*J. London Math. Soc.* (2) 68 (2003) 651–670 © 2003 London Mathematical Society
DOI: 10.1112/S0024610703004678

# FIELDS OF DEFINITION FOR DIVISION ALGEBRAS

M. LORENZ, Z. REICHSTEIN, L. H. ROWEN AND D. J. SALTMAN


## Abstract

Let $A$ be a finite-dimensional division algebra containing a base field $k$ in its center $F$. $A$ is defined over a subfield $F_0$ if there exists an $F_0$-algebra $A_0$ such that $A = A_0 \bigotimes_{F_0} F$. The following are shown. (i) In many cases $A$ can be defined over a rational extension of $k$. (ii) If $A$ has odd degree $n \geqslant 5$, then $A$ is defined over a field $F_0$ of transcendence degree $\leqslant \frac{1}{2}(n-1)(n-2)$ over $k$. (iii) If $A$ is a $\mathbb{Z}/m \times \mathbb{Z}/2$-crossed product for some $m \geqslant 2$ (and in particular, if $A$ is any algebra of degree 4) then $A$ is Brauer equivalent to a tensor product of two symbol algebras. Consequently, $\mathrm{M}_m(A)$ can be defined over a field $F_0$ such that $\mathrm{trdeg}_k(F_0) \leqslant 4$. (iv) If $A$ has degree 4 then the trace form of $A$ can be defined over a field $F_0$ of transcendence degree $\leqslant 4$. (In (i), (iii) and (iv) it is assumed that the center of $A$ contains certain roots of unity.)


## 1. *Introduction*

Let $F$ be a field and $A$ a finite-dimensional $F$-algebra. We say that $A$ is defined over a subfield $F_0 \subset F$ if there exists an $F_0$-algebra $A_0$ such that $A = A_0 \bigotimes_{F_0} F$.

Throughout this paper we will assume that $A$ is a finite-dimensional central simple $F$-algebra, and $F$ (and $F_0$) contain a base field $k$.

### 1.1. *Parameter reduction*

If $A$ is defined over $F_0$ and $\mathrm{trdeg}_k(F_0) < \mathrm{trdeg}_k(F)$ then the passage from $A$ to $A_0$ may be viewed as 'parameter reduction' in $A$. This leads to the natural question: What is the smallest value of $\mathrm{trdeg}_k(F_0)$ such that $A$ is defined over $F_0$?

This number is clearly finite; we shall denote it by $\tau(A)$. Of particular interest to us will be the case where $A = \mathrm{UD}(n)$ is the universal division algebra of degree $n$ and $F = Z(n)$ is the center of $\mathrm{UD}(n)$. Recall that $\mathrm{UD}(n)$ is the subalgebra of $\mathrm{M}_n(k(x_{ij}, y_{ij}))$ generated, as a $k$-division algebra, by two generic $n \times n$-matrices $X = (x_{ij})$ and $Y = (y_{ij})$, where $x_{ij}$ and $y_{ij}$ are $2n^2$ independent commuting variables over $k$; see, for example, [**20**, Section II.1; **27**, Section 3.2; **34**, Section 14]. We shall write $d(n)$ for $\tau(\mathrm{UD}(n))$. It is easy to show that $d(n) \geqslant \tau(A)$ for any central simple algebra $A$ of degree $n$ whose center contains $k$; see Remark 2.8 (cf. also [**22**, Lemma 9.2]). In other words, every central simple algebra of degree $n$ can be 'reduced to at most $d(n)$ parameters'.

To the best of our knowledge, the earliest attempt to determine the value of $d(n)$ is due to Procesi, who showed that $d(n) \leqslant n^2$; see [**20**, Theorem 2.1]. If $n = 2$, 3 or 6 and $k$ contains a primitive $n$th root of unity then $d(n) = 2$, because $\mathrm{UD}(n)$ is cyclic for these $n$ and we can take $A_0$ to be a symbol algebra; cf. [**22**, Lemma 9.4].

Received 4 January 2000; final revision 26 February 2003.

2000 *Mathematics Subject Classification* 16K20, 16W22, 20C10, 20J06, 11E81.

M. Lorenz was supported in part by NSF grant DMS-9988756. Z. Reichstein was supported in part by NSF grant DMS-9801675 and an NSERC research grant. L. H. Rowen was supported by the Israel Science Foundation, founded by the Israel Academy of Sciences and Humanities – Center of Excellence program 8007/99-3. D. J. Saltman was supported in part by NSF grant DMS-9970213.

Rost [**25**] recently proved that

$$d(4) = 5. \tag{1.1}$$

For other $n$ the exact value of $d(n)$ is not known. However, the following inequalities hold:

$$\begin{array}{ll} d(n) \leqslant n^2 - 3n + 1 & \text{if } n \geqslant 4 \text{ [14]}, \\ d(n) \leqslant d(nm) \leqslant d(n) + d(m) & \text{if } (n, m) = 1 \text{ [22, Section 9.4]}, \\ d(n^r) \geqslant 2r & \text{[21, Theorem 16.1]}, \\ d(n) \leqslant \frac{1}{2}(n-1)(n-2) + n & \text{for odd } n \text{ [28], cf. [22, Section 9.3]}. \end{array} \tag{1.2}$$

In this paper we will sharpen the last inequality by showing that

$$d(n) \leqslant \tfrac{1}{2}(n-1)(n-2)$$

for every odd $n \geqslant 5$. Moreover, in $\mathrm{UD}(n)$, reduction to this number of parameters can be arranged in a particularly nice fashion.

THEOREM 1.1. *Let $n \geqslant 5$ be an odd integer, $\mathrm{UD}(n)$ the universal division algebra of degree $n$, and $Z(n)$ its center. Then there exists a subfield $F$ of $Z(n)$ and a division algebra $D$ of degree $n$ with center $F$ such that*

(a) $\mathrm{UD}(n) = D \bigotimes_F Z(n)$;

(b) $\mathrm{trdeg}_k(F) = \frac{1}{2}(n-1)(n-2)$;

(c) *$Z(n)$ is a rational extension of $F$.*

*In particular, $d(n) \leqslant \frac{1}{2}(n-1)(n-2)$.*

We remark that in the language of [**22**], the last assertion of Theorem 1.1 can be written as

$$\mathrm{ed}(\mathrm{PGL}_n) \leqslant \tfrac{1}{2}(n-1)(n-2). \tag{1.3}$$

### 1.2. *Rational fields of definition*

Another natural question is whether or not a given central simple algebra $A$ can be defined over a rational extension of $k$. We give the following partial answer to this question.

THEOREM 1.2. *Let $A$ be a finite-dimensional central simple algebra with center $F$ and let $t_1, t_2, \ldots$ be algebraically independent central indeterminates over $F$.*

(a) *Assume that $\deg(A) = 2^i p_1 \ldots p_r$, where $i = 0$, 1 or 2 and $p_1, \ldots, p_r$ are distinct odd primes. Then for $s \gg 0$ the algebra $A(t_1, \ldots, t_s)$ is defined over a rational extension of $k$.*

(b) *Suppose that the center of $A$ contains a primitive $e$th root of unity, where $e$ is the exponent of $A$. Then there exists an $r \geqslant 1$ such that for $s \gg 0$ the algebra $\mathrm{M}_r(A)(t_1, \ldots, t_s)$ is defined over a rational extension of $k$. (Here we are imposing no restrictions on the degree of $A$.)*

Here $A(t_1, \ldots, t_s)$ stands for $A \bigotimes_F F(t_1, \ldots, t_s)$ and similarly for $\mathrm{M}_r(A)(t_1, \ldots, t_s)$. Note that part (b) may be interpreted as saying that for $s \gg 0$ the Brauer class of $A(t_1, \ldots, t_s)$ is defined over a rational extension of $k$.

1.3. *$\mathbb{Z}/m \times \mathbb{Z}/2$-crossed products*

As usual, we let $(a, b)_m$ denote the symbol algebra

$$F\{x, y\}/(x^m = a,\ y^m = b,\ xy = \zeta_m yx), \tag{1.4}$$

where $a, b \in F^*$ and $\zeta_m$ is a (fixed) primitive $m$th root of unity in $F$; cf. [**29**, pp. 194–197]. In Section 6 we will prove the following.

THEOREM 1.3. *Let $A$ be a $\mathbb{Z}/m \times \mathbb{Z}/2$-crossed product central simple algebra whose center $F$ contains a primitive $2m$th root of unity $\zeta_{2m}$. Then $A$ is Brauer equivalent (over $F$) to $(a, b)_m \bigotimes_F (c, d)_{2m}$ for some $a, b, c, d \in F^*$. (In other words, $\mathrm{M}_m(A)$ is isomorphic to $(a, b)_m \bigotimes_F (c, d)_{2m}$.)*

Note that Theorem 1.3 applies to any division algebra of degree 4, since, by a theorem of Albert [**1**], any such algebra is a $\mathbb{Z}/2 \times \mathbb{Z}/2$-crossed product (see also [**27**, Theorem 2.3.28; **29**, Theorem 7.1.45]). In this setting our argument yields, in particular, an elementary proof of [**37**, Theorem 2, p. 288]. We also remark that Theorem 1.3 may be viewed as an explicit form of the Merkurjev–Suslin theorem for $\mathbb{Z}/m \times \mathbb{Z}/2$-crossed products.

Letting $F_0 = k(\zeta_{2m}, a, b, c, d)$, we note that

$$(a, b)_m \bigotimes_F (c, d)_{2m} = ((a, b)_m \bigotimes_{F_0} (c, d)_{2m}) \bigotimes_{F_0} F.$$

Thus Theorem 1.3 shows that $\mathrm{M}_m(A)$ is defined over $F_0$. Since $\mathrm{trdeg}_k(F_0) \leqslant 4$, we obtain the following.

COROLLARY 1.4. *Let $A$ be a $\mathbb{Z}/m \times \mathbb{Z}/2$-crossed product central simple algebra whose center contains a primitive $2m$th root of unity. Then $\tau(\mathrm{M}_m(A)) \leqslant 4$. In particular, $\tau(\mathrm{M}_2(A)) \leqslant 4$ for every central simple algebra $A$ of degree 4 whose center contains a primitive 4th root of unity.*

Note that the last assertion complements, in a somewhat surprising way, the above-mentioned result of Rost (1.1). Indeed, suppose the base field $k$ contains a primitive 4th root of unity. Then for $A = \mathrm{UD}(4)$, Rost's theorem says that $\tau(A) = 5$, whereas Corollary 1.4 says that $\tau(\mathrm{M}_2(A)) \leqslant 4$.

1.4. *Fields of definition of quadratic forms*

A quadratic form $q : V \longrightarrow F$ on an $F$-vector space $V = F^n$ is said to be defined over a subfield $F_0$ of $F$ if $q = q_{F_0} \otimes F$, where $q_{F_0}$ is a quadratic form on $V_0 = F_0^n$.

In Section 7 we discuss fields of definition of quadratic forms. Of particular interest to us will be trace forms of central simple algebras of degree 4, recently studied by Rost, Serre and Tignol [**26**]. (Recall that the trace form of a central simple algebra $A$ is the quadratic form $x \mapsto \mathrm{Tr}(x^2)$ defined over the center of $A$.) We will use a theorem of Serre [**35**] (see our Proposition 7.3) to prove the following.

THEOREM 1.5. *Let $A$ be a central simple algebra of degree 4 whose center $F$ contains a primitive 4th root of unity. Then the trace form of $A$ is defined over a subfield $F_0 \subset F$ such that $\mathrm{trdeg}_k(F_0) \leqslant 4$.*

Note that Theorem 1.5 may also be viewed as complementing (1.1).

## 2. *Preliminaries*

### 2.1. *$\mathscr{G}$-lattices*

Throughout, $\mathscr{G}$ will denote a finite group and $\mathscr{H}$ will be a subgroup of $\mathscr{G}$. Recall that a *$\mathscr{G}$-module* is a (left) module over the integral group ring $\mathbb{Z}[\mathscr{G}]$. As usual, $\mathrm{Ext}_{\mathscr{G}}$ stands for $\mathrm{Ext}_{\mathbb{Z}[\mathscr{G}]}$. A *$\mathscr{G}$-lattice* is a $\mathscr{G}$-module that is free of finite rank over $\mathbb{Z}$. Further, a $\mathscr{G}$-lattice $M$ is called

(1) a *permutation lattice* if $M$ has a $\mathbb{Z}$-basis that is permuted by $\mathscr{G}$;
(2) *permutation projective* (or *invertible*) if $M$ is a direct summand of some permutation $\mathscr{G}$-lattice.

A $\mathscr{G}$-module $M$ is called *faithful* if no $1 \neq g \in \mathscr{G}$ acts as the identity on $M$.

The $\mathscr{G}/\mathscr{H}$-augmentation kernel $\omega(\mathscr{G}/\mathscr{H})$ is defined as the kernel of the natural augmentation map

$$\mathbb{Z}[\mathscr{G}/\mathscr{H}] = \mathbb{Z}[\mathscr{G}] \bigotimes_{\mathbb{Z}[\mathscr{H}]} \mathbb{Z} \longrightarrow \mathbb{Z}, \quad \overline{g} = g \otimes 1 \mapsto 1 \qquad (g \in \mathscr{G}).$$

Thus there is a short exact sequence of $\mathscr{G}$-lattices

$$0 \longrightarrow \omega(\mathscr{G}/\mathscr{H}) \longrightarrow \mathbb{Z}[\mathscr{G}/\mathscr{H}] \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{2.1}$$

$\omega(\mathscr{G}/\{1\})$ will be written as $\omega\mathscr{G}$; this is the ordinary augmentation ideal of the group ring $\mathbb{Z}[\mathscr{G}]$; cf. [**18**, Chapter 3].

### 2.2. *Extension sequences*

Exact sequences of $\mathscr{G}$-lattices of the form

$$\begin{gathered} 0 \longrightarrow M \longrightarrow P \longrightarrow \omega(\mathscr{G}/\mathscr{H}) \longrightarrow 0, \\ \text{with } P \text{ permutation and } M \text{ faithful} \end{gathered} \tag{2.2}$$

will play an important role in the sequel. In this subsection we introduce two such sequences, (2.3) and (2.4).

Let $d_{\mathscr{G}}(\omega(\mathscr{G}/\mathscr{H}))$ denote the minimum number of generators of $\omega(\mathscr{G}/\mathscr{H})$ as a $\mathbb{Z}[\mathscr{G}]$-module. Then for any $r \geqslant d_{\mathscr{G}}(\omega(\mathscr{G}/\mathscr{H}))$ there exists an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}[\mathscr{G}]^r \xrightarrow{f} \omega(\mathscr{G}/\mathscr{H}) \longrightarrow 0 \tag{2.3}$$

of $\mathscr{G}$-lattices.

LEMMA 2.1. *$M$ is a faithful $\mathscr{G}$-lattice if and only if $r \geqslant 2$ or $\mathscr{H} \neq \{1\}$.*

*Proof.* It is enough to show that $M \otimes \mathbb{Q}$ is $\mathscr{G}$-faithful; thus we may work over the semisimple algebra $\mathbb{Q}[\mathscr{G}]$. Since $f \otimes \mathbb{Q}$ splits, we have a $\mathbb{Q}[\mathscr{G}]$-isomorphism $(\omega(\mathscr{G}/\mathscr{H}) \otimes \mathbb{Q}) \oplus (M \otimes \mathbb{Q}) \simeq \mathbb{Q}[\mathscr{G}]^r$. Similarly, the canonical exact sequence $\mathbb{Z}[\mathscr{G}]\omega\mathscr{H} \rightarrowtail \mathbb{Z}[\mathscr{G}] \twoheadrightarrow \mathbb{Z}[\mathscr{G}/\mathscr{H}]$ gives $(\omega(\mathscr{G}/\mathscr{H}) \otimes \mathbb{Q}) \oplus \mathbb{Q} \oplus \mathbb{Q}[\mathscr{G}]\omega\mathscr{H} \simeq \mathbb{Q}[\mathscr{G}]$. Therefore,

$$M \otimes \mathbb{Q} \simeq \mathbb{Q}[\mathscr{G}]^{r-1} \oplus \mathbb{Q} \oplus \mathbb{Q}[\mathscr{G}]\omega\mathscr{H}.$$

If $r \geqslant 2$ then $\mathbb{Q}[\mathscr{G}]^{r-1}$ is $\mathscr{G}$-faithful, and if $\mathscr{H} \neq \{1\}$ then $\omega\mathscr{H} \otimes \mathbb{Q}$ is $\mathscr{H}$-faithful and so $\mathbb{Q}[\mathscr{G}]\omega\mathscr{H} \simeq (\omega\mathscr{H} \otimes \mathbb{Q})\uparrow_{\mathscr{H}}^{\mathscr{G}}$ is $\mathscr{G}$-faithful. In either case, $M \otimes \mathbb{Q}$ is faithful, as

desired. On the other hand, $r = 1$ and $\mathscr{H} = \{1\}$ leads to $M \otimes \mathbb{Q} \simeq \mathbb{Q}$ which is not faithful. □

LEMMA 2.2. *There is an exact sequence*

$$0 \longrightarrow \omega(\mathscr{G}/\mathscr{H})^{\otimes 2} \xrightarrow{m} P \longrightarrow \omega(\mathscr{G}/\mathscr{H}) \longrightarrow 0, \tag{2.4}$$

*where $P$ is the (permutation) sublattice $P = \bigoplus_{\overline{g_1} \neq \overline{g_2} \in \mathscr{G}/\mathscr{H}} \mathbb{Z}(\overline{g_1} \otimes \overline{g_2})$ of $\mathbb{Z}[\mathscr{G}/\mathscr{H}]^{\otimes 2}$. The lattice $\omega(\mathscr{G}/\mathscr{H})^{\otimes 2}$ is faithful if and only if $\mathscr{H}$ contains no normal subgroup $\neq \{1\}$ of $\mathscr{G}$ and $[\mathscr{G} : \mathscr{H}] \geqslant 3$.*

*Proof.* Tensoring sequence (2.1) with $\omega(\mathscr{G}/\mathscr{H})$ and putting $P' = \omega(\mathscr{G}/\mathscr{H}) \otimes \mathbb{Z}[\mathscr{G}/\mathscr{H}]$, we obtain an exact sequence

$$0 \longrightarrow \omega(\mathscr{G}/\mathscr{H})^{\otimes 2} \longrightarrow P' \longrightarrow \omega(\mathscr{G}/\mathscr{H}) \longrightarrow 0,$$

where $\otimes = \bigotimes_{\mathbb{Z}}$. The elements $(\overline{g_1} - \overline{g_2}) \otimes \overline{g_2}$ with $\overline{g_1} \neq \overline{g_2} \in \mathscr{G}/\mathscr{H}$ form a $\mathbb{Z}$-basis of $P'$, and the map

$$m : (\overline{g_1} - \overline{g_2}) \otimes \overline{g_2} \mapsto \overline{g_1} \otimes \overline{g_2}$$

is a $\mathscr{G}$-isomorphism $P' \simeq P$.

For the faithfulness assertion, note that $\mathscr{N} = \bigcap_{g \in \mathscr{G}} \mathscr{H}^g$ acts trivially on $\mathbb{Z}[\mathscr{G}/\mathscr{H}]$ and hence on $\omega(\mathscr{G}/\mathscr{H})^{\otimes 2}$; so $\mathscr{N} = \{1\}$ is surely required for faithfulness. Also, if $[\mathscr{G} : \mathscr{H}] \leqslant 2$ then $\mathscr{G}$ acts trivially on $\omega(\mathscr{G}/\mathscr{H})^{\otimes 2}$. Conversely, if $\mathscr{N} = \{1\}$ and $[\mathscr{G} : \mathscr{H}] \geqslant 3$ then it is easy to verify that $\omega(\mathscr{G}/\mathscr{H})^{\otimes 2}$ is indeed faithful. □

### 2.3. *Twisted multiplicative $\mathscr{G}$-fields*

Recall that a *$\mathscr{G}$-field* is a field $F$ on which the finite group $\mathscr{G}$ acts by automorphisms, written $f \mapsto g(f)$. Morphisms of $\mathscr{G}$-fields are $\mathscr{G}$-equivariant field homomorphisms. The $\mathscr{G}$-field $F$ is called *faithful* if every $1 \neq g \in \mathscr{G}$ acts non-trivially on $F$. If $K \subseteq F$ is a field extension and $V \subseteq F$ a subset of $F$ (not necessarily algebraically independent over $K$) then we let $K(V)$ denote the subfield of $F$ that is generated by $K$ and $V$.

LEMMA 2.3 (cf. [**36**, Appendix 3]). *Let $K \subseteq F$ be an extension of $\mathscr{G}$-fields with $K$ faithful. Assume that $F = K(V)$ for some $\mathscr{G}$-stable $K$-subspace $V \subseteq F$. Then*

(a) *$V = KV^{\mathscr{G}}$, where $V^{\mathscr{G}}$ denotes the $\mathscr{G}$-invariants in $V$;*
(b) *$F = K(V^{\mathscr{G}})$;*
(c) *$F^{\mathscr{G}} = K^{\mathscr{G}}(V^{\mathscr{G}})$.*

*Proof.* (a) Let $S = K\#\mathscr{G}$ denote the skew group ring for the given $\mathscr{G}$-action on $K$. The $\mathscr{G}$-action on $F$ and multiplication with $K$ make $F$ a (left) $S$-module, and $V$ is a submodule. Moreover, since $K$ is a faithful $\mathscr{G}$-field, $S$ is a simple ring; see, for example, [**10**, p. 473]. In particular, the element $t = \sum_{g \in \mathscr{G}} g \in S$ generates $S$ as a 2-sided ideal. Thus, $S = StS = KtK$ and consequently, $V = KtKV = KV^{\mathscr{G}}$.

(b) is an immediate consequence of (a).

(c) Let $E = K^{\mathscr{G}}(V^{\mathscr{G}})$. We want to show that $E = F^{\mathscr{G}}$. Clearly $E \subseteq F^{\mathscr{G}}$. To prove equality, note that $KE$ is a subring of $F$ containing $K$ and $V^{\mathscr{G}}$, and that $\dim_E KE \leqslant \dim_{K^{\mathscr{G}}} K = |\mathscr{G}|$. Thus, $KE$ is a field, and hence (b) implies that $KE = F$. Therefore, $\dim_E F \leqslant |\mathscr{G}| = \dim_{F^{\mathscr{G}}} F$. Since $E \subseteq F^{\mathscr{G}}$, this is only possible if $E = F^{\mathscr{G}}$. □

We recall a well-known construction of $\mathcal{G}$-fields; cf. [**32**]. Given a $\mathcal{G}$-field $E$, a $\mathcal{G}$-lattice $M$, and an extension class $\gamma \in \mathrm{Ext}_{\mathcal{G}}(M, E^*)$, the *twisted multiplicative $\mathcal{G}$-field* $E_\gamma(M)$ is constructed as follows. Form the group algebra $E[M]$ of $M$ over $E$; this is a commutative integral domain with group of units $\mathrm{U}(E[M]) = E^* \times M$. We shall use multiplicative notation for $M$ in this setting. Let $E(M)$ denote the field of fractions of $E[M]$. Choose an extension of $\mathcal{G}$-modules

$$1 \longrightarrow E^* \longrightarrow V \longrightarrow M \longrightarrow 1 \tag{2.5}$$

representing $\gamma$. Thus, as abelian groups, $V \simeq \mathrm{U}(E[M])$. Using this identification, we obtain a $\mathcal{G}$-action on $\mathrm{U}(E[M])$ inducing the given action on $E^*$. The action of $\mathcal{G}$ on $\mathrm{U}(E[M])$ extends uniquely to $E[M]$, and to $E(M)$; we will use $E_\gamma[M]$ and $E_\gamma(M)$ to denote $E[M]$ and $E(M)$ with the $\mathcal{G}$-actions thus obtained. For $\gamma = 1$, we will simply write $E[M]$ and $E(M)$. We remark that the choice of the sequence (2.5) representing a given $\gamma \in \mathrm{Ext}_{\mathcal{G}}(M, E^*)$ is insubstantial; a different choice leads to $\mathcal{G}$-isomorphic results.

For future reference, we record the following application of Lemma 2.3 essentially due to Masuda [**17**]; see also [**16**, Proposition 1.6; **34**, Lemma 12.8].

PROPOSITION 2.4. *Let $E$ be a faithful $\mathcal{G}$-field and let $P$ be a permutation $\mathcal{G}$-lattice. Then any twisted multiplicative $\mathcal{G}$-field $E_\gamma(P)$ can be written as*

$$E_\gamma(P) = E(t_1, \ldots, t_n)$$

*with $\mathcal{G}$-invariant transcendental (over $E$) elements $t_i$. In particular, $E_\gamma(P)^{\mathcal{G}} = E^{\mathcal{G}}(t_1, \ldots, t_n)$ is rational over $E^{\mathcal{G}}$.*

*Proof.* We have an extension of $\mathcal{G}$-modules $1 \longrightarrow E^* \longrightarrow \mathrm{U}(E_\gamma[P]) \longrightarrow P \longrightarrow 1$ representing $\gamma$, as in (2.5). Fix a $\mathbb{Z}$-basis, $X$, of $P$ that is permuted by the action of $\mathcal{G}$. For each $x \in X$, choose a preimage $x' \in \mathrm{U}(E_\gamma[P])$. Then $\{x'\}_{x \in X}$ is a collection of transcendental generators of $E_\gamma(P)$ over $E$, and $\mathcal{G}$ acts via $g(x') = g(x)'y$ for some $y = y(g, x) \in E^*$. Letting $V$ denote the $E$-subspace of $E_\gamma(P)$ that is generated by $\{x'\}_{x \in X}$, we conclude from Lemma 2.3 that $V$ has a basis consisting of $\mathcal{G}$-invariant elements, say $t_1, \ldots, t_n$, and $E_\gamma(P) = E(t_1, \ldots, t_n)$, $E_\gamma(P)^{\mathcal{G}} = E^{\mathcal{G}}(t_1, \ldots, t_n)$. The $t_i$ are transcendental over $E$, since $\mathrm{trdeg}_E E_\gamma(P) = \mathrm{rank}(P) = n$. □

## 2.4. *Rational specialization*

Let $A/F$ and $B/K$ be central simple algebras. We will call $B/K$ a *rational specialization* of $A/F$ if there exists a field $F'$ containing both $F$ and $K$ such that $F'/K$ is rational and

$$B \bigotimes_K F' \simeq A \bigotimes_F F'.$$

In other words, $B$ is a rational specialization of $A$ if $\deg A = \deg B$ and $A$ embeds in some $B(t_1, \ldots, t_n)$, where $t_1, t_2, \ldots$ are independent variables over $F$; cf. [**34**, p. 73].

*For the rest of this paper we fix an (arbitrary) base field $k$. All other fields are understood to contain a copy of $k$ and all maps (that is, inclusions) between fields are understood to restrict to the identity map on $k$.*

DEFINITION 2.5. Let $\Lambda$ be a class of central simple algebras. We shall say that an algebra $A \in \Lambda$ has the *rational specialization property* in the class $\Lambda$ if every $B \in \Lambda$

is a rational specialization of $A$. If $\Lambda$ is the class of all central simple algebras of degree $n = \deg(A)$ then we will omit the reference to $\Lambda$ and will simply say that $A$ has the rational specialization property.

EXAMPLE 2.6. By [**23**, Lemma 3.1], UD($n$) has the rational specialization property. This is also implicit in [**33**]. We remark that any central simple algebra $A/F$ with the rational specialization property is a division algebra. To see this, specialize $A$ to UD($n$), where $n = \deg(A)$.

Recall the definition of $\tau(A)$ given at the beginning of this paper.

LEMMA 2.7. *Let $A/F$ and $B/K$ be central simple algebras.*
(a) *If $A' \simeq A \bigotimes_F F'$ for some rational field extension $F'/F$ then $\tau(A) = \tau(A')$.*
(b) (cf. [**34**, Lemma 11.1]) *If $A$ is a rational specialization of $B$ then $\tau(A) \leqslant \tau(B)$.*

*Proof.* (a) The inequality $\tau(A') \leqslant \tau(A)$ is immediate from the definition of $\tau$. To prove the opposite inequality, suppose that $A' \simeq A_0 \bigotimes_{F_0} F'$, where $A_0$ is a central simple algebra over an intermediate field $k \subset F_0 \subset F'$, and $A_0$ is chosen so that $\operatorname{trdeg}_k(F_0) = \tau(A')$. In particular, $\operatorname{trdeg}_k(F_0) \leqslant \operatorname{trdeg}_k(F)$. Then by [**23**, Proposition 3.2], $A_0$ embeds in $A$, that is, $A \simeq A_0 \bigotimes_{F_0} F$ for some embedding $F_0 \hookrightarrow F$. Consequently, $\tau(A) \leqslant \operatorname{trdeg}_k(K) = \tau(A')$, as desired.

(b) We may assume that $B \bigotimes_K F' = A'$, as in (a). Clearly $\tau(B) \geqslant \tau(A')$, and part (a) tells us that $\tau(A') = \tau(A)$. □

REMARK 2.8. Combining Example 2.6 with Lemma 2.7(b), we see that $\tau(A) \leqslant d(n) = \tau(\mathrm{UD}(n))$ holds for every central simple algebra $A$ of degree $n$; cf. [**22**, Lemma 9.2].

## 3. $\mathcal{G}/\mathcal{H}$-crossed products

We shall call a central simple algebra $A/F$ an $(E, \mathcal{G}/\mathcal{H})$-*crossed product* if $A$ has a maximal subfield $L$ whose Galois closure $E$ over $F$ has the property that $\mathrm{Gal}(E/F) = \mathcal{G}$ and $\mathrm{Gal}(E/L) = \mathcal{H}$. (We adopt the convention that a *maximal subfield* of $A$ is a subfield $L$ that is maximal as a commutative subring; so $[L : F]$ is equal to the degree of $A$.) We will say that $A$ is a $\mathcal{G}/\mathcal{H}$-crossed product if it is an $(E, \mathcal{G}/\mathcal{H})$-crossed product for some faithful $\mathcal{G}$-field $E$. If $\mathcal{H} = \{1\}$ then a $\mathcal{G}/\mathcal{H}$-crossed product is just a $\mathcal{G}$-*crossed product* in the usual sense (see, for example, [**27**, Definition 3.1.23]).

EXAMPLE 3.1. Consider the universal division algebra UD($n$) generated by two generic matrices, $X$ and $Y$, over $k$. Denote the center of this algebra by $Z(n)$. Setting $L = Z(n)(X)$, we see that UD($n$) is an $\mathcal{S}_n/\mathcal{S}_{n-1}$-crossed product [**20**, Theorem 1.9]; see also Section 4 below. Here $\mathcal{S}_n$ denotes the symmetric group on $n$ letters.

Since the degree of a $\mathcal{G}/\mathcal{H}$-crossed product is equal to $[\mathcal{G} : \mathcal{H}]$, we see that isomorphism classes of $(E, \mathcal{G}/\mathcal{H})$-crossed products are in one-to-one correspondence with the relative Brauer group $\mathbf{B}(L/F)$, which, in turn, is naturally identified with the kernel of the restriction homomorphism $H^2(\mathcal{G}, E^*) \longrightarrow H^2(\mathcal{H}, E^*)$; cf. [**19**, 14.7].

A $\mathcal{G}$-module $M$ is called $H^1$-*trivial* if $H^1(\mathcal{H}, M) = 0$ holds for every subgroup $\mathcal{H} \leqslant \mathcal{G}$. Equivalently, $M$ is $H^1$-trivial if $\mathrm{Ext}_{\mathcal{G}}(P, M) = 0$ for all permutation projective $\mathcal{G}$-lattices $P$; see, for example, [**34**, Lemma 12.3].

LEMMA 3.2. *Given an exact sequence*

$$0 \longrightarrow M \longrightarrow P \longrightarrow \omega(\mathcal{G}/\mathcal{H}) \longrightarrow 0,$$

*of $\mathcal{G}$-lattices, with $P$-permutation, let $N$ be an $H^1$-trivial $\mathcal{G}$-module. Denote the kernel of the restriction homomorphism $H^2(\mathcal{G}, N) \longrightarrow H^2(\mathcal{H}, N)$ by $K(\mathcal{G}/\mathcal{H}, N)$. Then there is a natural isomorphism*

$$\phi_N : \mathrm{Hom}_{\mathcal{G}}(M, N)/\mathrm{Im}(\mathrm{Hom}_{\mathcal{G}}(P, N)) \xrightarrow{\simeq} K(\mathcal{G}/\mathcal{H}, N)$$

*which is functorial in $N$.*

Note that here, unlike in sequence (2.2), the $\mathcal{G}$-lattice $M$ is not required to be faithful.

*Proof of Lemma* 3.2. The lemma is a variant of [**34**, Theorem 12.10], where the same assertion is made for the sequence (2.4). The proof of [**34**, Theorem 12.10] goes through unchanged in our setting. □

In subsequent applications we will always take $N = E^*$, where $E$ is a faithful $\mathcal{G}$-field. Note that $E^*$ is $H^1$-trivial by Hilbert's Theorem 90. As we remarked above, $K(\mathcal{G}/\mathcal{H}, E^*)$ is naturally identified with $\mathbf{B}(L/F)$, where $L = E^{\mathcal{H}}$, and elements of $\mathbf{B}(L/F)$ are in one-to-one correspondence with $(E, \mathcal{G}/\mathcal{H})$-crossed products. We shall denote the $(E, \mathcal{G}/\mathcal{H})$-crossed product associated to a $\mathcal{G}$-homomorphism $f : M \longrightarrow E^*$ by $\mathrm{Alg}(f)$.

LEMMA 3.3. *Consider a sequence of $\mathcal{G}$-lattices of the form* (2.2). *Let $E$ be $\mathcal{G}$-field and $f : M \longrightarrow E^*$ be a homomorphism of $\mathcal{G}$-modules. If $k(f(M))$ is contained in a faithful $\mathcal{G}$-subfield $E_0$ of $E$ then* $\mathrm{Alg}(f)$ *is defined over $E_0^{\mathcal{G}}$.*

*Proof.* Since $f$ is the composition of $f_0 : M \longrightarrow E_0^*$ with the inclusion $E_0^* \hookrightarrow E^*$, Lemma 3.2 tells us that $A = \mathrm{Alg}(f_0) \bigotimes_{E_0^{\mathcal{G}}} E^{\mathcal{G}}$. □

REMARK 3.4. In the special case where the sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow \omega(\mathcal{G}/\mathcal{H}) \longrightarrow 0$$

is given by (2.4), $M = \omega(\mathcal{G}/\mathcal{H})^{\otimes 2}$ has a particularly convenient set of generators

$$y_{ijh} = (\overline{g_i} - \overline{g_j}) \otimes (\overline{g_j} - \overline{g_h}),$$

where $\mathcal{G}/\mathcal{H} = \{\overline{g_1}, \ldots, \overline{g_n}\}$ is the set of left cosets of $\mathcal{H}$ in $\mathcal{G}$ and $i$, $j$, $h$ range from 1 to $n = [\mathcal{G} : \mathcal{H}]$; cf. [**31**, Lemma 1.2]. If $f : \omega(\mathcal{G}/\mathcal{H})^{\otimes 2} \longrightarrow E^*$ is a $\mathcal{G}$-module homomorphism then the elements $c_{ijh} = f(y_{ijh})$ form a *reduced Brauer factor set* for $\mathrm{Alg}(f)$ in the sense of [**31**, p. 449]. Conversely, for any reduced Brauer factor set $(c_{ijh})$ in $E^*$, there exists a homomorphism $f : \omega(\mathcal{G}/\mathcal{H})^{\otimes 2} \longrightarrow E^*$ such that $f(y_{ijh}) = c_{ijh}$; see [**31**, Corollary 1.3]. Thus Lemma 3.3 takes the following form.

*Let $A$ be an $(E, \mathcal{G}/\mathcal{H})$-crossed product defined by a reduced Brauer factor set $(c_{ijh})$. Suppose that $(c_{ijh})$ is contained in a faithful $\mathcal{G}$-subfield $E_0$ of $E$. Then $A$ is defined over $E_0^{\mathcal{G}}$.*

The following theorem is a variant of [**34**, Theorem 12.11].

THEOREM 3.5. *Given the sequence* (2.2), *let $\mu : M \hookrightarrow k(M)^*$ be the natural inclusion. Then $D = \mathrm{Alg}(\mu)$ has the rational specialization property in the class of $\mathcal{G}/\mathcal{H}$-crossed products containing a copy of $k$ in their center. In particular, $\tau(A) \leqslant \mathrm{rank}(M)$ holds for any $\mathcal{G}/\mathcal{H}$-crossed product $A/F$ with $k \subset F$.*

*Proof.* Write $A = \mathrm{Alg}(f)$ for some $\mathcal{G}$-homomorphism $f : M \longrightarrow E^*$, where $E$ is a faithful $\mathcal{G}$-field with $E^{\mathcal{G}} = F$; see the remarks following Lemma 3.2. Furthermore, let $E(P)$ denote the fraction field of the group algebra $E[P]$, with the $\mathcal{G}$-action induced from the $\mathcal{G}$-actions on $E$ and $P$. By Proposition 2.4, there exists an $E$-isomorphism $j : E(P) \simeq E(\mathbf{t})$ of $\mathcal{G}$-fields, where $r = \mathrm{rank}(P)$ and $\mathbf{t} = (t_1, \ldots, t_r)$ is an $r$-tuple of indeterminates on which $\mathcal{G}$ acts trivially. Therefore, $E(P)^{\mathcal{G}} \simeq E^{\mathcal{G}}(\mathbf{t}) = F(\mathbf{t})$ is a rational extension of $F$. Let $f_{\mathbf{t}} : M \longrightarrow E(\mathbf{t})^*$ denote the composition of $f$ with the natural inclusion $E^* \hookrightarrow E(\mathbf{t})^*$. Then $\mathrm{Alg}(f_{\mathbf{t}}) = \mathrm{Alg}(f) \bigotimes_F F(\mathbf{t}) = A \bigotimes_F F(\mathbf{t})$. By Lemma 3.2, $\mathrm{Alg}(f_{\mathbf{t}}) \simeq \mathrm{Alg}(f_{\mathbf{t}} + g|_M)$ for any $g \in \mathrm{Hom}_{\mathcal{G}}(P, E(\mathbf{t})^*)$. Let $g$ be the composite $g : P \hookrightarrow E(P)^* \xrightarrow{\sim} E(\mathbf{t})^*$ and let $\varphi$ be the $\mathcal{G}$-module map $\varphi : M \longrightarrow E(\mathbf{t})^*$, $\varphi(m) = f_{\mathbf{t}}(m)g(m)$. We claim that $\varphi$ lifts to an embedding of $\mathcal{G}$-fields $k(M) \hookrightarrow E(\mathbf{t})$. Indeed, modulo $E^*$, $\varphi(m) \equiv g(m) \in P \subseteq E(\mathbf{t})^*$. Hence, $\{\varphi(m)\}_{m \in M}$ is an $E$-linearly independent subset of $E(\mathbf{t})$, and so the map $k[\varphi] : k[M] \longrightarrow E(\mathbf{t})$, $\sum_m k_m m \mapsto \sum_m k_m \varphi(m)$ is a $\mathcal{G}$-equivariant embedding of the group ring $k[M]$ into $E(\mathbf{t})$. This embedding lifts to an embedding of $\mathcal{G}$-fields $\phi : k(M) = Q(k[M]) \hookrightarrow E(\mathbf{t})$, as we have claimed. Thus $\phi \circ \mu = \varphi$, and hence $D \bigotimes_{k(M)^{\mathcal{G}}} F(\mathbf{t}) = \mathrm{Alg}(\phi \circ \mu) = \mathrm{Alg}(\varphi) \simeq \mathrm{Alg}(f_{\mathbf{t}}) = A \bigotimes_F F(\mathbf{t})$. This proves that $A$ is a rational specialization of $D$.

Lemmas 2.7(b) and 3.3 now imply that $\tau(A) \leqslant \tau(D) \leqslant \mathrm{trdeg}_k\, k(M)^{\mathcal{G}} = \mathrm{rank}(M)$. This completes the proof of the theorem. □

REMARK 3.6. Continuing with the notation used in Theorem 3.5, the rational specialization property of $D = \mathrm{Alg}(\mu)$ implies that $D$ is a division algebra of exponent $[\mathcal{G} : \mathcal{H}]$. Indeed, by [**5**, Appendix] there exists a $\mathcal{G}/\mathcal{H}$-crossed product division algebra of exponent $[\mathcal{G} : \mathcal{H}]$, and the above assertion can be proved by specializing $D$ to this algebra. Alternatively, the fact that $D$ is a division algebra of exponent $[\mathcal{G} : \mathcal{H}]$ can be checked directly by showing that the image of $\mu$ in $H^2(\mathcal{G}, k(M)^*)$ (see Lemma 3.2) has order $[\mathcal{G} : \mathcal{H}]$.

REMARK 3.7. The above construction applies in particular to sequences of the form (2.3). The following special type of sequence (2.3) has been particularly well explored. Write $\mathcal{G} = \langle \mathcal{H}, g_1, \ldots, g_r \rangle$ for suitable $g_i \in \mathcal{G}$; the minimal such $r$ is usually denoted by $d(\mathcal{G}/\mathcal{H})$. Then we can define an epimorphism of $\mathcal{G}$-lattices $f : \mathbb{Z}[\mathcal{G}]^r \twoheadrightarrow \omega(\mathcal{G}/\mathcal{H})$, $f(\alpha_1, \ldots, \alpha_r) = \sum_{i=1}^r \alpha_i \overline{(g_i - 1)}$, where $\overline{\ } : \mathbb{Z}[\mathcal{G}] \twoheadrightarrow \mathbb{Z}[\mathcal{G}/\mathcal{H}]$ is the canonical map; see [**18**, Lemma 3.1.1]. The kernel $R(\mathcal{G}/\mathcal{H}) = \mathrm{Ker}\, f$ is called a *relative relation module*; it has the following group theoretical description. Let $\mathcal{F}_r$ denote the free group on $r$ generators and consider the presentation

$$1 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F}_r * \mathcal{H} \longrightarrow \mathcal{G} \longrightarrow 1 \tag{3.1}$$

where $\mathscr{F}_r * \mathscr{H} \longrightarrow \mathscr{G}$ is the identity on $\mathscr{H}$ and sends the $r$ generators of $\mathscr{F}_r$ to the elements $g_1, \dots, g_r$. Then $R(\mathscr{G}/\mathscr{H}) \simeq \mathscr{R}^{\mathrm{ab}} = \mathscr{R}/[\mathscr{R}, \mathscr{R}]$, with $\mathscr{G}$ acting by conjugation; see [**11**] and [**9**] (for $\mathscr{H} = \{1\}$). Thus, we have the following version of sequence (2.3) with $M = \mathscr{R}^{\mathrm{ab}}$:

$$0 \longrightarrow \mathscr{R}^{\mathrm{ab}} \longrightarrow \mathbb{Z}[\mathscr{G}]^r \longrightarrow \omega(\mathscr{G}/\mathscr{H}) \longrightarrow 0. \tag{3.2}$$

When $\mathscr{H} = \{1\}$ and $r \geqslant 2$, the division algebra $D$ constructed via (3.2) in Theorem 3.5 is identical to the generic $\mathscr{G}$-crossed product of Snider [**37**]; see also Rosset [**24**]. Explicitly, we can state the following.

*Given a free presentation* $1 \longrightarrow \mathscr{R} \longrightarrow \mathscr{F}_r \longrightarrow \mathscr{G} \longrightarrow 1$ *of* $\mathscr{G}$ *with* $r \geqslant 2$, *let* $M = \mathscr{R}^{\mathrm{ab}} \leqslant \overline{\mathscr{F}_r} = \mathscr{F}_r/[\mathscr{R}, \mathscr{R}]$ *and let* $a \in H^2(\mathscr{G}, k(M)^*)$ *be the image of the extension class* $[1 \longrightarrow M \longrightarrow \overline{\mathscr{F}_r} \longrightarrow \mathscr{G} \longrightarrow 1] \in H^2(\mathscr{G}, M)$ *under the natural inclusion* $\mu : M \hookrightarrow k(M)^*$. *Then* $D = \mathrm{Alg}(\mu)$ *is the* $\mathscr{G}$*-crossed product* $(k(M), \mathscr{G}, a)$ *or, equivalently, the localization of the group algebra* $k[\overline{\mathscr{F}_r}]$ *at the nonzero elements of* $k[M]$.

COROLLARY 3.8. *Let* $A$ *be a* $\mathscr{G}/\mathscr{H}$*-crossed product and let* $d_{\mathscr{G}}(\omega(\mathscr{G}/\mathscr{H}))$ *be the minimal number of generators of* $\omega(\mathscr{G}/\mathscr{H})$ *as a* $\mathscr{G}$*-module. Then*

$$\tau(A) \leqslant r|\mathscr{G}| - [\mathscr{G} : \mathscr{H}] + 1,$$

*where*

$$r = \begin{cases} d_{\mathscr{G}}(\omega(\mathscr{G}/\mathscr{H})) & \text{if } \mathscr{H} \neq \{1\} \\ \max\{2, d_{\mathscr{G}}(\omega(\mathscr{G}/\mathscr{H}))\} & \text{if } \mathscr{H} = \{1\}. \end{cases}$$

*Proof.* Applying Theorem 3.5 to the exact sequence (2.3), we obtain

$$\tau(A) \leqslant \mathrm{rank}(M) = \mathrm{rank}(\mathbb{Z}[G]^r) - \mathrm{rank}(\omega(\mathscr{G}/\mathscr{H})) = r|\mathscr{G}| - [\mathscr{G} : \mathscr{H}] + 1,$$

as claimed. Note that for $r$ as above, Lemma 2.1 tells us that $M$ is faithful, so that Theorem 3.5 is, indeed, applicable. □

REMARK 3.9. As we pointed out in Remark 3.7, $d_{\mathscr{G}}(\omega(\mathscr{G}/\mathscr{H})) \leqslant d(\mathscr{G}/\mathscr{H})$. The difference $\mathrm{pr}(\mathscr{G}/\mathscr{H}) = d(\mathscr{G}/\mathscr{H}) - d_{\mathscr{G}}(\omega(\mathscr{G}/\mathscr{H})) \geqslant 0$ can be arbitrarily large, even if $\mathscr{H} = \{1\}$. In this case $d(\mathscr{G}) = d(\mathscr{G}/\{1\})$ is the minimal number of generators of $\mathscr{G}$, and $\mathrm{pr}(\mathscr{G}) = d(\mathscr{G}) - d_{\mathscr{G}}(\omega\mathscr{G})$ is usually called the *presentation rank* or *generation gap* of $\mathscr{G}$. All solvable groups $\mathscr{G}$ have presentation rank $\mathrm{pr}(\mathscr{G}) = 0$; see [**9**, Lectures 6 and 7]. Moreover, if the derived subgroup $[\mathscr{G}, \mathscr{G}]$ is nilpotent then $\mathrm{pr}(\mathscr{G}/\mathscr{H}) = 0$ holds for every subgroup $\mathscr{H}$ of $\mathscr{G}$; see [**11**].

COROLLARY 3.10. (a) *Suppose that a group* $\mathscr{G}$ *of order* $n$ *can be generated by* $r \geqslant 2$ *elements. Then* $\tau(A) \leqslant (r-1)n+1$ *for any* $\mathscr{G}$*-crossed product central simple algebra* $A$.

(b) $\tau(A) \leqslant (\lfloor \log_2(n) \rfloor - 1)n + 1$ *holds for any crossed product central simple algebra* $A$ *of degree* $n \geqslant 4$.

Here, as usual, $\lfloor x \rfloor$ denotes the largest integer $\leqslant x$.

*Proof of Corollary* 3.10. (a) is an immediate consequence of Corollary 3.8. (b) follows from (a), because any group of order $n$ can be generated by $r \leqslant \log_2(n)$ elements. (Indeed, $|\langle \mathscr{G}_0, g \rangle| \geqslant 2|\mathscr{G}_0|$ for any subgroup $\mathscr{G}_0$ of $\mathscr{G}$ and any $g \in \mathscr{G} \setminus \mathscr{G}_0$.) Note also that $\lfloor \log_2(n) \rfloor \geqslant 2$ for any $n \geqslant 4$. □

## 4. *Proof of Theorem* 1.1

For the next two sections we shall assume that $\mathscr{G} = \mathscr{S}_n$ and $\mathscr{H} = \mathscr{S}_{n-1}$. We will use the following standard notations for $\mathscr{S}_n$-lattices:

$$\mathbb{Z}[\mathscr{S}_n/\mathscr{S}_{n-1}] = U_n \quad \text{and} \quad \omega(\mathscr{S}_n/\mathscr{S}_{n-1}) = A_{n-1}. \tag{4.1}$$

The natural generators of $U_n$ will be denoted by $u_1, \dots, u_n$; the symmetric group $\mathscr{S}_n$ permutes them via $\sigma(u_i) = u_{\sigma(i)}$. $A_{n-1}$ is the sublattice of $U_n$ generated by $u_i - u_1$ as $i$ ranges from 2 to $n$.

Recall that the universal division algebra $\mathrm{UD}(n)$ is generated, as a $k$-division algebra, by a pair of generic $n \times n$-matrices $X$ and $Y$. We may assume without loss of generality that $X$ is diagonal. Following [**28**] we will denote the diagonal entries of $X$ by $\zeta'_{ii}$ and the entries of $Y$ by $\zeta_{ij}$, where $\zeta'_{ii}$ and $\zeta_{ij}$ are algebraically independent variables over $k$. The group $\mathscr{S}_n$ permutes these variables as follows:

$$\sigma(\zeta'_{ii}) = \zeta'_{\sigma(i)\sigma(i)} \quad \text{and} \quad \sigma(\zeta_{ij}) = \zeta_{\sigma(i)\sigma(j)}.$$

We identify the multiplicative group generated by $\zeta'_{ii}$ with the $\mathscr{S}_n$-lattice $U_n$ (via $\zeta'_{ii} \leftrightarrow u_i$), and the multiplicative group generated by $\zeta_{ij}$ with $U_n \otimes U_n$ (via $\zeta_{ij} \leftrightarrow u_i \otimes u_j$). Consider the exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow U_n \oplus U_n^{\otimes 2} \xrightarrow{f} A_{n-1} \longrightarrow 0 \tag{4.2}$$

of $\mathscr{S}_n$-lattices, where $f(u_i, u_j \otimes u_h) = u_j - u_h$. This sequence is the sequence (2.4) of Lemma 2.2 for $\mathscr{G} = \mathscr{S}_n$ and $\mathscr{H} = \mathscr{S}_{n-1}$, with two extra copies of $U_n$ added; the second copy of $U_n$ is the sublattice of $U_n^{\otimes 2}$ that is spanned by all elements $u_i \otimes u_i$. Both copies of $U_n$ belong to $\mathrm{Ker}(f)$; in fact,

$$\mathrm{Ker}(f) = U_n \oplus U_n \oplus A_{n-1}^{\otimes 2},$$

where $A_{n-1}^{\otimes 2}$ is identified with the sublattice of $U_n^{\otimes 2}$ that is spanned by all elements $(u_i - u_j) \otimes (u_l - u_m)$.

Let $E = k(\mathrm{Ker}(f))$ and $F = E^{\mathscr{S}_n}$. By a theorem of Formanek and Procesi, $F$ is naturally isomorphic to the center $Z(n)$ of $\mathrm{UD}(n)$; see [**6**, Theorem 3]. Note that $E = F(\zeta'_{11}, \dots, \zeta'_{nn})$ is generated over $F$ by the eigenvalues of the generic matrix $X$. Consequently, $\mathrm{UD}(n)$ is an $(E, \mathscr{S}_n/\mathscr{S}_{n-1})$-product, and $E^{\mathscr{S}_{n-1}}$ is isomorphic to the maximal subfield $Z(n)(X)$ of $\mathrm{UD}(n)$; see [**20**, Section II.1].

Theorem 1.1 is now a consequence of the following.

PROPOSITION 4.1. *Suppose that $n \geqslant 5$ is odd. Then*
(a) $\mathrm{UD}(n)$ *is defined over* $F_0 = k(\bigwedge^2 A_{n-1})^{\mathscr{S}_n}$;
(b) $Z(n) = k(\mathrm{Ker}(f))^{\mathscr{S}_n}$ *is rational over* $F_0 = k(\bigwedge^2 A_{n-1})^{\mathscr{S}_n}$.

Here, we view $\bigwedge^2 A_{n-1}$ as the sublattice of antisymmetric tensors in $A_{n-1}^{\otimes 2}$, that is, the $\mathbb{Z}$-span of all $a \wedge a' = a \otimes a' - a' \otimes a$ with $a, a' \in A_{n-1}$.

*Proof of Proposition* 4.1. We will deduce part (a) from Remark 3.4 by constructing a reduced Brauer factor set contained in $E_0 = k(\bigwedge^2 A_{n-1})$. First we note that the $\mathscr{S}_n$-action on $E_0$ is faithful, because $\bigwedge^2 A_{n-1}$ is a faithful $\mathscr{S}_n$-lattice for every $n \geqslant 4$. (Indeed, $\bigwedge^2 A_{n-1} \otimes \mathbb{Q}$ is the simple $\mathscr{S}_n$-representation corresponding to the partition $(n-2, 1^2)$ of $n$; cf. [**8**, Exercise 4.6].)

We now proceed with the construction of the desired Brauer factor set. The computation in [**28**, Section 2] shows that the elements

$$c_{ijh} = \zeta_{ij}\zeta_{jh}\zeta_{ih}^{-1} \in E^*$$

form a Brauer factor set for UD($n$). By [**28**, Theorem 4], if $n$ is odd, UD($n$) has a normalized (and, in particular, reduced) Brauer factor set $(c'_{ijh})$ given by

$$c'_{ijh} = (c_{ijh}/c_{hji})^{(n+1)/2} = \left(\zeta_{ij}\zeta_{ji}^{-1}\zeta_{jh}\zeta_{hj}^{-1}\zeta_{hi}\zeta_{ih}^{-1}\right)^{(n+1)/2}.$$

Now observe that $\zeta_{ij}\zeta_{ji}^{-1}\zeta_{jh}\zeta_{hj}^{-1}\zeta_{hi}\zeta_{ih}^{-1}$ is precisely the element of $U_n^{\otimes 2}$ we identified with $(u_i - u_j) \wedge (u_j - u_h)$. Thus every $c'_{ijh}$ lies in $\bigwedge^2 A_{n-1} \subset E_0$, as desired.

(b) The canonical exact sequence

$$0 \longrightarrow \bigwedge^2 A_{n-1} \longrightarrow A_{n-1}^{\otimes 2} \longrightarrow \mathsf{Sym}^2 A_{n-1} \longrightarrow 0$$

of $\mathscr{S}_n$-lattices gives rise to an exact sequence

$$0 \longrightarrow \bigwedge^2 A_{n-1} \longrightarrow A_{n-1}^{\otimes 2} \oplus U_n \oplus \mathbb{Z} \longrightarrow Q \longrightarrow 0,$$

where we have put $Q = \mathsf{Sym}^2 A_{n-1} \oplus U_n \oplus \mathbb{Z}$. The crucial fact here is that, by [**15**, Section 3.5], if $n$ is odd, $Q$ is a permutation lattice. Applying Proposition 2.4 to the extension of (faithful) $\mathscr{S}_n$-fields $E_0 = k(\bigwedge^2 A_{n-1}) \subseteq k(A_{n-1}^{\otimes 2} \oplus U_n \oplus \mathbb{Z}) \cong (E_0)_\gamma(Q)$, where $\gamma$ is the image of the class of the above extension in $\mathrm{Ext}_{\mathscr{S}}(Q, E_0^*)$, we conclude that

$$k\left(A_{n-1}^{\otimes 2} \oplus U_n \oplus \mathbb{Z}\right) \simeq k(\bigwedge^2 A_{n-1})(x_1, \ldots, x_m)$$

as $\mathscr{S}_n$-fields, where $m = n(n+1)/2 + 1$ and $\mathscr{S}_n$ acts trivially on the $x_i$. Similarly, putting $L_n = A_{n-1}^{\otimes 2} \oplus U_n^2$, the obvious sequence $0 \longrightarrow A_{n-1}^{\otimes 2} \oplus U_n \longrightarrow L_n \longrightarrow U_n \longrightarrow 0$ leads to $k(L_n) \simeq k(A_{n-1}^{\otimes 2} \oplus U_n)(t_1, \ldots, t_n)$ as $\mathscr{S}_n$-fields. Therefore,

$$\begin{aligned} k(L_n) &\simeq k\left(A_{n-1}^{\otimes 2} \oplus U_n\right)(t_1, \ldots, t_n) \\ &= k\left(A_{n-1}^{\otimes 2} \oplus U_n \oplus \mathbb{Z}\right)(t_1, \ldots, t_{n-1}) \\ &\simeq k(\bigwedge^2 A_{n-1})(x_1, \ldots, x_m, t_1, \ldots, t_{n-1}) \end{aligned}$$

as $\mathscr{S}_n$-fields, which implies that

$$Z(n) \simeq k(L_n)^{\mathscr{S}_n} \simeq k(\bigwedge^2 A_{n-1})^{\mathscr{S}_n}(x_1, \ldots, x_m, t_1, \ldots, t_{n-1});$$

so $Z(n)$ is rational over $F_0 = k(\bigwedge^2 A_{n-1})^{\mathscr{S}_n}$. □

## 5. *Proof of Theorem* 1.2

### 5.1. *Proof of part* (a)

REDUCTION 5.1. *Suppose an algebra $A_0$ of degree n has the rational specialization property* (*see Section* 2.4). *If Theorem* 1.2(a) *holds for $A_0$ then it holds for any central simple algebra $A$ of degree n.*

*Proof.* Suppose that for some $r \geqslant 1$, $A_0(t_1, \ldots, t_r)$ is defined over a rational extension $F_0$ of $k$. Let $A/F$ be an arbitrary central simple algebra of degree $n$. Then by the rational specialization property, $A_0$ embeds in $A(t_{r+1}, \ldots, t_s)$ for some $s \gg 0$; thus $A_0(t_1, \ldots, t_r)$ embeds in $A(t_1, \ldots, t_s)$. This shows that $A(t_1, \ldots, t_s)$ is defined over $F_0$, as desired. □

In particular, in proving Theorem 1.2(a), we may assume that $A$ is a division algebra of degree $n$; see Example 2.6. By primary decomposition (cf., for example, [**19**, p. 261]), we only need to consider the cases where $n = 2$, $n = 4$ and $n$ is an odd prime. This follows from the next reduction.

REDUCTION 5.2. *If the conclusion of Theorem* 1.2(a) *holds for central simple algebras* $A_1/F$ *and* $A_2/F$ (*for every choice of the base field* $k \subset F$) *then it also holds for* $A = A_1 \bigotimes_F A_2$.

*Proof.* After replacing $A_1$ and $A_2$ by, respectively, $A_1(t_1, \dots, t_s)$ and $A_2(t_1, \dots, t_s)$, we may assume that $A_1$ is defined over a subfield $F_1 \subset F$ such that $k \subset F_1$ and $F_1$ is rational over $k$. We will now think of $F_1$ (rather than $k$) as our new base field. After adding more indeterminates, we may assume that $A_2$ is defined over a subfield $F_2 \subset F$, where $F_1 \subset F_2$ and $F_2$ is rational over $F_1$. Now $F_2$ is rational over $k$, and since $A_1$ and $A_2$ are both defined over $F_2$, so is $A$. □

We are now ready to complete the proof of Theorem 1.2(a).

First, suppose that $n = 2$ or 4. Since UD($n$) has the rational specialization property, we may assume that $A = \mathrm{UD}(n)$; see Reduction 5.1. Since the center of UD($n$) is known to be rational for $n = 2$ (see [**20**, Theorem 2.2]) and $n = 4$ (see [**7**]), these algebras clearly satisfy the conclusion of Theorem 1.2(a). This completes the proof of the theorem for $n = 2$, 4. We remark that the same argument goes through for $n = 3$ (because the center of UD(3) is known to be rational; see [**6**]) and for $n = 5, 7$ (because the centers of UD(5) and UD(7) are known to be stably rational; see [**4**]), but we shall not need it in these cases.

From now on we will assume that $n = p$ is an odd prime. Then the $\mathscr{S}_n$-lattice $A_{n-1}^{\otimes 2}$ is faithful; see Lemma 2.2. Furthermore, by a theorem of Bessenrodt and LeBruyn [**4**, Proposition 3] (see also [**3**, Lemma 2.8] for a more explicit form of this result), $A_{n-1}^{\otimes 2}$ is permutation projective, that is, there exists an $\mathscr{S}_n$-lattice $L$ such that $P = A_{n-1}^{\otimes 2} \oplus L$ is a permutation. We can assume that $k(P)^{\mathscr{S}_n}$ is rational over $k$. Indeed, after adding a copy of $U_n$ if necessary, we have $P = U_n \oplus Q$ for some permutation lattice $Q$, and so $k(P) \simeq k(U_n)(Q)$. Proposition 2.4 implies that $k(P)^{\mathscr{S}_n}$ is rational over $k(U_n)^{\mathscr{S}_n}$, which in turn is rational over $k$.

Let

$$i : A_{n-1}^{\otimes 2} \hookrightarrow k\big(A_{n-1}^{\otimes 2}\big)^*$$

and

$$j : A_{n-1}^{\otimes 2} \hookrightarrow k\big(A_{n-1}^{\otimes 2} \oplus P\big)^*$$

be the natural embeddings of $\mathscr{S}_n$-modules. (Here, $j$ identifies $A_{n-1}^{\otimes 2}$ with the first summand of $A_{n-1}^{\otimes 2} \oplus P$.) Recall that by Lemma 3.2 these embeddings, in combination with the exact sequence (2.4) (for $\mathscr{G} = \mathscr{S}_n$ and $\mathscr{H} = \mathscr{S}_{n-1}$; see (4.1)), give rise to central simple algebras Alg($i$) and Alg($j$).

By Theorem 3.5, Alg($i$) has the rational specialization property in the class of $\mathscr{S}_n/\mathscr{S}_{n-1}$-crossed products. Thus, the universal division algebra UD($n$), being an $\mathscr{S}_n/\mathscr{S}_{n-1}$-crossed product (see Example 3.1), is a rational specialization of Alg($i$). Since UD($n$) has the rational specialization property in the class of all central simple algebras of degree $n$ (see Example 2.6), so does Alg($i$).

We claim that $\mathrm{Alg}(j)$ also has the rational specialization property in the class of central simple algebras of degree $n$. Indeed, by Lemma 3.2,

$$\mathrm{Alg}(j) = \mathrm{Alg}(i) \bigotimes_{F^{\mathscr{G}_n}} E^{\mathscr{G}_n},$$

where $F = k(A_{n-1}^{\otimes 2})$ and $E = k(A_{n-1}^{\otimes 2} \oplus P)$. Now Proposition 2.4 tells us that $E^{\mathscr{G}_n}$ is a rational extension of $F^{\mathscr{G}_n}$, and the claim follows.

By Reduction 5.1 it now suffices to prove that $\mathrm{Alg}(j)$ is defined over a purely transcendental extension of $k$. Put $E_0 = k(A_{n-1}^{\otimes 2} \oplus (0) \oplus L) \subseteq E$. Since the image of $j$ is contained in $E_0^*$, Lemma 3.3 tells us that $\mathrm{Alg}(j)$ is defined over $E_0^{\mathscr{G}_n}$; but $E_0 \simeq k(P)$ and so $E_0^{\mathscr{G}_n} \simeq k(P)^{\mathscr{G}_n}$ which is indeed rational over $k$. This completes the proof of Theorem 1.2(a).

### 5.2. *Proof of part* (b)

By the Merkurjev–Suslin theorem,

$$\mathrm{M}_r(A) = (a_1, b_1)_{n_1} \bigotimes_F \cdots \bigotimes_F (a_l, b_l)_{n_l},$$

for some $r, l \geqslant 1$, where $(a, b)_n$ denotes a symbol algebra; see (1.4).

Let $\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_l$ be $2l$ central variables, algebraically independent over $F$. We will write $\lambda$ in place of $(\lambda_1, \ldots, \lambda_l)$ and $\mu$ in place of $(\mu_1, \ldots, \mu_l)$. Then

$$\begin{aligned}
\mathrm{M}_r(A)(\lambda, \mu) &= (a_1, b_1)_{n_1} \bigotimes_{K(\lambda,\mu)} \cdots \bigotimes_{K(\lambda,\mu)} (a_l, b_l)_{n_l} \\
&= (a_1', b_1')_{n_1} \bigotimes_{K(\lambda,\mu)} \cdots \bigotimes_{K(\lambda,\mu)} (a_l', b_l')_{n_l} \\
&= \big((a_1', b_1')_{n_1} \bigotimes_{F_0} \cdots \bigotimes_{F_0} (a_l', b_l')_{n_l}\big) \otimes_{F_0} K(\lambda, \mu),
\end{aligned}$$

where $a_i' = a_i \lambda_i^{n_i}$ and $b_i' = b_i \mu_i^{n_i}$ for $i = 1, \ldots, l$ and $F_0 = k(a_1', b_1', \ldots, a_l', b_l')$. This shows that $\mathrm{M}_r(D)(\lambda, \mu)$ is defined over $F_0$. It remains to prove that $F_0$ is rational over $k$. The $2l$ elements $a_1', b_1', \ldots, a_l', b_l'$ are clearly algebraically independent over $F$. Hence, they are algebraically independent over $k$, and consequently, $F_0$ is rational over $k$, as claimed.

REMARK 5.3. Our proof of Theorem 1.2 can be used to deduce explicit lower bounds on $s$ in parts (a) and (b) from explicit lower bounds in theorems of Bessenrodt–LeBruyn [**4**, Proposition 3] (on $\mathrm{rank}(L)$) and Merkurjev–Suslin (on $r$). The lowest possible value of $r$ in part (b), called the *Merkurjev–Suslin number*, is of independent interest; see [**29**, Section 7.2].

## 6. *Proof of Theorem* 1.3

REDUCTION 6.1. *In the course of proving Theorem* 1.3*, we may assume without loss of generality that $A$ is a division algebra.*

Indeed, let $D = \mathrm{Alg}(\mu)$, as in Theorem 3.5, with $\mathscr{G} = \mathbb{Z}/m \times \mathbb{Z}/2$, and $\mathscr{H} = \{1\}$. Then $D$ is a division algebra (see Remark 3.6), and any other $\mathscr{G}$-crossed product $A/F$ is a rational specialization of $D$. Thus, if we know that Theorem 1.3 holds for $D$

then it holds for $A(t_1,\dots,t_s)$, where $t_1,\dots,t_s$ are independent variables over $F$. Using induction on $s$, we see that Reduction 6.1 is now a consequence of the following lemma (applied to $B = \mathrm{M}_m(A)$, with $r = 2$, $m_1 = m$ and $m_2 = 2m$).

LEMMA 6.2. *Let $B/K$ be a central simple algebra of degree $d = m_1 \dots m_r$ and let $t$ be an independent variable over $K$. Assume that $K$ contains a primitive root of unity of degree* $\mathrm{lcm}(m_1,\dots,m_r)$. *If*

$$B(t) = (a_1(t), b_1(t))_{m_1} \otimes \dots \otimes (a_r(t), b_r(t))_{m_r}$$

*for some $a_1(t), b_1(t), \dots, a_r(t), b_r(t) \in K(t)$ then*

$$B = (a_1', b_1')_{m_1} \otimes \dots \otimes (a_r', b_r')_{m_r}$$

*for some $a_1', b_1', \dots, a_r', b_r' \in K$.*

Our proof is based on a standard specialization argument; for the sake of completeness, we supply the details below.

*Proof of Lemma* 6.2. We may assume that $K$ is an infinite field; otherwise $B$ is a matrix algebra over $K$, and we can take, for example, $a_i' = 1$, $b_i' = -1$ for every $i$.

Choose generators $x_i(t)$ and $y_i(t)$ for the cyclic subalgebra $(a_i(t), b_i(t))_{m_i}$ of $B(t) = B \bigotimes_K K(t)$ such that $x_i(t)^{m_i} = a_i(t)$, $y_i(t)^{m_i} = b_i(t)$, and $x_i(t)y_i(t) = \zeta_{m_i} y_i(t)x_i(t)$, where $\zeta_{m_i}$ is a primitive root of unity of degree $m_i$ in $K$. Choose a $K$-basis $b_1,\dots,b_{d^2}$ of $B$ and write

$$x_i(t) = \sum_{j=1}^{d^2} \alpha_{ij}(t) b_i \quad \text{and} \quad y_i(t) = \sum_{j=1}^{d^2} \beta_{ij}(t) b_i, \tag{6.1}$$

for some $\alpha_{ij}(t), \beta_{ij}(t) \in K(t)$. Since $K$ is an infinite field, we can choose $t_0 \in K$ such that $\alpha_{ij}(t_0)$ and $\beta_{ij}(t_0)$ are well-defined and

$$x_i(t_0) = \sum_{j=1}^{d^2} \alpha_{ij}(t_0) b_i \quad \text{and} \quad y_i(t_0) = \sum_{j=1}^{d^2} \beta_{ij}(t_0) b_i$$

are nonzero. Let $B_i$ denote the subalgebra of $B$ that is generated by $x_i(t_0)$ and $y_i(t_0)$. Then $B_i = (a_i', b_i')_{m_i}$, where $a_i' = x_i(t_0)^{m_i} = a_i(t_0)$ and $b_i' = y_i(t_0)^{m_i} = b_i(t_0)$, and $B_1,\dots,B_r$ are commuting subalgebras of $B$ of degrees $m_1,\dots,m_r$. Hence, by the double centralizer theorem (cf., for example, [**19**, Theorem 12.7]), $B = B_1 \otimes \dots \otimes B_r$. This completes the proof of Lemma 6.2 and thus of Reduction 6.1. □

We now continue with the proof of Theorem 1.3. In the course of the proof we shall use the following notations. Write $\mathscr{G} = \mathbb{Z}/m \times \mathbb{Z}/2 = \langle \sigma_1, \sigma_2 \rangle$, where $\sigma_1^m = \sigma_2^2 = 1$. Let $K = F(\alpha_1, \alpha_2)$, be a maximal $\mathscr{G}$-Galois subfield of $A$, where $\alpha_1^m = a_1$ and $\alpha_2^2 = a_2$ are elements of $F$, and

$$\begin{aligned} &\sigma_1(\alpha_1) = \zeta_m \alpha_1, \quad &&\sigma_1(\alpha_2) = \alpha_2, \\ &\sigma_2(\alpha_1) = \alpha_1, \quad &&\sigma_2(\alpha_2) = -\alpha_2. \end{aligned} \tag{6.2}$$

Here $\zeta_m \in F$ is a primitive $m$th root of unity, so that $K$ is, indeed, a $\mathscr{G}$-Galois extension of $F$. Note that the statement of Theorem 1.3 assumes that $F$ contains not only a primitive $m$th root of unity $\zeta_m$ but also a primitive $2m$th root of unity $\zeta_{2m}$; we shall make use of $\zeta_{2m}$ later in the proof.

By the Skolem–Noether theorem, there exist units $z_1, z_2 \in A$ such that $z_i x z_i^{-1} = \sigma_i(x)$ for every $x \in K$ $(i = 1, 2)$. Set

$$z_1^m = b_1 \in F(\alpha_2)^*, \quad z_2^2 = b_2 \in F(\alpha_1)^*, \quad \text{and} \quad u = z_1 z_2 z_1^{-1} z_2^{-1} \in K^*. \tag{6.3}$$

By [**2**, Theorem 1.3], the algebra structure of $A$ can be recovered from the $\mathcal{G}$-field $K$ and the elements $u \in K^*$, $b_1 \in F(\alpha_2)^*$ and $b_2 \in F(\alpha_1)^*$. (These elements have to satisfy certain compatibility conditions; the exact form of these conditions shall not concern us in the sequel.) We will write $A = (K, \mathcal{G}, u, b_1, b_2)$.

LEMMA 6.3. *Let $A = (K, \mathcal{G}, u, b_1, b_2)$ and $A' = (K, \mathcal{G}, u', b_1', b_2')$ be $\mathcal{G}$-crossed products. Then $A \bigotimes_F A'$ is Brauer equivalent to $(K, \mathcal{G}, uu', b_1 b_1', b_2 b_2')$.*

*Proof.* The class of $A = (K, \mathcal{G}, u, b_1, b_2)$ in the relative Brauer group $\mathbf{B}(K/F) = H^2(\mathcal{G}, K^*)$ is given by a normalized 2-cocycle $a : \mathcal{G} \times \mathcal{G} \longrightarrow K^*$ so that

$$b_i = a(\sigma_i, \sigma_i) a\big(\sigma_i^2, \sigma_i\big) \ldots a\big(\sigma_i^{m_i - 1}, \sigma_i\big)$$

holds for $i = 1, 2$, where $m_1 = m$ and $m_2 = 2$, and

$$u = a(\sigma_1, \sigma_2) a(\sigma_s, \sigma_1)^{-1}.$$

Similarly, the class of $A'$ is given by a 2-cocycle $a'$. Then the class of $A \bigotimes_F A'$ is given by the cocycle $aa'$; see, for example, [**19**, Proposition 14.3]. This proves the lemma.

The following alternative ring-theoretic argument was suggested by the referee. Choose $z_1, z_2 \in A$, as in (6.3), and similarly for $z_1'$, $z_2'$ in $A'$. The subalgebra $S$ of $A \bigotimes_F A'$ generated by $K \otimes 1$, $z_1 \otimes z_1'$ and $z_2 \otimes z_2'$, is clearly isomorphic to $(K, \mathcal{G}, uu', b_1 b_1', b_2 b_2')$. Its centralizer $C_{A \otimes A'}(S)$ is an $F$-central simple algebra of degree $2m = [K : F]$, containing $(K \otimes K)^{\mathcal{G}}$, where $\mathcal{G}$ acts diagonally on $K \otimes K$. Since $\mathcal{G}$ is abelian, $(K \otimes K)^{\mathcal{G}} \simeq F \oplus \ldots \oplus F$, as an $F[\mathcal{G}]$-algebra. In particular, $C_{A \otimes A'}$ contains the idempotents of $K \otimes K$ and, hence, is split over $F$. We thus conclude that

$$A \bigotimes_F A' \simeq S \bigotimes_F C_{A \otimes_F A'}(S) \sim S \simeq (K, \mathcal{G}, uu', b_1 b_1', b_2 b_2'),$$

as claimed. (Here $\sim$ denotes Brauer equivalence over $F$.) □

We now proceed with the proof of Theorem 1.3, using the notations of (6.2) and (6.3). Since $b_1 = z_1^m \in K^{\sigma_1} = F(\alpha_2)$, we can write

$$b_1 = f_1 + f_2 \alpha_2, \tag{6.4}$$

for some $f_1, f_2 \in F$.

LEMMA 6.4. (a) *If $f_1 = 0$ then $A$ is cyclic.*
(b) *If $f_2 = 0$ then $A = (a, b)_m \otimes (c, d)_2$, for some $a, b, c, d \in F^*$.*

*Proof.* (a) If $f_1 = 0$ then $z_1^{2m} = b_1^2 = f_2^2 a_2 \in F^*$ but $z_1^m = f_2 \alpha_2 \notin F$. Since $F$ contains a primitive root of unity of degree $2m$, $F(z_1)$ is a cyclic maximal subfield of $A$ of degree $2m$; cf. [**13**, Theorem VIII.6.10(b)]. Thus $A$ is a cyclic algebra, as claimed.

(b) If $f_2 = 0$, that is, $b_1 \in F$, then the $F$-subalgebra $A_0$ of $A$ generated by $z_1$ and $\alpha_1$ is cyclic of degree $m$. By the double centralizer theorem, $A = A_m \otimes Q$, where $Q$ is a quaternion algebra, as claimed. □

We are now ready to finish the proof of Theorem 1.3. Lemma 6.4 tells us that Theorem 1.3 is immediate if $f_1 = 0$ or $f_2 = 0$. Thus from now on we shall assume that $f_1 f_2 \neq 0$.

Now let $A = (K, \mathscr{G}, u, b_1, b_2)$ and, for any $f \in F^*$, define $A_f = (K, \mathscr{G}, u, fb_1, b_2)$. Since $(a_1, f)_m \bigotimes_F \mathrm{M}_2(F) \simeq (K, \mathscr{G}, 1, f, 1)$, Lemma 6.3 tells us that $A_f \sim (a_1, f)_m \bigotimes_F A$, where $\sim$ denotes Brauer equivalence. In other words, $A \sim (f, a_1)_m \bigotimes_F A_f$. Thus it is enough to show that $A_f$ is cyclic, for some $f \in F^*$.

To prove the last assertion, observe that if we expand $(z_1 + \alpha_1)^m$ then all terms, other than $z_1^m$ and $\alpha_1^m$, will cancel. (For a simple proof of this fact, due to Bergman, see [**29**, p. 195]). Thus, if $\gamma = z_1 + \alpha_1$ then

$$\gamma^m = z_1^m + \alpha_1^m = fb_1 + a_1$$

in $A_f$. Setting $f = -a_1/f_1 \in F^*$, we obtain $\gamma^m = c\alpha_2$, where $c = -a_1 f_2/f_1 \in F^*$; see (6.4). Thus $\gamma^{2m} = c^2 a_2 \in F^*$ but $\gamma^m \notin F$. Since $F$ contains a primitive $2m$th root of unity, $F(\gamma)/F$ is a cyclic field extension of degree $2m$. In other words, $F(\gamma)$ is a cyclic maximal subfield of $A_f$, and $A_f$ is a cyclic algebra of degree $2m$, as claimed.

## 7. *The field of definition of a quadratic form*

### 7.1. *Preliminaries*

Let $V = F^n$ be an $F$-vector space, equipped with a quadratic form $q : V \longrightarrow F$. Recall that $q$ is said to be defined over a subfield $F_0$ of $F$ if $q = q_{F_0} \otimes F$, where $q_{F_0}$ is a quadratic form on $V_0 = F_0^n$. Is easy to see that $q$ is defined over $F_0$ if and only if $V$ has an $F$-basis $e_1, \ldots, e_n$ such that $b(e_i, e_j) \in F_0$, where $b : V \times V \longrightarrow F$ is the symmetric bilinear form associated to $q$ (that is, $q(v) = b(v, v)$).

We shall always assume that $\mathrm{char}(F) \neq 2$ and $F$ (and $F_0$) contain a base subfield $k$. As usual, we shall write $\langle a_1, \ldots, a_n \rangle$ for the diagonal form

$$a_n x_1^2 + \ldots + a_n x_n^2$$

and $\langle\langle a_1, \ldots, a_n \rangle\rangle$ for the Pfister form $\langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_n \rangle$. Given a quadratic form $q$ we shall ask the following questions.

(a) What is the smallest value of $\mathrm{trdeg}_k(F_0)$, where $q$ is defined over $F_0$? We shall denote this number by $\tau(q)$.

(b) Can $q$ be defined over a rational extension $F_0$ of $k$?

These are the same questions we asked for central simple algebras in the Introduction. In the case of quadratic forms our answers are more complete (and the proofs are easier).

PROPOSITION 7.1. *Let $V = F^n$ and let $q : V \longrightarrow F$ be a quadratic form on $V$. Then the following hold.*

(a) $\tau(q) \leqslant n$. *Moreover, if $a_1, \ldots, a_n$ are independent variables over $k$, $F = k(a_1, \ldots, a_n)$, and $q = \langle a_1, \ldots, a_n \rangle$ then $\tau(q) = n$.*

(b) *Let $t_1, \ldots, t_n$ be independent variables over $F$. Then $q' = q \bigotimes_F F(t_1, \ldots, t_n)$ is defined over a rational extension $F_0$ of $k$.*

*Proof.* Diagonalizing $q$, write $q = \langle a_1, \dots, a_n \rangle$ in the basis $e_1, \dots, e_n$.

(a) To prove the first assertion, set $F_0 = k(a_1, \dots, a_n)$. Then $q$ is defined over $F_0 = k(a_1, \dots, a_n)$ and $\operatorname{trdeg}_k(F_0) \leqslant n$, as desired. For the proof of the second assertion see [**22**, Proof of Theorem 10.3].

(b) Set $a_i' = t_i^2 a_i$. Then $q' = \langle a_1, \dots, a_n \rangle = \langle a_1', \dots, a_n' \rangle$ over $F(t_1, \dots, t_n)$. Hence, $q'$ is defined over $F_0 = k(a_1', \dots, a_n')$. We claim that $F_0$ is rational over $k$. Indeed, since the nonzero elements of $\{a_1', \dots, a_n'\}$ are algebraically independent over $F$, they are algebraically independent over $k$, and the claim follows. □

In the sequel we shall need the following analogue of Lemma 2.7.

LEMMA 7.2. *Let $q$ be a quadratic form defined over $F$, $t_1, \dots, t_r$ be independent variables over $F$, and $F' = F(t_1, \dots, t_r)$. Set $q' = q \bigotimes_F F'$. Then $\tau(q) = \tau(q')$.*

*Proof.* The inequality $\tau(q') \leqslant \tau(q)$ is obvious from the definition of $\tau(q)$. To prove the opposite inequality, we may assume that $F$ is an infinite field; otherwise $\tau(q) = 0$, and there is nothing to prove. We may also assume that $r = 1$; the general case then follows by induction on $r$. Let $b'$ be the symmetric bilinear form associated to $q'$ and choose a basis $b_1(t), \dots, b_n(t)$ of $(F')^n$ so that $\operatorname{trdeg}_k k(\alpha_{ij}(t)) = \tau(q')$, where $\alpha_{ij}(t) = b'(b_i(t), b_j(t))$. Since $F$ is an infinite field, we can find a $c \in F$ such that (i) the vectors $b_1(c), \dots, b_d(c)$ are well-defined and form a basis of $F^d$, and (ii) each $\alpha_{ij}(c)$ is well-defined. Now $q$ is defined over $k(\alpha_{ij}(c))$ and thus

$$\tau(q) \leqslant \operatorname{trdeg}_k(\alpha_{ij}(c)) \leqslant \operatorname{trdeg}_k(\alpha_{ij}(t)) = \tau(q'),$$

as claimed. □

### 7.2. *Proof of Theorem* 1.5

Let $F$ be a field containing a primitive 4th root of unity. Note that for the purpose of proving Theorem 1.5, we may assume that $A/F$ is a division algebra. Otherwise, $A$ is isomorphic to $\mathrm{M}_4(F)$ or to $\mathrm{M}_2(D)$, where $D = (a, b)_2$ is a quaternion algebra. Thus, $A$ is defined over $k$ or over the field $F_0 = k(a, b)$, respectively, and so is the trace form of $A$. Alternatively, a simple direct computation shows that the trace form of $\mathrm{M}_2(E)$ is trivial (and thus is defined over $k$) for any central simple algebra $E/F$.

From now on we will assume that $A/F$ is a division algebra of degree 4. By a theorem of Albert [**1**], $A$ is a $\mathscr{G}$-crossed product, with $\mathscr{G} = \mathbb{Z}/2 \times \mathbb{Z}/2$. Let $K$ be a $\mathscr{G}$-Galois maximal subfield. Using the notations introduced in Section 6 (with $m = 2$), we write $\mathscr{G} = \langle \sigma_1, \sigma_2 \rangle$, $K = F(\alpha_1, \alpha_2)$, $\alpha_i^2 = a_i \in F$ and $A = (K, \mathscr{G}, u, b_1, b_2)$ for some $u \in K^*$, $b_1 \in F(\alpha_2) = K^{\sigma_1}$, and $b_2 \in F(\alpha_1) = K^{\sigma_2}$. Set $\sigma_3 = \sigma_1\sigma_2 \in \operatorname{Gal}(K/F)$; $z_3 = (z_1 z_2)^{-1}$, $\alpha_3 = \alpha_1\alpha_2$, $a_3 = \alpha_3^2 = a_1 a_2$, $b_3 = z_3^2$ (so that $b_i = z_i^2$ for $i = 1, 2, 3$), and

$$t_i = \tfrac{1}{2} \operatorname{Tr}_{K^{\sigma_i}/F}\left(z_i^2\right), \quad n_i = N_{K^{\sigma_i}/F}\left(z_i^2\right),$$

for $i = 1, 2, 3$.

Our proof of Theorem 1.5 is based on the following result of Serre [**35**], [**26**].

PROPOSITION 7.3. *Suppose $z_1$ and $z_2$ are chosen so that $t_i \neq 0$ and $n_i^2 - t_i \neq 0$ for any $i = 1, 2, 3$. Then the trace form $q$ of $A$ is Witt-equivalent (over $F$) to $q_2 \oplus q_4$, where*

$$q_2 = \left\langle\!\left\langle n_1 - t_1^2,\ n_2 \right\rangle\!\right\rangle$$

*is a 2-fold Pfister form and*

$$q_4 = \langle\langle t_1 - n_1^2,\ (n_2 - t_2^2)n_2,\ t_1t_2,\ t_2t_3 \rangle\rangle$$

*is a 4-fold Pfister form.*

We claim that for the purpose of proving Theorem 1.5, we may assume without loss of generality that $t_i \neq 0$ and $n_i^2 - t_i \neq 0$ for any $i = 1, 2, 3$. Indeed, in view of Lemma 7.2 it suffices to prove Theorem 1.5 for a single division algebra $A$ which has the rational specialization property in the class of algebras of degree 4, for example, for $A = \mathrm{UD}(4)$; see Remark 2.8. Thus we only need to show that in this algebra $t_i \neq 0$ and $n_i - t_i^2 \neq 0$ for any choice of $z_1, z_2$.

We may assume without loss of generality that $i = 1$ (the cases where $i = 2$ and 3 will then follow by symmetry). Write $b_1 = f_1 + f_2\alpha_2$ for some $f_1, f_2 \in F$, where $t_1 = f_1$ and $n_1 - t_1^2 = f_2^2 a_2$. Lemma 6.4 shows that if $t_1 = 0$ then $A$ is cyclic and if $n_1 - t_1^2 = 0$ then $A$ is biquaternion. However since our algebra $A$ has the rational specialization property, it is neither cyclic nor biquaternion. We conclude that $t_1(n_1 - t_1^2) \neq 0$, as claimed.

We now proceed to simplify the form given by Proposition 7.3. After expanding $q_2$ and $q_4$, cancelling the common term $\langle 1, t_1^2 - n_1 \rangle$ (which can be done, since we are assuming that $\sqrt{-1} \in F$) and dividing some of the entries by elements of $(F^*)^2$, we see that the trace form of $A$ is Witt equivalent to the 16-dimensional form

$$q = \left\langle 1,\ 1 - \frac{n_1}{t_1^2} \right\rangle \otimes \left( \left\langle \frac{n_2}{t_2^2} \right\rangle \oplus \left\langle\!\!\left\langle \left(1 - \frac{n_2}{t_2^2}\right) \frac{n_2}{t_2^2},\ t_1t_2,\ t_2t_3 \right\rangle\!\!\right\rangle_0 \right), \tag{7.1}$$

where $\langle\langle \lambda_1, \ldots, \lambda_r \rangle\rangle_0$ is defined as a $(2^r - 1)$-dimensional form such that $\langle\langle \lambda_1, \lambda_2, \ldots, \lambda_r \rangle\rangle_0 \oplus \langle 1 \rangle$ is the $n$-fold Pfister form $\langle\langle \lambda_1, \lambda_2, \ldots, \lambda_r \rangle\rangle$.

Note that since $q$ and the trace form of $A$ are Witt equivalent 16-dimensional forms, the Witt decomposition theorem implies that they are, in fact, the same (that is, isometric). We now observe that all entries of $q$ lie in the subfield $F_0 = k(n_1/t_1^2, n_2/t_2^2, t_1t_2, t_2t_3)$ of $F$. Thus the trace form of $A$ is defined over $F_0$. Since $F_0$ is generated by four elements over $k$, we have $\mathrm{trdeg}_k\ F_0 \leqslant 4$. This completes the proof of Theorem 1.5.

*Acknowledgements.* The authors would like to thank the referee of this paper for a number of helpful and constructive comments and for catching several mistakes in an earlier version of the paper.

## *References*


1. A. A. Albert, 'Structure of algebras', American Mathematical Society Colloquium Publications 24 (American Mathematical Society, Providence, RI, 1939).
2. S. A. Amitsur and S. J. Saltman, 'Generic abelian crossed products and $p$-algebras', *J. Algebra* 51 (1978) 76–87.
3. E. Beneish, 'Induction theorems on the stable rationality of the center of the ring of generic matrices', *Trans. Amer. Math. Soc.* 350 (1998) 3571–3585.
4. C. Bessenrodt and L. LeBruyn, 'Stable rationality of certain $\mathrm{PGL}_n$-quotients', *Invent. Math.* 104 (1991) 179–199.
5. B. Fein, D. J. Saltman and M. Schacher, 'Embedding problems for finite dimensional division algebras', *J. Algebra* 167 (1994) 588–626.
6. E. Formanek, 'The center of the ring of $3 \times 3$ generic matrices', *Linear and Multilinear Algebra* 7 (1979) 203–212.
7. E. Formanek, 'The center of the ring of $4 \times 4$ generic matrices', *J. Algebra* 62 (1980) 304–319.
8. W. Fulton and J. Harris, *Representation theory, a first course* (Springer, New York, 1991).

**9.** K. W. Gruenberg, 'Relation modules of finite groups', CBS Regional Conference Series in Mathematics 25 (American Mathematical Society, Providence, RI, 1976).
**10.** N. Jacobson, *Basic algebra* II, 2nd edn (W. H. Freeman, New York, 1989).
**11.** W. Kimmerle, 'Relative relation modules as generators for integral group rings of finite groups', *Math. Z.* 172 (1980) 143–156.
**12.** T.-Y. Lam, *The algebraic theory of quadratic forms* (W. A. Benjamin, 1973).
**13.** S. Lang, *Algebra* (Addison Wesley, 1965).
**14.** N. Lemire, 'Essential dimension of algebraic groups and integral representations of Weyl groups', Preprint, http://www.math.uwo.ca/~nlemire/preprints.html.
**15.** N. Lemire and M. Lorenz, 'On certain lattices associated with generic division algebras', *J. Group Theory* 3 (2000) 385–405.
**16.** H. W. Lenstra, 'Rational functions invariant under a finite abelian group', *Invent. Math.* 25 (1974) 299–325.
**17.** K. Masuda, 'On a problem of Chevalley', *Nagoya Math. J.* 8 (1955) 59–63.
**18.** D. S. Passman, *The algebraic structure of group rings* (John Wiley, New York, 1977).
**19.** R. S. Pierce, *Associative algebras* (Springer, New York, 1982).
**20.** C. Procesi, 'Non-commutative affine rings', *Atti Accad. Naz. Lincei* VIII (1967) 239–255.
**21.** Z. Reichstein, 'On a theorem of Hermite and Joubert', *Canad. J. Math.* 51 (1999) 69–95.
**22.** Z. Reichstein, 'On the notion of essential dimension for algebraic groups', *Transfor. Groups* 5 (2000) 265–304.
**23.** Z. Reichstein and N. Vonessen, 'An embedding property of universal division algebras', *J. Algebra* 177 (1995) 451–462.
**24.** S. Rosset, 'Group extensions and division algebras', *J. Algebra* 53 (1978) 297–303.
**25.** M. Rost, 'Computation of some essential dimensions', Preprint, http://www.math.ohio-state.edu/~rost/ed.html.
**26.** M. Rost, J.-P. Serre and J.-P. Tignol, 'The trace form of a central simple algebra of degree four', in preparation.
**27.** L. H. Rowen, *Polynomial identities in ring theory* (Academic Press, 1980).
**28.** L. H. Rowen, 'Brauer factor sets and simple algebras', *Trans. Amer. Math. Soc.* 282 (1984) 765–772.
**29.** L. H. Rowen, *Ring theory – Vol.* 2 (Academic Press, 1988).
**30.** L. H. Rowen, 'Division algebras over $C_2$- and $C_3$-fields', *Proc. Amer. Math. Soc.* 130 (2002) 1607–1610.
**31.** L. H. Rowen and D. J. Saltman, 'Normalized Brauer factor sets', *J. Algebra* 198 (1997) 446–468.
**32.** D. J. Saltman, 'Twisted multiplicative field invariants, Noether's problem, and Galois extensions', *J. Algebra* 131 (1990) 535–558.
**33.** D. J. Saltman, 'A note on generic division algebras', *Contemp. Math.* 130 (1992) 385–394.
**34.** D. J. Saltman, 'Lectures on division algebras', CBS Regional Conference Series in Mathematics 94 (American Mathematical Society, Providence, RI, 1999).
**35.** J.-P. Serre, personal communication 16 November and 24 November 1998.
**36.** I. R. Shafarevich, *Basic algebraic geometry – Vol.* 1, 2nd edn (Springer, 1994).
**37.** R. L. Snider, 'Is the Brauer group generated by cyclic algebras?', *Ring Theory, Waterloo*, 1978, Lecture Notes in Mathematics 734 (Springer, 1979) 279–301.

*M. Lorenz*
*Department of Mathematics*
*Temple University*
*Philadelphia*
*PA 19122-6094*
*USA*

lorenz@math.temple.edu

*Z. Reichstein*
*Department of Mathematics*
*University of British Columbia*
*Vancouver*
*BC*
*Canada V6T 1Z2*

reichst@math.ubc.ca

*L. H. Rowen*
*Department of Mathematics and Computer Science*
*Bar-Ilan University*
*Ramat-Gan 52900*
*Israel*

rowen@macs.biu.ac.il

*D. J. Saltman*
*Department of Mathematics*
*University of Texas*
*Austin*
*TX 78712*
*USA*

saltman@math.utexas.edu