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On Euler classes of abelian-by-finite groups

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Abstract. Let Γ be a finitely generated abelian-by-finite group and k a field of characteristic $p \geqslant 0$. We show that the Euler class of Γ over k has finite order if and only if every p-regular element of Γ has infinite centralizer in Γ. We also give a lower bound for the order of the Euler class in terms of suitable finite subgroups of Γ. This lower bound is derived from a more general result on finite-dimensional representations of smash products of Hopf algebras.

Introduction

The Euler class of a group Γ over a commutative ring k is defined, under suitable hypotheses, as the class of the trivial $k\Gamma$-module k_Γ in the Grothendieck group $G_0(k\Gamma)$. Here, $k\Gamma$ denotes the group ring of Γ over k and k_Γ equals k, with every element of Γ acting as the identity. The Grothendieck group $G_0(k\Gamma)$ is K_0 of the category of $k\Gamma$-modules of type FP_∞, which have finite projective dimension as a k-module. Recall that a module is said to be of *type* FP_∞ if it has a resolution, possibly of infinite length, by finitely generated projective modules. If the trivial $k\Gamma$-module k_Γ has such a resolution, one can define the Euler class $[k_\Gamma] \in G_0(k\Gamma)$.

Euler classes are traditionally considered under the stronger hypothesis that k_Γ be of *type* FP, that is, it admits a resolution of *finite length* by finitely generated projectives over $k\Gamma$. This entails that $G_0(k\Gamma) \simeq K_0(k\Gamma)$, the Grothendieck group of the category of finitely generated projective $k\Gamma$-modules; so one can view $[k_\Gamma] \in K_0(k\Gamma)$. For our purposes, however, this setting is too restrictive. Indeed, we study Euler classes in order to learn more about $G_0(k\Gamma)$, especially its torsion, for certain groups Γ. In this note, we concentrate on finitely generated abelian-by-finite groups and we shall work over a base field k. In particular, the group algebra $k\Gamma$ will be noetherian, and hence $k\Gamma$-modules of type FP_∞ coincide with finitely generated $k\Gamma$-modules. Our main result on Euler characteristics is the following

Theorem. *Let Γ be a finitely generated abelian-by-finite group and k a field of characteristic $p \geqslant 0$. Then*

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(a) *the Euler class* $[k_\Gamma]$ *has finite order if and only if every p-regular element of* Γ *has infinite centralizer in* Γ.

(b) *Assume that* $p > 0$ *and let G be a finite p-subgroup of* Γ. *If every* $g \in G \backslash \{1\}$ *has finite centralizer in* Γ *then the order of G divides the order of* $[k_\Gamma]$.

Recall that *p-regular elements* are elements of finite order not divisible by p; so 0-regular just means torsion. In (b), it is understood that every integer divides ∞. The restriction to positive characteristics p and p-groups G is justified by the fact that otherwise the order of $[k_\Gamma]$ would be infinite for $G \neq \{1\}$, by (a). For $k = \mathbb{Q}$, assertion (a) follows from results of Brown [4] and Moody [14], even for polycyclic-by-finite groups Γ; this has been observed by Kropholler and Moselle [9]. Arbitrary base fields k appear to require a different approach.

The proof of (a), given in Section 1, depends on techniques from [6], specifically a base change map $G_0(k\Gamma) \to G_0(k\overline{\Gamma})$ into a carefully selected finite quotient $\overline{\Gamma}$ of Γ. Part (b), on the other hand, is an immediate application of a more general result on finite-dimensional representations of smash products of Hopf algebras. This result, Corollary 3.3, is proved using a construction from [12]. We have recast this construction, called *stable restriction* here, in the language of stable module categories.

As is obvious from the foregoing, our approach to Euler classes is resolutely algebraic, due in part to our ulterior interest in the structure of $G_0(k\Gamma)$. For a recent article on Euler classes from a more topological perspective providing some background from homological group theory and topology, I recommend [10]. Finally, much of the material presented here can presumably be pushed to polycyclic-by-finite groups at least. A more pressing issue, however, is a sharpening of part (b) of the Theorem. To this end, it might be helpful to look at relative versions of the stable restriction maps considered in this note.

Notations and conventions. Throughout, Γ will denote a finitely generated abelian-by-finite group and k will be a commutative field of characteristic $p \geqslant 0$. Our notation concerning the Grothendieck group G_0 follows [2].

1 Proof of the Theorem, part (a)

Let A be any torsion-free abelian normal subgroup of Γ having finite index in Γ and let $\mathbf{C}_\Gamma(A)$ denote the centralizer of A in Γ. Consider the canonical map $^- : \Gamma \to \overline{\Gamma} = \Gamma / \mathbf{C}_\Gamma(A)$. We claim:

> every p-regular element of Γ has infinite centralizer in Γ if and only if every p-regular element of $\overline{\Gamma}$ has a non-trivial fixed point in A.

To see this, first note that the centralizer $\mathbf{C}_\Gamma(g)$ of any $g \in \Gamma$ is infinite if and only if $\bar{g}$ has a non-trivial fixed point in A. Since $^-$ sends p-regular elements of Γ to p-regular elements of $\overline{\Gamma}$, the condition on $\overline{\Gamma}$ is certainly sufficient. For the converse, let $x \in \overline{\Gamma}$ be a p-regular element. Fix $g \in \Gamma$ so that $\bar{g} = x$. If g has infinite order then $\mathbf{C}_\Gamma(g)$ is certainly infinite and so x has a non-trivial fixed point in A. If, on the other hand, g

has finite order then we may write $g = g_p g_{p'}$ with g_p a p-element ($=1$ if $p = 0$), $g_{p'}$ p-regular, and $g_p g_{p'} = g_{p'} g_p$. Inasmuch as $x = \overline{g_p}\,\overline{g_{p'}}$ is p-regular, we must have $\overline{g_p} = 1$. Thus $x = \overline{g_{p'}}$ and since $g_{p'}$ has infinite centralizer, x has a non-trivial fixed point.

We may choose A above at our convenience. In particular, by [6, Lemma 1.7 and proof of Proposition 1.8], we may assume that the base change map

$$G_0(k\Gamma) \to G_0(k[\Gamma/A]), \qquad [V] \mapsto \sum_{i \geqslant 0} (-1)^i [H_i(A, V)]$$

has kernel the torsion subgroup of $G_0(k\Gamma)$. Thus $[k_\Gamma]$ has finite order if and only if $\sum_{i \geqslant 0} (-1)^i [H_i(A, k)] = 0$ holds in $G_0(k[\Gamma/A])$. Since $\mathbf{C}_\Gamma(A)$ acts trivially on each $H_i(A, k)$, the element $[H_i(A, k)] \in G_0(k[\Gamma/A])$ actually belongs to the image of the inflation monomorphism $G_0(k\overline{\Gamma}) \hookrightarrow G_0(k[\Gamma/A])$. Hence

$[k_\Gamma]$ has finite order if and only if $\alpha_k := \sum_{i \geqslant 0} (-1)^i [H_i(A, k)] = 0$ in $G_0(k\overline{\Gamma})$.

Now, as $k[\overline{\Gamma}]$-modules,

$$H_i(A, k) \simeq \Big(\bigwedge^i A\Big) \otimes_{\mathbb{Z}} k,$$

where $\bigwedge^i A$ denote the ith exterior power of A; see [5, Theorem V(6.4)]. Put

$$\alpha_{\mathbb{Q}} = \sum_{i \geqslant 0} (-1)^i \Big[\bigwedge^i A \otimes \mathbb{Q}\Big] \in G_0(\mathbb{Q}\overline{\Gamma}).$$

Then $\alpha_k = d(\alpha_{\mathbb{Q}})$, where $d : G_0(\mathbb{Q}\overline{\Gamma}) \to G_0(k\overline{\Gamma})$ is the scalar extension map $(\cdot) \otimes_{\mathbb{Q}} k$ if $p = 0$, and the decomposition map $G_0(\mathbb{Q}\overline{\Gamma}) \to G_0(\mathbb{F}_p\overline{\Gamma})$ followed by scalar extension $(\cdot) \otimes_{\mathbb{F}_p} k$ if $p > 0$. Moreover, $\alpha_{\mathbb{Q}}$ has character

$$\chi_{\alpha_{\mathbb{Q}}}(x) = \det(1 - x_A) \qquad (x \in \overline{\Gamma}),$$

where $x_A \in \mathrm{GL}(A)$ denotes the action of x on A. Indeed, the characteristic polynomial of any $f \in \mathrm{End}(A)$ is given by

$$\det(X\,\mathrm{Id}_A - f) = \sum_i (-1)^i \operatorname{trace}\Big(\bigwedge^i f\Big) X^{n-i}.$$

Identifying elements of $G_0(k\overline{\Gamma})$ with (certain) complex-valued functions on the set $\overline{\Gamma}_{p'}$ of p-regular elements of $\overline{\Gamma}$ by means of (Brauer) characters, the element $\alpha_k \in G_0(k\overline{\Gamma})$ is simply the restriction of $\chi_{\alpha_{\mathbb{Q}}}$ from $\overline{\Gamma}$ to $\overline{\Gamma}_{p'}$; cf. [16, Section 18.3]. Thus $\alpha_k = 0$ if and only if $\det(1 - x_A) = 0$ holds for every p-regular element $x \in \overline{\Gamma}$. Since the latter condition is equivalent with x having a non-trivial fixed point in A, the proof of part (a) of the Theorem is complete.

2 Stably finitely generated modules

Throughout this section, S will denote a QF-ring, that is, a ring whose projective and injective modules coincide; cf. [1, Theorem 31.9].

2.1 Stable module categories. We briefly review some pertinent facts concerning stable module categories; see [8] for details.

Let $\mathsf{Mod}(S)$ denote the category of all left S-modules, $\mathsf{mod}(S)$ the full subcategory of finitely generated modules, and let $\mathsf{StMod}(S)$ and $\mathsf{Stmod}(S)$ denote the corresponding stable module categories: the objects of $\mathsf{StMod}(S)$ and $\mathsf{Stmod}(S)$ are the same as those of $\mathsf{Mod}(S)$ and $\mathsf{mod}(S)$, respectively, but morphisms are equivalence classes of S-module homomorphisms, where two homomorphisms $\alpha, \beta : M \to N$ are called equivalent if $\alpha - \beta$ factors through a projective module. Thus the set of morphisms from M to N in $\mathsf{StMod}(S)$ is the abelian group

$$\underline{\mathrm{Hom}}_S(M,N) := \mathrm{Hom}_S(M,N)/\mathrm{PHom}_S(M,N),$$

where $\mathrm{Hom}_S(M,N)$ is the group of S-module homomorphisms and $\mathrm{PHom}_S(M,N)$ is the subgroup of homomorphisms that factor through a projective module. We will write $\underline{\alpha}$ for the equivalence class of a homomorphism α, and $\underline{M}$ when explicitly viewing the module M in the stable category. Thus $\underline{M} \simeq \underline{N}$ in $\mathsf{StMod}(S)$ if and only if $M \oplus P \simeq N \oplus Q$ holds in $\mathsf{Mod}(S)$ with suitable projectives P and Q. The categories $\mathsf{StMod}(S)$ and $\mathsf{Stmod}(S)$, while no longer abelian, are at least triangulated. In particular, one can define the Grothendieck group

$$\underline{G_0}(S) := K_0(\mathsf{Stmod}(S))$$

as in [8, p. 95]: each triangle $\underline{U} \xrightarrow{u} \underline{V} \xrightarrow{v} \underline{W} \xrightarrow{w}$ in $\mathsf{Stmod}(S)$ yields an equation $[\underline{V}] = [\underline{U}] + [\underline{W}]$ in $\underline{G_0}(S)$. It is not hard to show that

$$\underline{G_0}(S) \simeq G_0(S)/c(K_0(S)),$$

where $c : K_0(S) \to G_0(S)$ is the Cartan homomorphism; see [18, Proposition 1].

2.2 Stably finitely generated modules. We will call an S-module M *stably finitely generated* if $\underline{M}$ is isomorphic in $\mathsf{StMod}(S)$ to a finitely generated module, say M'. In this case, we put

$$\theta(M) := [\underline{M'}] \in \underline{G_0}(S).$$

We remark that stably finitely generated modules were called almost injective in [12]. The following lemma is identical with [12, Theorem 1.2]; we give a new proof in the framework of triangulated categories following [8, Chapter 1] for notation, terminology and axioms.

Lemma. *Let* $0 \to U \to V \to W \to 0$ *be an exact sequence in* $\mathsf{Mod}(S)$. *If two of* $\{U, V, W\}$ *are stably finitely generated then all three are. In this case,*

$$\theta(V) = \theta(U) + \theta(W)$$

holds in $\underline{G_0}(S)$.

Proof. The given exact sequence yields a triangle

$$\underline{U} \xrightarrow{u} \underline{V} \xrightarrow{v} \underline{W} \xrightarrow{w}$$

in $\mathsf{StMod}(S)$. Therefore, in order to prove the first assertion, it suffices to prove the following: given a triangle $\Delta = (\underline{U} \xrightarrow{u} \underline{V} \xrightarrow{v} \underline{W} \xrightarrow{w})$ in $\mathsf{StMod}(S)$ such that two of $\{\underline{U}, \underline{V}, \underline{W}\}$ are isomorphic to objects of $\mathsf{Stmod}(S)$, then all three are. Moreover, by the rotation axiom (TR2), it suffices to consider the case where the two modules in question are $\underline{U}$ and $\underline{V}$, say $f : \underline{U} \xrightarrow{\simeq} \underline{U'}$ and $g : \underline{V} \xrightarrow{\simeq} \underline{V'}$ are isomorphisms in $\mathsf{StMod}(S)$ with U' and V' finitely generated. By axiom (TR1) for $\mathsf{Stmod}(S)$, the morphism $u' = g \circ u \circ f^{-1} : \underline{U'} \to \underline{V'}$ in $\mathsf{Stmod}(S)$ embeds into a triangle $\Delta' = (\underline{U'} \xrightarrow{u'} \underline{V'} \xrightarrow{v'} \underline{W'} \xrightarrow{w'})$ in $\mathsf{Stmod}(S)$ (and hence in $\mathsf{StMod}(S)$). So W' is finitely generated. By axiom (TR3) for $\mathsf{StMod}(S)$, there is a morphism $h : \underline{W} \to \underline{W'}$ so that $(f, g, h) : \Delta \to \Delta'$ is a morphism of triangles in $\mathsf{StMod}(S)$. Finally, the 5-Lemma [8, Proposition 1.2(c)] implies that h is an isomorphism, proving that W is stably finitely generated.

Finally, start with the given exact sequence and its associated triangle Δ in $\mathsf{StMod}(S)$, and assume that there are isomorphisms

$$f : \underline{U} \xrightarrow{\simeq} \underline{U'}, \quad g : \underline{V} \xrightarrow{\simeq} \underline{V'}, \quad h : \underline{W} \xrightarrow{\simeq} \underline{W'}$$

in $\mathsf{StMod}(S)$ with U', V', W' finitely generated. Then

$$\Delta' = (\underline{U'}, \underline{V'}, \underline{W'}, g \circ u \circ f^{-1}, h \circ u \circ g^{-1}, Tf \circ u \circ h^{-1})$$

is a sextuple in $\mathsf{StMod}(S)$ that is isomorphic to Δ via (f, g, h). By (Tr1), Δ' is a triangle in $\mathsf{StMod}(S)$, and hence in $\mathsf{Stmod}(S)$. From this triangle, we obtain the desired equation $[\underline{V'}] = [\underline{U'}] + [\underline{W'}]$ in $\underline{G_0}(S)$.

2.3 QF-algebras. Assume now that the QF-ring S is a finite-dimensional algebra over a field k. Then we have the following characterization of stably finitely generated S-modules.

Lemma. *The following are equivalent for an S-module M*:

(i) *M is stably finitely generated*;

(ii) *for all finitely generated S-modules V, $\underline{\mathrm{Hom}}_S(V, M)$ is finite-dimensional over k*;

(iii) *for all simple S-modules V, $\underline{\mathrm{Hom}}_S(V, M)$ is finite-dimensional over k.*

Proof. Only (iii) $\Rightarrow$ (i) needs a proof. For this, write $M = M_{\mathrm{pf}} \oplus P$, where P is projective and M_{pf}, the *projective-free part* of M, has no non-zero projective submodules; cf. [12, Lemma 1.1] or [15, Lemma 3.1]. Note that $\mathrm{PHom}_S(V, M_{\mathrm{pf}}) = 0$ holds for every simple S-module V. Thus

$$\underline{\mathrm{Hom}}_S(V, M) \simeq \underline{\mathrm{Hom}}_S(V, M_{\mathrm{pf}}) \simeq \mathrm{Hom}_S(V, M_{\mathrm{pf}}).$$

Now (iii) entails that M_{pf} has a finite-dimensional socle, and hence M_{pf} is finite-dimensional itself. Since $\underline{M} \simeq \underline{M_{\mathrm{pf}}}$ in $\mathsf{StMod}(S)$, we conclude that M is stably finitely generated.

Remark. For QF-algebras, another proof of the first assertion of Lemma 2.2 can be based on the above characterization and the fact that $\underline{\mathrm{Hom}}_S(V, .)$ is a 'cohomological functor'; see [8, Proposition 1.2].

2.4 Stable restriction. Let $S \to T$ be a ring homomorphism. Assume that the following hypothesis is satisfied:

$$\text{all finitely generated } T\text{-modules are stably finitely generated as } S\text{-modules.} \quad (*)$$

Then Lemma 2.2 allows us to define a homomorphism

$$\underline{\mathrm{res}}_{T,S} : G_0(T) \to \underline{G_0}(S), \quad [M] \mapsto \theta(M_S),$$

which we will call *stable restriction* from T to S. Of course, if M is actually finitely generated over S then $\theta(M_S)$ is just the image of the ordinary restricted module M_S in $\underline{G_0}(S) = G_0(S)/c(K_0(S))$.

3 Stable restriction for Hopf algebras

Throughout this section, H will denote a finite-dimensional Hopf algebra over the field k, with counit ε, antipode s, and comultiplication Δ. The latter will be written $\Delta(h) = \sum h_1 \otimes h_2$ for $h \in H$. Finally, $\Lambda \in H$ denotes a fixed non-zero left integral for H.

3.1 Homomorphisms. Let M and N be left H-modules. Then $\mathrm{Hom}_k(N, M)$ can be made into an H-module by defining

$$(hf)(n) = \sum h_1 f(s(h_2)m) \quad (h \in H, n \in N, f \in \mathrm{Hom}_k(N, M)).$$

The H-invariants

$$\mathrm{Hom}_k(N, M)^H = \{f \in \mathrm{Hom}_k(N, M) \mid hf = \varepsilon(h)f \text{ for all } h \in H\}$$

coincide with the H-module maps $\mathrm{Hom}_H(N, M)$; cf. [20, Lemma 1]. Taking $N = k_\varepsilon$, the trivial H-module, evaluation at $1 \in k$ yields an H-module isomorphism $\mathrm{Hom}_k(k_\varepsilon, M) \simeq M$.

Recall that H is a Frobenius algebra; cf. [13, Theorem 2.1.3]. So H can play the role of S in Section 2.

Lemma. *Let V and M be H-modules with V finitely generated. Then*

$$\underline{\mathrm{Hom}}_H(V, M) \simeq \mathrm{Hom}_H(V, M)/\Lambda\,\mathrm{Hom}_k(V, M).$$

In particular, $\underline{\mathrm{Hom}}_H(k_\varepsilon, M) \simeq M^H/\Lambda M$.

Proof. We must show that $\Lambda\,\mathrm{Hom}_k(V,M)$ coincides with the space $\mathrm{PHom}_H(V,M)$ of H-module maps $V \to M$ that factor through some projective. But all these maps factor through the free H-module $\mathrm{ind}_k^H(M) = H \otimes M|_k$ via the epimorphism $\pi : \mathrm{ind}_k^H(M) \twoheadrightarrow M$, $h \otimes m \mapsto hm$. So $\mathrm{PHom}_H(V,M)$ consists of all H-module maps of the form $\pi \circ \varphi$ for some $\varphi \in \mathrm{Hom}_H(V, \mathrm{ind}_k^H(M))$.

Given $f \in \mathrm{Hom}_k(V,M)$, define $\tilde{f} : V \to \mathrm{ind}_k^H(M)$ by $\tilde{f}(v) = \sum \Lambda_1 \otimes f(s(\Lambda_2)v)$. Note that $\tilde{f} = \Lambda(\mu \circ f)$, where $\mu : M \to \mathrm{ind}_k^H(M))$ is the k-linear map given by $m \mapsto 1 \otimes m$. Hence $\tilde{f} \in \mathrm{Hom}_H(V, \mathrm{ind}_k^H(M))$. We claim:

$$\mathrm{Hom}_k(V,M) \simeq \mathrm{Hom}_H(V, \mathrm{ind}_k^H(M)) \quad \text{via } f \mapsto \tilde{f}.$$

Indeed, as H-modules,

$$X := \mathrm{Hom}_k(V, \mathrm{ind}_k^H(M)) \simeq \mathrm{ind}_k^H(M) \otimes V^*$$

(see e.g. [11, §2.1]) and, by the Fundamental Theorem of Hopf Modules (cf. [13, Theorem 1.9.4]), $\mathrm{ind}_k^H(M) \otimes V^* \simeq H \otimes \mathrm{Hom}_k(V,M)$ is a free H-module. Thus $X^H = \Lambda X$, which implies our claim. Since $\pi \circ \tilde{f} = \Lambda f$, we conclude that $\mathrm{PHom}_H(V,M) = \Lambda\,\mathrm{Hom}_k(V,M)$, as desired.

3.2 Smash products. Let R be a left H-module algebra and $T = R \# H$ the associated smash product; see [13, (4.1.1), (4.1.3)]. Thus R is a k-algebra that also is a left H-module subject to certain conditions, and $T = R \otimes H$, made into a k-algebra by means of the multiplication

$$(r \# h)(r' \# h') = \sum r(h_1 r') \# h_2 h'.$$

Here, as is usual, $r \# h$ stands for the element $r \otimes h$ of T. Both R and H are subalgebras of T via the natural identifications $r = r \# 1$ and $h = 1 \# h$.

We will need the following facts about T-modules. First, R is a left T-module via $(r \# h) \cdot r' = r(hr')$. Next, given a left T-module M and a left H-module V, the space $\mathrm{Hom}_k(V,M)$ becomes a left T-module by the rule

$$(r \# h) f(v) = \sum (r \# h_1) f(s(h_2)v).$$

When restricted to H, this action is the one considered in 3.1. If both M and V are finitely generated then so is $\mathrm{Hom}_k(V,M)$. Indeed, M is finitely generated over R and V is finite-dimensional in this case, and as R-modules, $\mathrm{Hom}_k(V,M) \simeq M^{(\dim_k V)}$.

The following proposition gives a criterion for hypothesis $(*)$ in 2.4 to be satisfied in our present setting. The result is [12, Theorem 1.7], transplanted into a Hopf algebra setting.

Proposition. *Assume that R is noetherian as left module over the subalgebra R^H of H-invariants. Then all finitely generated T-modules are stably finitely generated as H-modules if and only if $R^H / \Lambda R$ is finite-dimensional over k.*

Proof. In view of Lemmas 2.3 and 3.1, the condition on $R^H/\Lambda R$ is surely necessary, even for just the T-module R to be stably finitely generated over H.

Conversely, assume the condition is satisfied and let M be a finitely generated T-module. We will show that M is stably finitely generated over H by checking condition (ii) in Lemma 2.3. To this end, let V be any finitely generated H-module. Then, as we have observed above, $\mathrm{Hom}_k(V, M)$ is a finitely generated T-module, and hence it is finitely generated over R as well. Our noetherian hypothesis allows us to conclude that $\mathrm{Hom}_k(V, M)$ is noetherian over R^H. Hence the R^H-submodule $\mathrm{Hom}_H(V, M)$ is also finitely generated. Our condition on $R^H/\Lambda R$ further entails that $\mathrm{Hom}_H(V, M)/\Lambda \,\mathrm{Hom}_k(V, M)$ is finite-dimensional over k. Condition (ii) in Lemma 2.3 now follows by invoking Lemma 3.1.

Remark. The question as to when exactly the noetherian hypothesis in the above proposition holds is largely unresolved at present. By a result of Ferrer-Santos ([7], cf. [13, §4.2]), one knows however that the hypothesis is satisfied whenever R is affine commutative and H is cocommutative. This will be sufficient for our purposes.

3.3 Orders of finite-dimensional classes. We continue with the notations and hypotheses of 3.2; in particular, R will be assumed left noetherian over R^H, and so T is left noetherian. The following result gives a lower bound for the orders of finite-dimensional classes $[M] \in G_0(T)$.

Corollary. *Assume that R is left noetherian over R^H and that $R^H/\Lambda R$ is finite-dimensional over k. Let δ denote the greatest common divisor of the dimensions of all finitely generated projective H-modules. Then, for every T-module M with* $\dim_k M < \infty$, *the order of* $[M] \in G_0(T)$ *is divisible by* $\delta/\gcd(\delta, \dim_k M)$.

Proof. By virtue of Proposition 3.2, we may define stable restriction from T to H,

$$\underline{\mathrm{res}}_{T,H} : G_0(T) \to \underline{G_0}(H),$$

as in 2.4. We further have a homomorphism

$$\underline{G_0}(H) = G_0(H)/cK_0(H) \to \mathbb{Z}/\delta\mathbb{Z}$$

sending $[\underline{V}]$ to the residue class of $\dim_k V$. The composite map $G_0(T) \to \mathbb{Z}/\delta\mathbb{Z}$ sends the class $[M] \in G_0(T)$ to the residue class of $\dim_k M$, an element of $\mathbb{Z}/\delta\mathbb{Z}$ having order $\delta/\gcd(\delta, \dim_k M)$. The corollary follows.

Remark. The number δ in the above corollary is divisible by

(i) the dimension of any local Hopf subalgebra of H, and

(ii) $p = \mathrm{char}\, k$, if H is involutory (that is, $s^2 = \mathrm{Id}$) and not semisimple.

This follows from [11, Lemma 2.4 and Theorem 2.3(b)], respectively.

4 Proof of the Theorem, part (b)

Let Γ be a finitely generated abelian-by-finite group and assume that $p = \operatorname{char} k > 0$. Let G a finite p-subgroup of Γ such that every $g \in G \backslash \{1\}$ has finite centralizer in Γ. We wish to show that $|G|$ divides the order of $[k_\Gamma] \in G_0(k\Gamma)$.

As in Section 1, we fix a torsion-free abelian normal subgroup A of Γ having finite index in Γ. Then the subgroup $\Gamma_1 = \langle A, G \rangle$ of Γ has finite index in Γ, and so (ordinary) restriction of modules from Γ to Γ_1 defines a homomorphism $G_0(k\Gamma) \to G_0(k\Gamma_1)$ sending the Euler class of Γ to the one of Γ_1. Thus we may assume that

$$\Gamma = \Gamma_1 = A \rtimes G.$$

Hence $k\Gamma$ is a smash product $R \# H$ with $R = kA$ and $H = kG$. Moreover, our centralizer hypothesis says that $\mathbf{C}_G(a) = \langle 1 \rangle$ holds for every $a \in A \backslash \{1\}$, which in turn translates into $\dim_k R/\Lambda R = 1$. Here, $\Lambda = \sum_{g \in G} g$ is the standard integral of $H = kG$. Thus Corollary 3.3 applies. Since every projective kG-module is free, we conclude that $|G|$ divides the order of $[k_\Gamma]$. This proves part (b) of the Theorem.

5 Some comments and examples

5.1 Fix a torsion-free abelian normal subgroup A of Γ having finite index in Γ and put $A^\Gamma = A \cap \operatorname{center}(\Gamma)$. Then, by an easy argument involving the transfer map,

$$A^\Gamma \neq \{1\} \text{ if and only if there is an epimorphism } \Gamma \twoheadrightarrow \mathbb{Z}.$$

In this case, we may consider the inflation homomorphism $G_0(k\mathbb{Z}) \to G_0(k\Gamma)$. This map sends $[k_\mathbb{Z}] = 0$ to k_Γ; so $[k_\Gamma] = 0$.

5.2 Reference [6] contains some results on the general nature of torsion in $G_0(k\Gamma)$ which in particular limit the possibilities for the order of $[k_\Gamma]$; see [6, Theorem 3.1]:

(i) If k is a splitting field for all finite subgroups of Γ then $G_0(k\Gamma)$ can have non-trivial q-torsion only for primes q so that Γ has non-trivial q-torsion.

(ii) If $p = \operatorname{char} k > 0$ then the orders of the p-elements of $G_0(k\Gamma)$ are bounded by the largest order of a p-subgroup of Γ.

5.3 Split crystallographic groups. By definition, these are semidirect products of the form $\Gamma = A \rtimes G$, where $A \simeq \mathbb{Z}^n$ is a free abelian group of finite rank n and G is a finite subgroup of $\mathrm{GL}(A) \simeq \mathrm{GL}_n(\mathbb{Z})$. Fixing this notation, and assuming $G \neq \{1\}$ throughout, we consider the following special cases.

5.3.1 Fixed-point-free actions. Assume that G acts fixed-point-freely on A, that is, $[a, g] \neq 1$ holds for all non-identity elements $a \in A$ and $g \in G$. Then part (a) of the Theorem implies:

$$[k_\Gamma] \text{ has finite order if and only if } G \text{ is a } p\text{-group (where } p = \operatorname{char} k).$$

In this case, from part (b) of the Theorem, we further obtain that $|G|$ divides the order of $[k_\Gamma]$. On the other hand, by 5.2, all torsion in $G_0(k\Gamma)$ is annihilated by $|G|$. Hence

$$\text{if } G \text{ is a } p\text{-group then } [k_\Gamma] \text{ has order } |G|.$$

We remark that a p-group that acts fixed-point-freely must be either cyclic or a generalized quaternion 2-group; see [19, Theorems 5.3.1, 5.3.2].

5.3.2 Groups G of prime order. If G has prime order then either $A^\Gamma \neq \{1\}$ or G acts fixed-point-freely. Thus, in view of 5.1 and 5.3.1, we conclude that the Euler class $[k_\Gamma]$ is trivial if and only if $A^\Gamma \neq \{1\}$; in case $A^\Gamma = \{1\}$ the order of the Euler class equals $|G|$ if $|G| = \operatorname{char} k$, and ∞ otherwise.

5.3.3 The case $n = 2$. A non-identity matrix $g \in \mathrm{GL}(A) = \mathrm{GL}_2(\mathbb{Z})$ has determinant 1 if and only if g has no non-trivial fixed point in A. So $G_1 = G \cap \mathrm{SL}(A)$ acts fixed-point-freely on A. Therefore, by part (a) of the Theorem, if $[k_\Gamma]$ has finite order then G_1 must be a p-group (where $p = \operatorname{char} k$). Conversely, if G_1 is a p-group, then all non-identity p'-elements of G have determinant $\neq 1$, and hence they have a non-trivial fixed point in A. Thus, by part (a) of the Theorem again, $[k_\Gamma]$ has finite order. So:

$$[k_\Gamma] \text{ has finite order if and only if } G_1 = G \cap \mathrm{SL}(A) \text{ is a } p\text{-group (where } p = \operatorname{char} k).$$

If $G_1 = \{1\}$ then $A^\Gamma \neq \{1\}$, and so $[k_\Gamma]$ is trivial, by 5.1. On the other hand, if G_1 is a non-trivial p-group then $|G_1|$ divides the order of $[k_\Gamma]$, by part (b) of the Theorem. Thus

$$[k_\Gamma] = 0 \text{ if and only if } G_1 = G \cap \mathrm{SL}(A) = \{1\}.$$

If G is a p-subgroup of $\mathrm{SL}(A)$, then by the foregoing and 5.2, the order of $[k_\Gamma]$ equals $|G|$.

For another example, suppose that $G = \langle -\mathrm{Id}, g \rangle$ for some involution g of determinant -1. The Euler class $[k_\Gamma]$ has finite order only if $p = 2$, and then the order is either $2 = |G_1|$ or $4 = |G|$. To resolve this ambiguity, note that $N = \langle A^{\langle g \rangle}, -g \rangle$ is a normal subgroup of Γ so that $\Gamma/N \simeq D_\infty$, the infinite dihedral group. The Euler class k_{D_∞} has order 2; see 5.3.1. An inflation argument as in 5.1 now shows that $2[k_\Gamma] = 0$. Thus $[k_\Gamma]$ has order 2.

Up to conjugacy, the above remarks leave three finite subgroups

$$G \subseteq \mathrm{GL}(A) = \mathrm{GL}_2(\mathbb{Z})$$

where the order of the Euler class of $\Gamma = A \rtimes G$ has not been completely determined. In standard crystallographic notation, the corresponding groups Γ are the symmetry groups of the wallpaper patterns of types $p4m$, $p3m1$, and $p31m$. We consider these

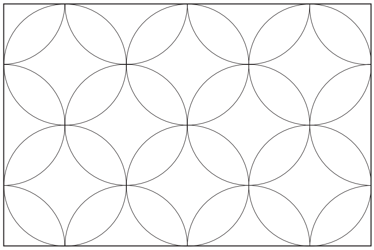

Figure 1. type $p4m$

groups in turn below, in each case expressing the Euler class $[k_\Gamma]$ by means of the cellular chain complex of a corresponding wallpaper pattern.

*Type p*4*m*. Here $\Gamma \simeq \mathbb{Z}^2 \rtimes D_4$; so G_1 is cyclic of order 4. By the foregoing, $[k_\Gamma]$ has finite order precisely for $p = 2$, and in this case, we know that the order is a multiple of 4. We will show that the order of $[k_\Gamma]$ is actually equal to 4. To this end, we view the $p4m$-wallpaper in Figure 1 as a Γ-stable CW-structure on $\mathbb{R}^2$, with 0-cells the points of intersection of the lines in the pattern, 1-cells the arcs between 0-cells, and 2-cells the enclosed regions. There is one Γ-orbit of 0-cells; we denote the stabilizer of a representative 0-cell by $D_4^{(1)}$, a dihedral group of order 8 inside Γ. Furthermore, there is one orbit of 1-cells, with stabilizer D_1, and two orbits of 2-cells with stabilizers D_2 and $D_4^{(2)}$, respectively. We may assume that $D_1 \subseteq D_2$ and that $D_2 \cap D_4^{(i)}$ has order 2 for $i = 1, 2$. The augmented cellular chain complex has the form

$$0 \to k{\uparrow}_{D_4^{(2)}}^{\Gamma} \oplus k{\uparrow}_{D_2}^{\Gamma} \to k{\uparrow}_{D_1}^{\Gamma} \to k{\uparrow}_{D_4^{(1)}}^{\Gamma} \to k_\Gamma \to 0$$

where $\cdot {\uparrow}_D^\Gamma = k\Gamma \otimes_{kD} \cdot$ denotes the induced $k\Gamma$-module; see [17, pp. 93–4]. (Note that the orientation $\pm$-sign is irrelevant here, since we are working in characteristic $p = 2$.) This gives the following formula for the Euler class of Γ:

$$[k_\Gamma] = [k{\uparrow}_{D_4^{(1)}}^{\Gamma}] - [k{\uparrow}_{D_1}^{\Gamma}] + [k{\uparrow}_{D_2}^{\Gamma}] + [k{\uparrow}_{D_4^{(2)}}^{\Gamma}].$$

Here

$$[k{\uparrow}_{D_2}^{\Gamma}] = [k{\uparrow}_{D_1}^{\Gamma}] \quad \text{and} \quad 2[k{\uparrow}_{D_2}^{\Gamma}] = 4[k{\uparrow}_{D_4^{(i)}}^{\Gamma}]$$

for $i = 1, 2$. From this we obtain $4[k_\Gamma] = 0$. Therefore the Euler class $[k_\Gamma]$ has order 4.

*Types p*3*m*1 *and p*31*m*. In both cases, $\Gamma \simeq \mathbb{Z}^2 \rtimes D_3$ and G_1 is cyclic of order 3. So $[k_\Gamma]$ has finite order precisely for $p = 3$, and then the order is a multiple of 3. Again, it turns out that the order of $[k_\Gamma]$ is equal to 3 when $p = 3$. We consider the case $p3m1$ in some detail, leaving the verification in type $p31m$ to the reader. The $p3m1$-wallpaper in Figure 2 has one Γ-orbit of 0-cells, with isotropy group D_3, one orbit of 1-cells, with isotropy D_1, and two orbits of 2-cells ('black' and 'white'), with iso-

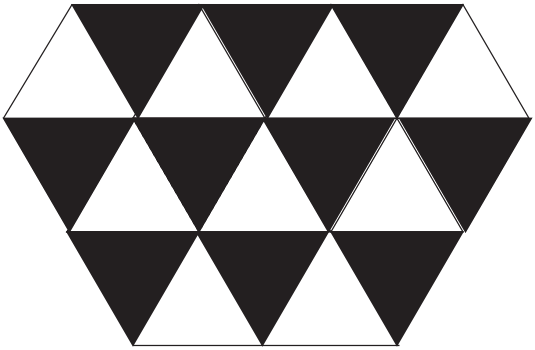

Figure 2. type $p3m1$

tropy groups $D_3^{(\pm)}$, two further copies of the dihedral group of order 6 inside Γ. We may assume that D_1 is contained in D_3 and in both $D_3^{(\pm)}$. Thus the cellular chain complex gives the following equation for $[k_\Gamma]$:

$$[k_\Gamma] = [k^-\uparrow_{D_3^{(+)}}^\Gamma] + [k^-\uparrow_{D_3^{(-)}}^\Gamma] - [k^-\uparrow_{D_1}^\Gamma] + [k\uparrow_{D_3}^\Gamma].$$

Here k^- denotes the sign-representation of the dihedral group in question: reflections act as -1 (orientation-reversing), rotations as $+1$ (orientation-preserving). In order to show that $3[k_\Gamma] = 0$, we note that

$$3[k\uparrow_{D_3}^\Gamma] + 3[k^-\uparrow_{D_3}^\Gamma] = [k\Gamma],$$

and similarly for $D_3^{(\pm)}$. Using this, and the equation

$$[k^-\uparrow_{D_1}^{D_3}] = 2[k_{D_3}^-] + [k_{D_3}],$$

one obtains the formula

$$3[k^-\uparrow_{D_3}^\Gamma] = 3[k^-\uparrow_{D_1}^\Gamma] - [k\Gamma],$$

and similarly for $D_3^{(\pm)}$. These formulas together easily yield $3[k_\Gamma] = 0$, as required.

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