sketch the proof of the second identity. Fix  $\mathbb{k}$ -linear splittings  $\sigma: W \rightarrow V$  and  $\tau: V \rightarrow U$  with  $\tau \circ \mu = \text{Id}_U$ ,  $\pi \circ \sigma = \text{Id}_W$  and  $\sigma \circ \pi + \mu \circ \tau = \text{Id}_V$ . Under the canonical isomorphism  $V \otimes V^* \cong \text{End}_{\mathbb{k}}(V)$ , evaluation  $\langle \cdot, \cdot \rangle$  becomes the trace map  $\text{trace}_V: \text{End}_{\mathbb{k}}(V) \rightarrow \mathbb{k}$ . For any  $\varphi \in \text{End}_{\mathbb{k}}(V)$ , we have  $\text{trace}_V(\varphi) = \text{trace}_U(\tau \circ \varphi \circ \mu) + \text{trace}_W(\pi \circ \varphi \circ \sigma)$ . The equation  $\delta_V \circ \mu = (\text{Id}_B \otimes \mu) \circ \delta_U$  gives  $\delta_U = (\text{Id}_B \otimes \tau) \circ \delta_V \circ \mu$ . Similarly,  $\delta_W = (\text{Id}_B \otimes \pi) \circ \delta_V \circ \sigma$ . Therefore,

$$\chi_V = (\text{Id}_B \otimes \text{trace}_V)(\delta_V) = (\text{Id}_B \otimes \text{trace}_U)(\delta_U) + (\text{Id}_B \otimes \text{trace}_W)(\delta_W) = \chi_U + \chi_W,$$

as desired. □

### 3.3. Application to Koszul algebras

Returning to the case of a Koszul algebra  $A$  over a field  $\mathbb{k}$ , we apply the foregoing to the bialgebra  $B = \underline{\text{end}} A$ . By Lemma 1, each of the (exact) Koszul complexes  $\mathbb{K}^{\ell, \bullet}(A)$  for  $\ell > 0$  gives an equation in the Grothendieck ring  $\mathcal{R}_B$ :

$$\sum_i (-1)^i [A_i^! *] [A_{\ell-i}] = 0.$$

Defining the Poincaré series  $P_A(t) = \sum_i [A_i] t^i$  and  $P_{A^!}(t) = \sum_i [A_i^! *] t^i$  in the power series ring  $\mathcal{R}_B[[t]]$ , these equations are equivalent to the following duality formula.

PROPOSITION 3. *For any Koszul algebra  $A$ , the identity*

$$P_A(t) P_{A^!}(-t) = 1$$

*holds in  $\mathcal{R}_B[[t]]$ , where  $B = \underline{\text{end}} A$ .*

Applying the ring homomorphism  $\chi[[t]]: \mathcal{R}_B[[t]] \rightarrow B[[t]]$  (Lemma 2), the duality formula in Proposition 3 takes the following form in  $B[[t]]$ .

COROLLARY 4 (Koszul MMT).  $(\sum_{\ell \geq 0} \chi_{A_\ell} t^\ell) \cdot (\sum_{m \geq 0} (-1)^m \chi_{A_m^!} t^m) = 1.$

Since the coactions  $\delta_{A_\ell}$  and  $\delta_{A_m^!}$  in (2.5) and (2.9) respect the grading, both factors actually belong to the subring  $\prod_{n \geq 0} B_n t^n$  of  $B[[t]]$ .

### 4. Proof of the quantum MacMahon master theorem

The quantum MacMahon master theorem (see [9, Theorem 1] and [15, Theorem 1.2]) is the special case of Corollary 4 where  $A$  is quantum affine space. We need to evaluate the characters  $\chi_{A_\ell}$  and  $\chi_{A_m^!}$  for  $A = A_{\mathbf{q}}^{n|0}$ .

Choose generators  $x_1, \dots, x_n$  for  $A$  as in Section 2.1. For each  $n$ -tuple  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$ , put

$$x^{\mathbf{m}} := x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \in A.$$

Then the homogeneous component  $A_\ell$  has a  $\mathbb{k}$ -basis consisting of the elements  $x^{\mathbf{m}}$  with  $|\mathbf{m}| = m_1 + m_2 + \dots + m_n = \ell$ . (This is the PBW-basis of  $A_{\mathbf{q}}^{n|0}$  that was referred to earlier.) With

respect to this basis, the coaction  $\delta_A$  of  $B = \underline{\text{end}} A$  on  $A_\ell$  in (2.5) has the form

$$\delta_A(x^{\mathbf{m}}) = \delta_A(x_1)^{m_1} \delta_A(x_2)^{m_2} \cdots \delta_A(x_n)^{m_n} = \sum_{\substack{\mathbf{r} \in \mathbb{Z}_{\geq 0}^n \\ |\mathbf{r}| = \ell}} b_{\mathbf{r}, \mathbf{m}} \otimes x^{\mathbf{r}}$$

for uniquely determined  $b_{\mathbf{r}, \mathbf{m}} \in B_\ell$ . In particular,  $G(\mathbf{m}) := b_{\mathbf{m}, \mathbf{m}} \in B_{|\mathbf{m}|}$  has the same meaning as in [9] and [15]. From equation (3.1) we obtain the formula

$$\chi_{A_\ell} = \sum_{\mathbf{m}: |\mathbf{m}| = \ell} G(\mathbf{m}). \tag{4.1}$$

It remains to calculate the character  $\chi_{A_m^!}$ . To this end, we identify  $A_m^!*$  with the subspace  $R_m(A^!)^\perp$  of  $\mathbb{T}(V)_m = V^{\otimes m}$  as in (2.8), and we think of  $\mathbb{T}(V)$  as the free algebra  $\mathbb{k}\langle \tilde{x}_1, \dots, \tilde{x}_n \rangle$  as in Section 2.1. Then  $A_m^!*$  has a  $\mathbb{k}$ -basis consisting of the elements

$$\wedge \tilde{x}_J := \sum_{\pi \in \mathfrak{S}_m} w(\pi) \tilde{x}_{j_{\pi 1}} \otimes \tilde{x}_{j_{\pi 2}} \otimes \cdots \otimes \tilde{x}_{j_{\pi m}},$$

where  $J = (j_1 < j_2 < \dots < j_m)$  is an  $m$ -element subset of  $\{1, \dots, n\}$ ,  $\mathfrak{S}_m$  is the symmetric group on  $\{1, \dots, m\}$  and

$$w(\pi) = \prod_{i < j, \pi i > \pi j} (-q_{\pi j, \pi i})^{-1}$$

as in [15]. Indeed, using the generators of  $R(A^!)$  exhibited in (2.2) and (2.3), it is straightforward to check that  $\wedge \tilde{x}_J$  vanishes on  $R_m(A^!) = R(A^!) \cap \mathbb{T}(V^*)_m$ . Hence,  $\wedge \tilde{x}_J$  belongs to  $A_m^!*$ . Furthermore, the elements  $\wedge \tilde{x}_J$  for distinct  $m$ -element subsets  $J \subseteq \{1, \dots, n\}$  are obviously  $\mathbb{k}$ -linearly independent and their number is  $\binom{n}{m}$ , which is equal to the dimension of  $A_m^!*$ . Therefore, we obtain the claimed basis of  $A_m^!*$ .

Consider the basis vector  $\wedge \tilde{x}_{[n]}$  corresponding to the subset  $J = \{1, 2, \dots, n\} =: [n]$ . The coaction  $\delta'$  of  $B = \underline{\text{end}} A$  on  $A_n^!*$  in equation (2.9) satisfies

$$\delta'(\wedge \tilde{x}_{[n]}) = \det_{\mathbf{q}}(Z) \otimes \wedge \tilde{x}_{[n]}, \tag{4.2}$$

where  $Z = (z_i^j)$  is the generic  $n \times n$  right-quantum  $\mathbf{q}$ -matrix as in Section 2.3 and

$$\det_{\mathbf{q}}(Z) = \sum_{\pi \in \mathfrak{S}_n} w(\pi) z_{\pi 1}^1 z_{\pi 2}^2 \cdots z_{\pi n}^n \tag{4.3}$$

denotes the multiparameter quantum determinant of  $Z$  as defined in [2] (see also [15]). To prove (4.2), note that the element  $\wedge \tilde{x}_{[n]}$  spans the one-dimensional  $B$ -subcomodule  $A_n^!*$  of  $V^{\otimes n}$ . Hence, we certainly have

$$\delta'(\wedge \tilde{x}_{[n]}) = D \otimes \wedge \tilde{x}_{[n]} \tag{4.4}$$

for some group-like element  $D \in B$ ; cf. [18, 8.2]. We need to show that  $D = \det_{\mathbf{q}}(Z)$ . But equation (2.9) readily implies that

$$\delta'(\tilde{x}_{\pi 1} \otimes \cdots \otimes \tilde{x}_{\pi n}) = z_{\pi 1}^1 \cdots z_{\pi n}^n \otimes \tilde{x}_1 \otimes \cdots \otimes \tilde{x}_n + \text{other terms},$$

where each of the ‘other terms’ belongs to some  $B \otimes \tilde{x}_{i_1} \otimes \cdots \otimes \tilde{x}_{i_n}$  with  $(i_1, \dots, i_n) \neq (1, \dots, n)$ . Multiplying by  $w(\pi)$  and summing over  $\pi$ , we see that

$$\delta'(\wedge \tilde{x}_{[n]}) = \det_{\mathbf{q}}(Z) \otimes \tilde{x}_1 \otimes \cdots \otimes \tilde{x}_n + \text{other terms}.$$

In view of (4.4), this implies that (4.2) holds.

For a general  $m$ -element subset  $J = (j_1 < j_2 < \dots < j_m) \subseteq \{1, \dots, n\}$ , let  $Z_J$  denote the submatrix  $(z_i^j)_{i, j \in J}$  of  $Z$ ; this is also a generic right-quantum matrix. Equation (4.2) implies that

$$\delta'(\wedge \tilde{x}_J) = \det_{\mathbf{q}}(Z_J) \otimes \wedge \tilde{x}_J + \rho,$$

where  $\rho \in \sum_{I \neq J} B \otimes \wedge \tilde{x}_I$  and

$$\det_{\mathbf{q}}(Z_J) = \sum_{\pi \in \mathfrak{S}_m} w(\pi) z_{j_{\pi 1}}^{j_1} z_{j_{\pi 2}}^{j_2} \dots z_{j_{\pi m}}^{j_m} \tag{4.5}$$

in accordance with (4.3). By (3.1), the character of  $A_m^*$  is therefore given by

$$\chi_{A_m^*} = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=m}} \det_{\mathbf{q}}(Z_J). \tag{4.6}$$

To summarize, we rephrase Corollary 4 with (4.1) and (4.6) using the notation of [9].

**THEOREM 5 [9, 15].** *Let  $A$  denote the  $\mathbb{k}$ -algebra generated by  $x_1, \dots, x_n$  subject to the relations (2.1), let  $B$  be the  $\mathbb{k}$ -algebra generated by  $z_1^1, \dots, z_n^n$  subject to (2.6) and (2.7), and let  $Z = (z_i^j)_{n \times n}$  denote the generic right-quantum  $\mathbf{q}$ -matrix. Consider the elements*

$$X_i = \sum_j z_i^j \otimes x_j \in B \otimes A = \bigoplus_{\mathbf{m}} B \otimes x^{\mathbf{m}},$$

where  $\mathbf{m}$  runs over the  $n$ -tuples  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$  and  $x^{\mathbf{m}} := x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$ . Define power series in  $B[[t]]$  by

$$\text{Bos}(Z) := \sum_{\ell \geq 0} \sum_{\mathbf{m}: |\mathbf{m}|=\ell} G(\mathbf{m}) t^\ell,$$

where  $G(\mathbf{m})$  is the  $B$ -coefficient of  $x^{\mathbf{m}}$  in  $X^{\mathbf{m}} = X_1^{m_1} X_2^{m_2} \dots X_n^{m_n}$ , and

$$\text{Ferm}(Z) := \sum_{m \geq 0} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=m}} (-1)^m \det_{\mathbf{q}}(Z_J) t^m$$

with  $\det_{\mathbf{q}}(Z_J)$  as in (4.5). Then:

$$\text{Bos}(Z) \cdot \text{Ferm}(Z) = 1.$$

### 5. Modifying the MacMahon identity

Applying endomorphisms of  $B[[t]]$  to the Koszul MacMahon identity in Corollary 4, we obtain new versions of this identity. In this section, we discuss a particular example for the case  $A = A_{\mathbf{q}}^{n|0}$ . As usual, we let  $B = \underline{\text{end}} A$ .

The algebraic torus  $(T \times T)/\mathbb{k}^*$ , with  $T = (\mathbb{k}^*)^n$  and  $\mathbb{k}^*$  diagonally embedded in  $T \times T$ , acts on  $B$  (and hence on  $B[[t]]$ ) via

$$(\tau, \tau')(z_i^j) = c_i d_j^{-1} z_i^j$$

for  $\tau = (c_1, \dots, c_n), \tau' = (d_1, \dots, d_n) \in T$ . Indeed,  $(\tau, \tau')$  respects the relations (2.6) and (2.7) of  $B$ , and hence  $(\tau, \tau')$  defines a graded  $\mathbb{k}$ -algebra automorphism of  $B$ . Note that the diagonal subgroup  $\{(\tau, \tau) \mid \tau \in T\}$  acts by bialgebra automorphisms while the subgroups  $T \times \{1\}$  and  $\{1\} \times T$  act by left and right  $B$ -comodule automorphisms, respectively. For any subset  $J \subseteq \{1, \dots, n\}$ , we have

$$(\tau, \tau')(\det_{\mathbf{q}}(Z_J)) = \mu_J(\tau \tau'^{-1}) \det_{\mathbf{q}}(Z_J) \quad \text{with } \mu_J(c_1, \dots, c_n) = \prod_{j \in J} c_j,$$

and for  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$ ,

$$(\tau, \tau')(G(\mathbf{m})) = \mu_{\mathbf{m}}(\tau \tau'^{-1}) G(\mathbf{m}) \quad \text{with } \mu_{\mathbf{m}}(c_1, \dots, c_n) = \prod_i c_i^{m_i}.$$



Therefore, applying  $(\tau, \tau')$  to the characters of  $A_\ell$  and  $A_m^{!*}$  as determined in (4.1) and (4.6), we obtain

$$(\tau, \tau')(\chi_{A_m^{!*}}) = \sum_{\substack{J \subseteq \{1, \dots, n\} \\ |J|=m}} \mu_J(\tau\tau'^{-1}) \det_{\mathbf{q}}(Z_J); \tag{5.1}$$

$$(\tau, \tau')(\chi_{A_\ell}) = \sum_{|\mathbf{m}|=\ell} \mu_{\mathbf{m}}(\tau\tau'^{-1}) G(\mathbf{m}). \tag{5.2}$$

The particular choice

$$\tau = (q^{n-1}, q^{n-3}, \dots, q^{1-n}) \tag{5.3}$$

and  $\tau' = 1$  leads to the following version of the quantum MacMahon master theorem:

$$\widetilde{\text{Bos}}(Z) \cdot \widetilde{\text{Ferm}}(Z) = 1, \tag{5.4}$$

where

$$\widetilde{\text{Bos}}(Z) := \sum_{\ell \geq 0} \sum_{\mathbf{m}: |\mathbf{m}|=\ell} q^{\ell(n+1)-2 \sum im_i} G(\mathbf{m}) t^\ell; \tag{5.5}$$

$$\widetilde{\text{Ferm}}(Z) := \sum_{m \geq 0} \sum_{J=(j_1 < \dots < j_m)} (-1)^m q^{m(n+1)-2(j_1+j_2+\dots+j_m)} \det_{\mathbf{q}}(Z_J) t^m. \tag{5.6}$$

REMARK. Let  $H := \underline{\text{gl}}(A)$  be the coordinate ring on the quantum general linear group (cf. [18, 8.5]). Then  $H$  is a Hopf algebra which is in fact coquasitriangular or cobraided (cf. [14, VIII.5]). Thus, for any finite-dimensional comodule  $X$ , there exists a canonical isomorphism  $X \rightarrow X^{**}$  of  $H$ -comodules, given in terms of the braiding. This isomorphism is in general not compatible with the tensor product. We note that the category of  $H$ -comodules also possesses a ribbon [14, XIV.6]. By composing the above canonical isomorphism with the ribbon, one obtains a functorial isomorphism  $\tau_X : X \rightarrow X^{**}$  which is compatible with the tensor product in the sense that

$$\tau_{X \otimes Y} = \tau_X \otimes \tau_Y.$$

Using  $\tau_X$  we can define a new type of character of  $X$ , called a quantum character, as follows (cf. [11]).  $\chi_{q,X}$  is the image of  $1 \in \mathbb{k}$  under the map

$$\mathbb{k} \xrightarrow{\text{db}} X^* \otimes X \xrightarrow{\text{Id} \otimes \tau_X} X^* \otimes X^{**} \longrightarrow H \otimes X^* \otimes X^{**} \xrightarrow{\text{ev}} H$$

where db is the ‘dual base’ map  $1 \mapsto \sum_i x^i \otimes x_i$ . As for the ordinary character, one can show that the quantum character is multiplicative with respect to the tensor product, and additive with respect to exact sequences. Applying the quantum character to the Koszul complex in Section 2.4, we obtain an identity analogous to Corollary 4 for  $\chi_{q,A_\ell}$  and  $\chi_{q,A_m^{!*}}$ .

Explicit computation shows that, for  $X = V = A_1$ ,

$$\tau_V = \text{diag}(q^{n-1}, q^{n-3}, \dots, q^{1-n}).$$

Further, one can check that

$$\chi_{q,A_\ell} = \tau(\chi_{A_\ell}); \quad \chi_{q,A_m^{!*}} = \tau(\chi_{A_m^{!*}}).$$

This explains the origin of the choice of  $\tau$  in (5.3).

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Phùng Hồ Hai  
 Mathematik  
 University of Duisburg-Essen  
 Germany  
 and  
 Institute of Mathematics  
 Hanoi  
 Vietnam

[hai.phung@uni-duisburg-essen.de](mailto:hai.phung@uni-duisburg-essen.de)

Martin Lorenz  
 Department of Mathematics  
 Temple University  
 Philadelphia, PA 19122-6094  
 USA

[lorenz@temple.edu](mailto:lorenz@temple.edu)