

On Cohen–Macaulay Rings of Invariants¹

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We investigate the transfer of the Cohen–Macaulay property from a commutative ring to a subring of invariants under the action of a finite group. Our point of view is ring theoretic and not a priori tailored to a particular type of group action. As an illustration, we discuss the special case of multiplicative actions, that is, actions on group algebras $k[\mathbb{Z}^n]$ via an action on \mathbb{Z}^n . © 2001 Academic Press

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INTRODUCTION

This article addresses the following question: to what extent does the Cohen–Macaulay property pass from a commutative ring R to a subring R^G of invariants under the action of a finite group G on R ? As is well known, the Cohen–Macaulay property is indeed inherited by R^G whenever the trace map $\text{tr}_G: R \rightarrow R^G$, $r \mapsto \sum_{g \in G} g(r)$, is surjective ([HE]; see also Corollary 3.2 below). In general, however, the property does not transfer, even in the particular case of linear actions, that is, G -actions on polynomial algebras $R = k[X_1, \dots, X_n]$ by linear substitutions of the variables. The Cohen–Macaulay problem for linear invariants has been rather thoroughly explored without, at present, being anywhere near a final solution.

Our focus in this article will not be on linear G -actions on polynomial algebras nor, for the most part, on any other kind of group action on affine algebras over a field. Rather, in Sections 1–4, we work entirely in the setting of commutative noetherian rings. Besides being more general, this approach has resulted in a number of simplifications of results previously obtained by Kemper [Ke₁, Ke₂] in a geometric setting using geometric

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methods. Nevertheless, the article owes a great deal to Kemper's insights and originated from a study of his work.

A short outline of the contents is as follows. Section 1 is devoted to relative trace maps. We determine the height of their image, an ideal of R^G , and use this result to give a lower bound for the height of annihilators in R^G of certain cohomology classes. Section 2 reviews basic material on Cohen–Macaulay rings and local cohomology. In particular, we describe a pair of spectral sequences constructed by Ellingsrud and Skjelbred [ES]. These quickly yield certain depth estimates. Section 3 develops the main technical tools of this article. We use the Ellingsrud–Skjelbred spectral sequences to derive a depth formula for modules of invariants. This formula underlies virtually all our subsequent applications to the Cohen–Macaulay property of rings of invariants R^G . These applications, in the main, concern the case where R has characteristic $p > 0$ and focus on the role played by the Sylow p -subgroup of G . For the precise statements of our results, we refer the reader to Section 4 where they are presented. The final Section 5 initiates the study of the Cohen–Macaulay property in the special setting of multiplicative actions. These are defined to be G -actions on Laurent polynomial algebras $R = k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ stabilizing the lattice of monomials $\langle X_1, \dots, X_n \rangle \cong \mathbb{Z}^n$; so we may think of G as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. We show that if G maps onto some nontrivial p -group and has a cyclic Sylow p -subgroup, P , then R^G is Cohen–Macaulay if and only if P is generated by a bireflection, that is, a matrix $g \in \mathrm{GL}_n(\mathbb{Z})$ so that $g - 1_{n \times n}$ has rank at most 2. In this case, P must have order 2, 3, or 4. A more detailed study of the Cohen–Macaulay property for multiplicative invariants will form the subject of the second author's Ph.D. thesis.

Notations and Conventions. Throughout, G will denote a finite group and R will be a commutative ring (with 1) on which G acts by ring automorphisms, $r \mapsto g(r)$. The subring of G -invariant elements of R will be denoted by R^G and the skew group ring of G over R by RG . Thus, RG is the free left R -module with basis the elements of G , made into a ring by means of the multiplication rule $rg \cdot r'g' = rg(r')gg'$ for $r, r' \in R, g, g' \in G$. The ring R is a module over RG via $rg \cdot r' = rg(r')$. All modules are understood to be left modules.

1. THE RELATIVE TRACE MAP

1.1. Throughout this section, H denotes a subgroup of G . The *relative trace map* $\mathrm{tr}_{G/H}: R^H \rightarrow R^G$ is defined by

$$\mathrm{tr}_{G/H}(r) = \sum_{g \in G/H} g(r) \quad (r \in R^H).$$

Here, g runs over any transversal for the cosets gH of H in G . Since $\text{tr}_{G/H}$ is R^G -linear, the image of $\text{tr}_{G/H}$ is an ideal of R^G which we shall denote by

$$R_H^G.$$

1.2. *Covering Primes.* The proof of the following lemma was communicated to us by Don Passman. The special case where R is an affine algebra over a field is covered by [Ke₂, Satz 4.7]. As usual, we will write ${}^sH = gHg^{-1}$ ($g \in G$) and $I_G(\mathfrak{Q}) = \{g \in G \mid (g - 1)(R) \subset \mathfrak{Q}\}$ denotes the *inertia group* of an ideal \mathfrak{Q} of R . We mention that in the geometric context (i.e., when R is the coordinate ring of an affine variety X and \mathfrak{Q} comes from a point $x \in X$), the inertia group $I_G(\mathfrak{Q})$ is just the isotropy group G_x .

LEMMA 1.1. *For any prime ideal \mathfrak{Q} of R ,*

$$\mathfrak{Q} \supseteq R_H^G \Leftrightarrow [I_G(\mathfrak{Q}) : I_{{}^sH}(\mathfrak{Q})] \in \mathfrak{Q} \quad \text{for all } g \in G.$$

Proof. The implication \Leftarrow follows from the straightforward formula

$$\text{tr}_{G/H}(r) \equiv \sum_{g \in I_G(\mathfrak{Q}) \setminus G/H} [I_G(\mathfrak{Q}) : I_{{}^sH}(\mathfrak{Q})]g(r) \pmod{\mathfrak{Q}}$$

for all $r \in R^H$. For \Rightarrow , assume that $\mathfrak{Q} \supseteq R_H^G$. It suffices to show that

$$[I_G(\mathfrak{Q}) : I_H(\mathfrak{Q})] \in \mathfrak{Q}.$$

Indeed, $R_H^G = R_{{}^sH}^G$, since $\text{tr}_{G/H}(r) = \text{tr}_{G/{}^sH}(g(r))$ holds for all $r \in R^H$ and $g \in G$.

To simplify notation, put $I = I_G(\mathfrak{Q})$ and let P denote a Sylow p -subgroup of $I \cap H = I_H(\mathfrak{Q})$, where $p \geq 0$ is the characteristic of the commutative domain R/\mathfrak{Q} . (Here $P = \{1\}$ if $p = 0$.) Then our desired conclusion, $[I : I \cap H] \in \mathfrak{Q}$, is equivalent to

$$[I : P] \in \mathfrak{Q}.$$

Furthermore, our assumption $\mathfrak{Q} \supseteq R_H^G$ entails that $\mathfrak{Q} \supseteq R_P^G$, because $\text{tr}_{G/P} = \text{tr}_{G/H} \circ \text{tr}_{H/P}$. Thus, leaving H for P , we may assume that $H = P$ is a p -subgroup of I . Let $D = \{g \in G \mid g(\mathfrak{Q}) = \mathfrak{Q}\}$ denote the decomposition group of \mathfrak{Q} ; so $I \leq D$. We claim that

$$\mathfrak{Q} \supseteq R_P^D.$$

To see this, choose $r \in R$ so that $r \in g(\mathfrak{Q})$ for all $g \in G \setminus D$ but $r \notin \mathfrak{Q}$. Then $s = \prod_{g \in D} g(r)$ also belongs to $\bigcap_{g \in G \setminus D} g(\mathfrak{Q})$ but not to \mathfrak{Q} and, in addition, $s \in R^D$. Now assume that, contrary to our claim, there exists an element $f \in R^P$ so that $\text{tr}_{D/P}(f) \notin \mathfrak{Q}$. Then $\text{tr}_{D/P}(sf) = s \text{tr}_{D/P}(f) \in \bigcap_{g \in G \setminus D} g(\mathfrak{Q}) \setminus \mathfrak{Q}$, and hence $\text{tr}_{G/P}(sf) \notin \mathfrak{Q}$, a contradiction.

By the claim, we may replace G by D , thereby reducing to the case where \mathfrak{Q} is G -stable. (Note that I is unaffected by this replacement.) So G acts on R/\mathfrak{Q} with kernel I , P is a p -subgroup of I , and $R_P^G \subseteq \mathfrak{Q}$. Thus, $0 \equiv \text{tr}_{G/P}(r) \equiv [I : P] \cdot \sum_{g \in G/I} g(r) \pmod{\mathfrak{Q}}$ holds for all $r \in R^P$. Our desired conclusion, $[I : P] \in \mathfrak{Q}$, will follow if we can show that $\sum_{g \in G/I} g(r) \notin \mathfrak{Q}$ holds for some $r \in R^P$. But $\sum_{g \in G/I} g$ induces a nonzero endomorphism on R/\mathfrak{Q} , by linear independence of automorphisms of $K = \text{Fract}(R/\mathfrak{Q})$; so $\sum_{g \in G/I} g(s) \notin \mathfrak{Q}$ holds for some $s \in R$. Putting $r = \prod_{h \in P} h(s)$, we have $r \in R^P$ and $r \equiv s^{|P|} \pmod{\mathfrak{Q}}$. Since $|P|$ is 1 or a power of $p = \text{char } K$, we obtain $\sum_{g \in G/I} g(r) \equiv \sum_{g \in G/I} g(s^{|P|}) \equiv (\sum_{g \in G/I} g(s))^{|P|} \not\equiv 0 \pmod{\mathfrak{Q}}$, as required. ■

1.3. *Height Formula.* For any collection \mathscr{H} of subgroups of G , we define the ideal $R_{\mathscr{H}}^G$ of R^G by

$$R_{\mathscr{H}}^G = \sum_{H \in \mathscr{H}} R_H^G.$$

Inasmuch as $R_D^G \subseteq R_H^G = R_{gH}^G$ holds for all $D \leq H \leq G$ and $g \in G$, there is no loss in assuming that \mathscr{H} is closed under G -conjugation and under taking subgroups.

Moreover, for any subgroup $H \leq G$, we define

$$I_R(H) = \sum_{h \in H} (h - 1)(R)R.$$

Thus, $I_R(H)$ is an ideal of R , and $\mathfrak{Q} \supseteq I_R(H)$ is equivalent with $H \leq I_G(\mathfrak{Q})$. When R is the coordinate ring of an affine variety X the ideal $I_R(H)$ defines the subvariety X^H of H -fixed points in X .

LEMMA 1.2. *Assume that R has characteristic p , a positive prime, and let \mathscr{H} be a collection of subgroups of G that is closed under G -conjugation and under taking subgroups. Then*

$$\text{height } R_{\mathscr{H}}^G = \inf\{\text{height } I_R(P) \mid P \text{ is a } p\text{-subgroup of } G, P \notin \mathscr{H}\}.$$

Proof. One has

$$\text{height } R_{\mathscr{H}}^G = \inf_{\mathfrak{q}} \text{height } \mathfrak{q} = \inf_{\mathfrak{Q}} \text{height } \mathfrak{Q},$$

where \mathfrak{q} runs over the prime ideals of R^G containing $R_{\mathscr{H}}^G$ and \mathfrak{Q} runs over the primes of R containing $R_{\mathscr{H}}^G$. Here, the first equality is just the definition of height, while the second equality is a consequence of the standard relations between the primes of R and R^G ; see, e.g., [Bou, Théorème 2, p. 42].

By Lemma 1.1,

$$\mathfrak{Q} \supseteq R_{\mathcal{L}}^G \Leftrightarrow p \mid [I_G(\mathfrak{Q}) : I_H(\mathfrak{Q})] \quad \text{for all } H \in \mathcal{L}.$$

Since $I_H(\mathfrak{Q}) = I_G(\mathfrak{Q}) \cap H$ belongs to \mathcal{L} for $H \in \mathcal{L}$, the latter condition just says that the Sylow p -subgroups of $I_G(\mathfrak{Q})$ do not belong to \mathcal{L} or, equivalently, some p -subgroup $P \leq I_G(\mathfrak{Q})$ does not belong to \mathcal{L} . Therefore,

$$\mathfrak{Q} \supseteq R_{\mathcal{L}}^G \Leftrightarrow \mathfrak{Q} \supseteq \bigcap_{P \leq G \text{ a } p\text{-subgroup, } P \notin \mathcal{L}} I_R(P),$$

which implies the asserted height formula. ■

1.4. *Annihilators of Cohomology Classes.* Let M be a module over the skew group ring RG . Then, for each $r \in R^G$, the map $\rho: M \rightarrow M$, $m \mapsto rm$, is G -equivariant, and hence ρ induces a map on cohomology $\rho_*: H^*(G, M) \rightarrow H^*(G, M)$. Letting r act on $H^*(G, M)$ via ρ_* we make $H^*(G, M)$ into an R^G -module.

The following lemma generalizes [Ke₃, Corollary 2.4].

LEMMA 1.3. *The ideal R_H^G of R^G annihilates the kernel of the restriction map $\text{res}_H^G: H^*(G, M) \rightarrow H^*(H, M)$.*

Proof. The action of $R^G = H^0(G, R)$ on $H^*(G, M)$ can also be interpreted as coming from the cup product

$$H^0(G, R) \times H^*(G, M) \xrightarrow{\cup} H^*(G, R \otimes_{\mathbb{Z}} M) \xrightarrow{\cdot} H^*(G, M),$$

where the map denoted by \cdot comes from the G -equivariant map $R \otimes_{\mathbb{Z}} M \rightarrow M$, $r \otimes m \mapsto rm$; see e.g., [Br, Exercise 1, p. 114]. Furthermore, the relative trace map $\text{tr}_{G/H}: R^H \rightarrow R^G$ is identical with the corestriction map $\text{cor}_H^G: H^0(H, R) \rightarrow H^0(G, R)$; cf. [Br, p. 81]. Thus, the transfer formula for cup products [Br, (3.8), p. 112] gives, for $s \in R^H$ and $x \in H^*(G, M)$,

$$\text{tr}_{G/H}(s)x = \cdot(\text{tr}_{G/H}(s) \cup x) = \cdot(\text{cor}_H^G(s \cup \text{res}_H^G(x))).$$

Therefore, if $\text{res}_H^G(x) = 0$ then $\text{tr}_{G/H}(s)x = 0$. ■

We summarize the material of this section in the following proposition. For convenience, we write $\text{res}_P^G(\cdot) = \cdot|_P$.

PROPOSITION 1.4. *Assume that R has characteristic p , and let M be an RG -module. Then, for any $x \in H^*(G, M)$,*

$$\text{height ann}_{R^G}(x) \geq \inf\{\text{height } I_R(P) \mid P \text{ a } p\text{-subgroup of } G, x|_P \neq 0\}.$$

Proof. Let \mathcal{X} denote the splitting data of x , that is, $\mathcal{X} = \{H \leq G \mid x|_H = 0\}$; cf. [CoR]. By Lemma 1.3, $\text{ann}_{R^G}(x) \supseteq R_{\mathcal{X}}^G$, and by Lemma 1.2, $\text{height } R_{\mathcal{X}}^G = \inf\{\text{height } I_R(P) \mid P \text{ is a } p\text{-subgroup of } G, x|_P \neq 0\}$. The proposition follows. ■

2. DEPTH

2.1. In this section, A denotes a commutative noetherian ring, α is an ideal of A , and M is a finitely generated module over the group ring $A[G]$.

2.2. *Depth and Local Cohomology.* Let H_{α}^i denote the i th local cohomology functor with respect to α , that is, the i th right derived functor of the α -torsion functor

$$\Gamma_{\alpha}(M) = H_{\alpha}^0(M) = \{m \in M \mid m \text{ is annihilated by some power of } \alpha\}.$$

Then

$$\text{depth}(\alpha, M) = \inf\{i \mid H_{\alpha}^i(M) \neq 0\}$$

(where $\inf \emptyset = \infty$); see [BS, Theorem 6.2.7].

Recall from Section 1.4 (with $A = R^G$) that $H^*(G, M)$ is a module over A . Our hypotheses on A and M entail that M is a noetherian A -module, and hence so are all $H^*(G, M)$. Therefore,

$$\text{depth}(\alpha, H^*(G, M)) = \inf\{i \mid H_{\alpha}^i(H^q(G, M)) \neq 0\}.$$

All $H_{\alpha}^i(M)$ are $A[G]$ -modules, via the action of $A[G]$ on M .

2.3. *The Ellingsrud–Skjelbred Spectral Sequences.* The above A -modules $H_{\alpha}^p(H^q(G, M))$ feature as the $E_2^{p,q}$ -terms of a certain spectral sequence due to Ellingsrud and Skjelbred [ES]. In fact, two related spectral sequences are constructed in [ES] in the following manner.

The α -torsion functor Γ_{α} and the G -fixed point functor $(\cdot)^G = H^0(G, \cdot)$ clearly commute: $\Gamma_{\alpha}(M^G) = (\Gamma_{\alpha}(M))^G$. Moreover, if the $A[G]$ -module M is injective, then one checks that $\Gamma_{\alpha}(M)$ is also injective as $A[G]$ -module (as in [BS, Proposition 2.1.4]) and M^G is injective as an A -module. Therefore, $H^i(G, \Gamma_{\alpha}(M)) = 0$ and $H_{\alpha}^i(M^G) = 0$ holds for all $i > 0$ if M is injective. We obtain two Grothendieck spectral sequences converging to $H_{\alpha}^*(G, M) := R^*(\Gamma_{\alpha}(\cdot)^G)(M) = R^*((\cdot)^G \Gamma_{\alpha})(M)$, for any $A[G]$ -module M ;

e.g., [Ro, Theorem 11.38]:

$$\begin{aligned} E_2^{p,q} = H_\alpha^p(H^q(G, M)) &\implies H_\alpha^{p+q}(G, M) \\ \mathcal{E}_2^{p,q} = H^p(G, H_\alpha^q(M)) &\implies H_\alpha^{p+q}(G, M) \end{aligned} \tag{2.1}$$

2.4. *Depth Estimates.* The depth formulas in Section 2.2 combined with the spectral sequences (2.1) yield the following estimates for $\text{depth}(\alpha, M^G)$.

LEMMA 2.1. (a) **lower bound:** $\text{depth}(\alpha, M^G) \geq \min\{\text{depth}(\alpha, M), h_\alpha + 1\}$, where $h_\alpha = \inf_{q > 0}\{q + \text{depth}(\alpha, H^q(G, M))\}$.

(b) **upper bound:** Assume that $H_\alpha^{p_0}(H^{q_0}(G, M)) \neq 0$ for some $p_0 \geq 0, q_0 > 0$ with $s := p_0 + q_0 < \text{depth}(\alpha, M)$. Assume further that $H_\alpha^{s+1-\ell}(H^\ell(G, M)) = 0$ holds for $\ell = 1, \dots, q_0 - 1$ and $H_\alpha^{s-1-\ell}(H^\ell(G, M)) = 0$ holds for $\ell > q_0$. Then $\text{depth}(\alpha, M^G) \leq s + 1$.

Proof. Put $m = \text{depth}(\alpha, M)$. Then $H_\alpha^q(M) = 0$ for $q < m$, and so the \mathcal{E} -sequence in (2.1) implies that $H_\alpha^n(G, M) = 0$ for $n < m$. Therefore, the E -sequence satisfies

$$E_\infty^{p,q} = 0 \quad \text{if } p + q < m. \tag{2.2}$$

Furthermore, $E_2^{p,0} = H_\alpha^p(M^G)$; so

$$\text{depth}(\alpha, M^G) = \inf\{p \mid E_2^{p,0} \neq 0\}.$$

Finally,

$$h_\alpha = \inf\{p + q \mid q > 0, E_2^{p,q} \neq 0\}.$$

To prove (a), assume that $p < \min\{m, h_\alpha + 1\}$. Then $E_\infty^{p,0} = 0$, by (2.2), and $E_r^{i,j} = 0$ for $j > 0, i + j < p, r \geq 2$. Recall that the differential d_r of E_r has bidegree $(r, 1 - r)$. Thus, $E_r^{p,0}$ has no nontrivial boundaries and consists entirely of cycles. This shows that $E_2^{p,0} = E_3^{p,0} = \dots = E_\infty^{p,0}$, and hence $E_2^{p,0} = 0$. Thus, (a) is proved.

For (b), we check that $E_2^{s+1,0} \neq 0$. Our hypotheses imply that, at position (p_0, q_0) , all incoming differentials d_r ($r \geq 2$) are 0 as well as all outgoing d_r ($r \geq 2, r \neq q_0 + 1$). Therefore, $E_{q_0+1}^{p_0, q_0} = E_2^{p_0, q_0}$ and $E_\infty^{p_0, q_0} = E_{q_0+2}^{p_0, q_0} = \text{Ker}(d_{q_0+1}^{p_0, q_0})$. The former implies that $E_{q_0+1}^{p_0, q_0} \neq 0$, by hypothesis on (p_0, q_0) , and the latter shows that $d_{q_0+1}^{p_0, q_0}$ is injective, because $E_\infty^{p_0, q_0} = 0$ by (2.2). Thus, $d_{q_0+1}^{p_0, q_0}$ embeds $E_{q_0+1}^{p_0, q_0}$ into $E_{q_0+1}^{s+1,0}$, forcing the latter to be nonzero. Hence, $E_2^{s+1,0}$ is nonzero as well, as desired. ■

In this article, we will only apply the above estimates in a very limited way, namely with $p_0 = 0$ in the notation of part (b). This case yields

estimates that could also be derived by other means, e.g., by using Koszul complexes as done by Kemper in [Ke₁, Ke₂].

2.5. *Cohen–Macaulay Rings.* For any finitely generated A -module V , one defines $\dim V = \dim(A/\text{ann}_A V)$ and

$$\text{height}(\alpha, V) = \text{height}(\alpha + \text{ann}_A V/\text{ann}_A V);$$

so $\dim V = \sup_{\alpha \supseteq \text{ann}_A V} \text{height}(\alpha, V)$. Always,

$$\text{depth}(\alpha, V) \leq \text{height}(\alpha, V);$$

see [BH, Exercise 1.2.22(a)]. The A -module V is called *Cohen–Macaulay* if equality holds for all ideals α of A . In order to show that V is Cohen–Macaulay, it suffices to check that $\text{depth}(\alpha, V) \geq \text{height}(\alpha, V)$ holds for all maximal ideals α of A with $\alpha \supseteq \text{ann}_A V$.

3. MODULES OF INVARIANTS

3.1. Throughout this section, R^G is assumed noetherian and α denotes an ideal of R^G . Moreover, M denotes an RG -module that is finitely generated as an R^G -module. Our finiteness assumptions hold, for example, whenever R is an affine algebra over some noetherian subring $k \subseteq R^G$ and M is a finitely generated RG -module; see [Bou, Théorème 2, p. 33].

3.2. *The Problem and a Sufficient Condition.* Assuming ${}_R M$ to be Cohen–Macaulay, we are interested in the question under what circumstances ${}_{R^G} M^G$ will be Cohen–Macaulay as well. We remark that ${}_R M$ is Cohen–Macaulay if and only if ${}_{R^G} M$ is; see [Ke₂, Proposition 1.17].

For future reference, we record the following simple lemma.

LEMMA 3.1. *Assume that ${}_R M$ is Cohen–Macaulay and that $\sqrt{\alpha} \supseteq \text{ann}_{R^G} M^G$. Then $\text{depth}(\alpha, M) = \text{height}(\alpha, M) \geq \text{height}(\alpha, M^G)$.*

Proof. Note that $\sqrt{\alpha} \supseteq \text{ann}_{R^G} M^G \supseteq \text{ann}_{R^G} M$ entails that $\text{height}(\alpha, M) \geq \text{height}(\alpha, M^G)$. Further, $\text{height}(\alpha, M) = \text{depth}(\alpha, M)$, because ${}_{R^G} M$ is Cohen–Macaulay. The lemma follows. ■

We now give a sufficient condition for ${}_{R^G} M^G$ to be Cohen–Macaulay. We note that $\dim_R M = \dim_{R^G} M$, by the familiar relations between the primes of R and of R^G .

COROLLARY 3.2. *Assume that ${}_R M$ is Cohen–Macaulay. If $H^q(G, M) = 0$ holds for $0 < q < \dim_R M - 1$ then ${}_{R^G} M^G$ is Cohen–Macaulay as well. In particular, this holds whenever the trace map $\text{tr}_G = \text{tr}_{G/\{1\}}: R \rightarrow R^G$ is surjective.*

Proof. Let α be an ideal of R^G with $\alpha \supseteq \text{ann}_{R^G} M^G$. Our hypothesis on $H^q(G, M)$ entails that the value of h_α in Lemma 2.1 satisfies $h_\alpha \geq \dim_R M - 1$. Also, $\dim_R M = \dim_{R^G} M \geq \text{height}(\alpha, M) \geq \text{height}(\alpha, M^G)$, by Lemma 3.1. Thus, Lemma 2.1(a) gives $\text{depth}(\alpha, M^G) \geq \text{height}(\alpha, M^G)$, which proves that ${}_{R^G} M^G$ is Cohen–Macaulay.

For the last assertion, just note that Lemma 1.3 implies that $H^q(G, M) = 0$ holds for all $q > 0$ when $R_{\{1\}}^G = R^G$. ■

Note that the condition $H^q(G, M) = 0$ for $0 < q < \dim_R M - 1$ is vacuous for $\dim_R M \leq 2$. For $\dim_R M = 3$ it becomes $H^1(G, M) = 0$. The latter holds, for example, whenever M is a G -permutation module without $|G|$ -torsion; explicitly, as a G -module, $M \cong \bigoplus_H (\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M(H))$, where H runs over certain subgroups of G and each $M(H)$ is an H -submodule of M^H so that $|H|m = 0, m \in M(H)$ implies $m = 0$.

3.3. *Example: Multiplicative Invariants over Cohen–Macaulay Rings.* Let $M = R = k[A]$ be the group ring of a free abelian group $A \cong \mathbb{Z}^n$ over a Cohen–Macaulay ring k ; so R is Cohen–Macaulay as well. Let G act on A by group automorphisms. By k -linear extension to R we obtain a G -action on R by algebra automorphism which is indeed a permutation action. Thus, the foregoing implies immediately that R^G is Cohen–Macaulay when $n + \dim k \leq 3$ and k has no $|G|$ -torsion. We will return to this type of group action, called multiplicative, in greater detail in Section 5, focusing on the case where k is a field whose characteristic divides $|G|$.

3.4. *Depth Formula.* In view of Corollary 3.2, we may concentrate on the case where M has non-vanishing positive G -cohomology. The following proposition is a version of results of Kemper, see [Ke₁, Corollary 1.6; Ke₂, Kor. 1.18].

PROPOSITION 3.3. *Assume that ${}_R M$ is Cohen–Macaulay and that $\sqrt{\alpha} \supseteq \text{ann}_{R^G} M^G$. Furthermore, assume that, for some $r \geq 0$, $H^q(G, M) = 0$ holds for $0 < q < r$ but $\alpha x = 0$ for some $0 \neq x \in H^r(G, M)$. Then*

$$\text{depth}(\alpha, M^G) = \min\{r + 1, \text{depth}(\alpha, M)\}.$$

Remark. $\text{height}(\alpha, M) = \text{depth}(\alpha, M)$ holds in the above formula; see Lemma 3.1.

Proof of Proposition 3.3. Our hypothesis $\alpha x = 0$ for some $0 \neq x \in H^r(G, M)$ is equivalent with $H_\alpha^0(H^r(G, M)) \neq 0$; so $\text{depth}(\alpha, H^r(G, M)) = 0$. The asserted equality is trivial for $r = 0$, since $\text{depth}(\alpha, M^G) = \text{depth}(\alpha, M) = 0$ holds in this case. Thus we assume that $r > 0$. Then, in

the notation of Lemma 2.1, we have $r = h_\alpha$, and part (a) of the lemma gives the inequality \geq .

To prove the reverse inequality, note that Lemma 3.1 gives $\text{depth}(\alpha, M) \geq \text{depth}(\alpha, M^G)$. Therefore, it suffices to show that $\text{depth}(\alpha, M^G) \leq r + 1$ if $\text{depth}(\alpha, M) > r + 1$. For this, we quote Lemma 2.1(b) with $p_0 = 0$ and $q_0 = r$ (so $s = r$). ■

4. THE SYLOW SUBGROUP OF G

4.1. In this section, we focus on rings of invariants R^G . If not explicitly mentioned otherwise, R is assumed to be noetherian as a R^G -module and to have characteristic p , a positive prime. We let P denote a Sylow p -subgroup of G .

4.2. *A Necessary Condition.* Put

$$\mu = \mu(G, R) = \inf\{r > 0 \mid H^r(G, R) \neq 0\}.$$

PROPOSITION 4.1. *Put $\mathcal{P} = \{P' \leq P \mid \text{height } I_R(P') \leq \mu + 1\}$. If R and R^G are both Cohen–Macaulay and $\mu < \infty$ then the restriction map*

$$\text{res}_{\mathcal{P}}^G: H^\mu(G, R) \rightarrow \prod_{P' \in \mathcal{P}} H^\mu(P', R)$$

is injective.

Proof. Let $0 \neq x \in H^\mu(G, R)$ be given and put $\alpha = \text{ann}_{R^G}(x)$. Then, by Proposition 1.4,

$$\text{height } \alpha \geq \inf\{\text{height } I_R(P') \mid P' \text{ a } p\text{-subgroup of } G, x|_{P'} \neq 0\}.$$

Since R^G is Cohen–Macaulay, $\text{height } \alpha = \text{depth } \alpha$. Finally, Proposition 3.3 with $M = R$ gives $\text{depth } \alpha \leq \mu + 1$. Thus, there exists a p -subgroup P' of G with $x|_{P'} \neq 0$ and $\text{height } I_R(P') \leq \mu + 1$. Note that both the condition $x|_{P'} \neq 0$ and the value of $\text{height } I_R(P')$ are preserved upon replacing P' by a conjugate ${}^gP'$ with $g \in G$. Therefore, we may assume that $P' \leq P$, which proves the proposition. ■

4.3. *Galois and Almost Galois Actions.* Recall that the G -action on a commutative ring R is *Galois*, in the sense of Auslander and Goldman [AG], if every maximal ideal of R has trivial inertia group in G or, equivalently, $I_R(H) = R$ holds for all subgroups $1 \neq H \leq G$. Thus, a G -action on the coordinate ring of an affine variety X is Galois precisely if the fixed point subvarieties X^H are empty.

We will say that a G -action on R is *almost Galois* if height $I_R(H) \geq \dim R$ holds for all $1 \neq H \leq G$ (where height $R = \infty$). In the geometric setting, this means that the fixed point subvarieties X^H are finite. For linear actions, X is a nonzero vector space on which G acts linearly; so each X^H is a subspace. Thus, linear actions are never Galois (when $G \neq 1$), and they are not even almost Galois in the modular case (i.e., when $p \neq 1$). Similarly, multiplicative actions (cf. Section 3.3) are never Galois but they are often almost Galois; see Section 5.2. Part (a) of the following proposition holds for any commutative ring R ; it is included for the sake of completeness.

PROPOSITION 4.2. (a) *If the G -action on R is Galois then R^G is Cohen–Macaulay if and only if R is.*

(b) *Assume that R is Cohen–Macaulay and that the action of the Sylow p -subgroup P on R is almost Galois. Then R^G is Cohen–Macaulay if and only if $\dim R \leq \mu + 1$.*

Proof. (a) By [CHR, Lemma 1.6 and Theorem 1.3], the trace map $\text{tr}_G: R \rightarrow R^G$ is surjective for Galois actions and R is finitely generated projective as an R^G -module. Thus, R is faithfully flat as an R^G -module. Moreover, for any prime \mathfrak{Q} of R , the fibre $R_{\mathfrak{Q}}/(\mathfrak{Q} \cap R^G)R_{\mathfrak{Q}}$ has dimension 0. Therefore, by [BH, 2.1.23], R^G is Cohen–Macaulay if and only if R is.

(b) The implication \Leftarrow follows from Corollary 3.2 with $M = R$. For the converse, let R^G be Cohen–Macaulay and assume, without loss, that $\mu < \infty$. Then Proposition 4.1 implies that there is a subgroup $1 \neq P' \leq P$ with height $I_R(P') \leq \mu + 1$. On the other hand, by hypothesis on the P -action, height $I_R(P') \geq \dim R$; so $\dim R \leq \mu + 1$. ■

4.4. Example: Affine Actions on Polynomial Algebras. A G -action on the polynomial algebra $R = k[X_1, \dots, X_n]$ over a field k is called *affine* if G stabilizes the subspace $L = k + \sum_i kX_i$ of polynomials of degree at most 1. Since G acts trivially on k , the quotient $V = L/k$ inherits a $k[G]$ -module structure. Let $\gamma \in \text{Ext}_{k[G]}(V, k)$ denote the extension class of $0 \rightarrow k \rightarrow L \rightarrow V \rightarrow 0$. We claim

the G -action on R is Galois if and only if the restrictions $\gamma|_H \in \text{Ext}_{k[H]}(V, k)$ are nonzero for all subgroups $1 \neq H \leq G$.

Indeed, if $\gamma|_H = 0$ for some H then H acts linearly on R , and hence the action won't be Galois. Conversely, assume that all $\gamma|_H$ are nonzero. Then G is a p -group, where $p = \text{char } k$. In order to show that all $I_R(H) = R$, it suffices to consider cyclic subgroups $1 \neq H = \langle h \rangle$. But then the restriction

map $\text{Ext}_{k[H]}(V, K) \rightarrow \text{Ext}_{k[H]}(V^H, k)$ can be identified with the canonical map $\text{ann}_{V^*} \sum h^i / (h-1)V^* \rightarrow (V^H)^* \cong V^* / (h-1)V^*$, and hence it is injective. Thus, $\gamma|_H$ has a nonzero image in $\text{Ext}_{k[H]}(V^H, k)$, that is, $0 \neq h(l) - l \in k$ holds for some $l \in L$. Since $h(l) - l \in I_R(H)$, we conclude that $I_R(H) = R$, as desired.

In the special case where the G -action on V is trivial, $\text{Ext}_{k[G]}(V, k)$ can be identified with $\text{Hom}(G, V^*)$ and the above Galois condition just says that γ is an injection $G \hookrightarrow V^*$. Geometrically, this case corresponds to an action of G on $\text{Hom}(R, k) \cong V^*$ by translations, $g(f) = f - \gamma(g)$.

Returning to general affine actions on R , we remark that if the action of the Sylow p -subgroup P is almost Galois then it is actually Galois. For, as we have remarked above, no nonidentity subgroup of P can act linearly, and so γ must restrict nontrivially to all these subgroups. In this case, the P -trace tr_P is surjective, and hence so is tr_G which in turn entails that $\mu = \infty$. Hence Proposition 4.2(b) says

$$\gamma|_H \neq 0 \text{ for all subgroups } 1 \neq H \leq P \Rightarrow R^G \text{ is Cohen-Macaulay.}$$

4.5. *Bireflections.* Following [Ke₂], we will call an element $g \in G$ a *bireflection* on R if height $I_R(\langle g \rangle) \leq 2$.

COROLLARY 4.3. *Assume that R and R^G are Cohen-Macaulay. Let H denote the subgroup of G that is generated by all p' -elements of G and all bireflections in P . Then $R^G = R_H^G$.*

Proof. First note that H is a normal subgroup of G and G/H is a p -group. Thus, if $R^G \neq R_H^G$ or, equivalently, $\widehat{H}^0(G/H, R^H) \neq 0$ then also $H^1(G/H, R^H) \neq 0$; see [Br, Theorem VI.8.5]. In view of the exact sequence

$$0 \rightarrow H^1(G/H, R^H) \rightarrow H^1(G, R) \xrightarrow{\text{res}_H^G} H^1(H, R)$$

(see [Ba, 35.3]) we further obtain $H^1(G, R) \neq 0$. Thus, $\mu = 1$ holds in Proposition 4.1 and every $P' \in \mathcal{P}$ consists of bireflections. Therefore $P' \subseteq H$ and Proposition 4.1 implies that $\text{res}_H^G: H^1(G, R) \rightarrow H^1(H, R)$ is injective, contradicting the above exact sequence. Therefore, we must have $R^G = R_H^G$. ■

In case $\mathbb{F}_p = \mathbb{Z} \cdot 1_R$ is a G -module direct summand of R , the conclusion $R^G = R_H^G$ of the above corollary is equivalent with $G = H$. In this form, the corollary is essentially [Ke₂, Korollar 4.10] (at least for affine R).

4.6. *The Case $|P| = p$.* Put

$$\mu_p(G) = \mu(G, \mathbb{F}_p) = \inf\{r > 0 \mid H^r(G, \mathbb{F}_p) \neq 0\}.$$

We will determine this number in the case where the order of G is divisible by p but not by p^2 ; in other words, $|P| = p$. As usual $\mathbb{N}_G(P)$ and $\mathbb{C}_G(P)$ will denote the normalizer and the centralizer, respectively, of P in G . Thus, $\mathbb{N}_G(P)/\mathbb{C}_G(P)$ is a subgroup of $\text{Aut}(P) = \text{Aut}(\mathbb{Z}/p) \cong \mathbb{F}_p^*$, and hence it is cyclic of order dividing $p - 1$.

COROLLARY 4.4. *Assume that $|P| = p$. Then $\mu_p(G) = 2[\mathbb{N}_G(P) : \mathbb{C}_G(P)] - 1$. Moreover, if \mathbb{F}_p is a G -module direct summand of R and R and R^G are both Cohen–Macaulay then height $I_R(P) \leq 2[\mathbb{N}_G(P) : \mathbb{C}_G(P)]$.*

Proof. Put $N = \mathbb{N}_G(P)$, $C = \mathbb{C}_G(P)$, and $r = 2[N : C] - 1$. In order to prove that $\mu_p(G) = r$, we use the fact that $H^*(G, \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p)^{N/C}$ holds for $* > 0$; see [Be, Corollary 3.6.19]. If $p = 2$ then $N = C$ and so $r = 1$. Moreover, $H^*(P, \mathbb{F}_p)^{N/C} \cong H^*(\mathbb{Z}/2, \mathbb{F}_2)$ equals \mathbb{F}_2 in all degrees. This proves the assertion for $p = 2$; so we assume p odd from now on. In this case, $H^*(\mathbb{Z}/p, \mathbb{F}_p) \cong \mathbb{F}_p[v_1, b_2]/(v_1^2, v_1b_2 - b_2v_1)$ with $\deg v_1 = 1$ and $\deg b_2 = 2$; see [AM, Corollary II.4.2]. Moreover, identifying $\text{Aut}(\mathbb{Z}/p)$ with \mathbb{F}_p^* , the action of $\text{Aut}(\mathbb{Z}/p)$ on $H^*(\mathbb{Z}/p, \mathbb{F}_p)$ becomes scalar multiplication, $v_1 \mapsto \ell v_1$, $b_2 \mapsto \ell b_2$, where $\ell \in \mathbb{F}_p^*$. Taking ℓ to be a generator for the subgroup of \mathbb{F}_p^* corresponding to N/C , we see that

$$H^*(P, \mathbb{F}_p)^{N/C} \cong \bigoplus_{i \geq 0} \mathbb{F}_p b_2^{i[N:C]} \oplus \bigoplus_{i > 0} \mathbb{F}_p v_1 b_2^{i[N:C]-1};$$

see [AM, pp. 104–105]. The smallest positive degree where $H^*(P, \mathbb{F}_p)^{N/C}$ does not vanish is therefore indeed $2([N : C] - 1) + 1 = r$.

Now assume that \mathbb{F}_p is a G -module direct summand of R and R and R^G are both Cohen–Macaulay. The former hypothesis implies that $H^r(G, R) \neq 0$ and hence $\mu \leq r$. Moreover, our hypothesis on $|P|$ implies that $\mathcal{S} \ni P$ holds in Proposition 4.1, because otherwise \mathcal{S} would consist of the identity subgroup alone. Therefore, height $I_R(P) \leq \mu + 1 \leq r + 1$, as desired. ■

In the case of a linear action, the upper bound for height $I_R(P)$ given in the above corollary also follows from [Ke₃, Theorem 3.1].

4.7. The Non-Cohen–Macaulay Locus. We finish this section by recording an elementary observation independent of the local cohomology methods used thus far and valid for any commutative ring R .

By definition, the non-Cohen–Macaulay locus of R^G consists of those prime ideals \mathfrak{q} of R^G so that the localization $(R^G)_{\mathfrak{q}}$ is not Cohen–Macaulay. Thus, R^G is Cohen–Macaulay if and only if its non-Cohen–Macaulay locus is empty. Here, we point out a general bound for the non-Cohen–Macaulay locus in terms of relative trace maps. More detailed results for affine algebras over a field can be found in [Ke₂, Kapitel 5]. Recall the notation $R_{\mathcal{S}}^G$ from Section 1.3.

PROPOSITION 4.5. *Let \mathcal{EM} denote the set of subgroups H of G so that R^H is Cohen–Macaulay. Then, for every prime ideal \mathfrak{q} of R^G so that $\mathfrak{q} \not\supseteq R_{\mathcal{EM}}^G$, the localization $(R^G)_{\mathfrak{q}}$ is Cohen–Macaulay.*

Proof. By hypothesis, $\mathfrak{q} \not\supseteq R_H^G$ for some $H \in \mathcal{EM}$. Let $R_{\mathfrak{q}}$ denote the localization of R at the multiplicative subset $R^G \setminus \mathfrak{q}$. Then the G -action on R extends to $R_{\mathfrak{q}}$ and $(R_{\mathfrak{q}})^G = (R^G)_{\mathfrak{q}}$. Similarly $(R_{\mathfrak{q}})^H = (R^H)_{\mathfrak{q}}$; so $(R_{\mathfrak{q}})^H$ is Cohen–Macaulay. By choice of \mathfrak{q} the relative trace map $\text{tr}_{G/H}: (R_{\mathfrak{q}})^H \rightarrow (R_{\mathfrak{q}})^G$ is onto. Fix an element $c \in (R_{\mathfrak{q}})^H$ so that $\text{tr}_{G/H}(c) = 1$ and define $\rho: (R_{\mathfrak{q}})^H \rightarrow (R_{\mathfrak{q}})^G$ by $\rho(x) = \text{tr}_{G/H}(cx)$. This map is a “Reynolds operator,” i.e., ρ is $(R_{\mathfrak{q}})^G$ -linear and restricts to the identity on $(R_{\mathfrak{q}})^G$. Since $(R_{\mathfrak{q}})^H$ is integral over $(R_{\mathfrak{q}})^G$, a result of Hochster and Eagon [HE; BH, Theorem 6.4.5] implies that $(R_{\mathfrak{q}})^G$ is Cohen–Macaulay, which proves the proposition. ■

As an application, we note that if G has subgroups H_i so that each R^{H_i} is Cohen–Macaulay and the indices $[G : H_i]$ are coprime in R^G then R^G is Cohen–Macaulay as well. Indeed, writing $1 = \sum_i [G : H_i] r_i$ with $r_i \in R^G$, we obtain $1 = \sum_i \text{tr}_{G/H_i}(r_i) \in R_{\mathcal{EM}}^G$; so the non-Cohen–Macaulay locus of R^G is empty.

5. MULTIPLICATIVE ACTIONS

5.1. In this section, we focus on a particular type of group action often called multiplicative actions. These arise from G -actions on lattices $A \cong \mathbb{Z}^n$ by extending the action k -linearly to the group algebra $R = k[A] \cong k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Here, we assume k to be a field such that $p = \text{char } k$ divides the order of G ; otherwise the invariant subalgebra R^G would certainly be Cohen–Macaulay because R is; see Proposition 4.5. There is no loss in assuming G to be faithfully embedded in $\text{GL}(A) \cong \text{GL}_n(\mathbb{Z})$, and we will do so. The above notations will remain valid throughout this section.

5.2. The action of G on $R = k[A]$ is almost Galois (see Section 4.3) if and only if G acts fixed-point-freely on A , that is, no $1 \neq g \in G$ has an eigenvalue 1 on A . Furthermore, an element $g \in G$ is a bireflection on R (Section 4.5) if and only if the endomorphism $g - 1 \in \text{End}(A) \cong M_n(\mathbb{Z})$ has rank at most 2. Both these observations are consequences of the following

LEMMA 5.1. *For any subgroup $H \leq G$, height $I_R(H) = n - \text{rank } A^H$.*

Proof. By definition, the ideal $I_R(H)$ of R is generated by the elements $h(a) - a = (h(a)a^{-1} - 1)a$ for $h \in H$, $a \in A$. Thus, $R/I_R(H) \cong$

$k[A/[H, A]]$, where we have put $[H, A] = \langle h(a)a^{-1} \mid h \in H, a \in A \rangle \leq A$. Consequently, $\text{height } I_R(H) = \dim R - \dim R/I_R(H) = n - \text{rank } A/[H, A]$. Finally, since the group algebra $\mathbb{Q}[H]$ is semisimple, $A \otimes \mathbb{Q} = (A^H \otimes \mathbb{Q}) \oplus ([H, A] \otimes \mathbb{Q})$; so $\text{rank } A/[H, A] = \text{rank } A^H$. ■

5.3. Since G permutes the k -basis A of R , the Eckmann–Shapiro Lemma [Br, VI(5.2)] implies that

$$H^*(G, R) \cong \bigoplus_{a \in A/G} H^*(G_a, k),$$

where G_a denotes the isotropy group of a in G . In particular, using the notations of Sections 4.2 and 4.6, we have

$$\mu = \inf_{a \in A} \mu_p(G_a). \quad (5.1)$$

5.4. *Example: Inversion.* Let $G = \langle g = -\mathbb{1}_{n \times n} \rangle$ act on $R = k[X_1^\pm, \dots, X_n^\pm]$ via $g(X_i) = X_i^{-1}$. This action is fixed-point-free on $A = \mathbb{Z}^n$. Moreover, assuming $p = 2$, we have $\mu = \mu_2(G) = 1$ by (5.1). Therefore, Proposition 4.2(b) gives

R^G is Cohen–Macaulay if and only if $n \leq 2$.

5.5. *Example: Reflection Groups.* An element $g \in G$ is called a *reflection* on R if $\text{height } I_R(\langle g \rangle) \leq 1$ or, equivalently, if the endomorphism $g - 1 \in \text{End}(A) \cong M_n(\mathbb{Z})$ has rank at most 1; see Lemma 5.1. If G is generated by reflections then R^G is an affine normal semigroup algebra over k ; see [Lo₁]. Therefore, R^G is Cohen–Macaulay in this case, for any field k ; see [BH, Theorem 6.3.5]. This is in contrast with the situation for finite group actions on polynomial algebras by linear substitutions of the variables, where (modular) reflection groups need not lead to Cohen–Macaulay invariants [Nak].

5.6. *Cyclic Sylow Subgroups.* As before, we let P denote a fixed Sylow p -subgroup of G . Moreover, $O^p(G)$ denotes the intersection of all normal subgroups N of G so that G/N is a p -group.

THEOREM 5.2. *Assume that $O^p(G) \neq G$ and that P is cyclic. Then R^G is Cohen–Macaulay if and only if P is generated by a bireflection. In this case, P has order 2, 3, or 4.*

Proof. Our hypothesis $O^p(G) \neq G$ is equivalent with $\mu_p(G) = 1$; so $\mu = 1$ holds as well, by (5.1). Assuming R^G to be Cohen–Macaulay, Corollary 4.3 and the subsequent remark imply that $G = H$. Since all p' -elements of G belong to $O^p(G)$, it follows that $G/O^p(G) = P/P \cap O^p(G)$ is generated by the images of the bireflections in P . Since P is cyclic, it follows that P is generated by a bireflection. Now, P acts faithfully on the lattice A/A^p of rank at most 2. Thus, P is isomorphic to

a cyclic p -subgroup of $\mathrm{GL}_2(\mathbb{Z})$, and these are easily seen to have orders 2, 3, or 4.

The converse follows from the more general lemma below which does not depend on cyclicity of P or nontriviality of $G/O^p(G)$. ■

LEMMA 5.3. *If rank $A/A^P \leq 2$ then R^G is Cohen–Macaulay.*

Proof. By Proposition 4.5, it suffices to show that R^P is Cohen–Macaulay; so we may assume that $G = P$ is a p -group. Note that G acts faithfully on $\bar{A} = A/A^G$. If G acts as a reflection group on \bar{A} then it does so on A as well, and hence the invariants R^G will be Cohen–Macaulay; see Section 5.5. Thus we may assume that \bar{A} has rank 2 and G acts on $\bar{A} \cong \mathbb{Z}^2$ as a non-reflection p -group. By the well-known classification of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ (e.g., [Lo₂, 2.7]), this leaves the cases $G \cong \mathbb{Z}/q$ with $q = 2, 3$, or 4 to consider.

The cases $q = 2$ or 3 can be dealt with along similar lines. Indeed, for both values of q , the only indecomposable G -lattices, up to isomorphism, are \mathbb{Z} , $\mathbb{Z}[G]$, and $\mathbb{Z}[G]/(\hat{G})$, where $\hat{G} = \sum_{g \in G} g$; see [CR, Exercise 4, p. 514/5]. Thus, $A \cong \mathbb{Z}^m \oplus (\mathbb{Z}[G]/(\hat{G}))^r \oplus \mathbb{Z}[G]^s$, and $R^G \cong k[B]^G[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$, where we have put $B = (\mathbb{Z}[G]/(\hat{G}))^r \oplus \mathbb{Z}[G]^s$. Since R^G is Cohen–Macaulay if and only if $k[B]^G$ is, we may assume that $m = 0$. Now, $\bar{A} \cong (\mathbb{Z}[G]/(\hat{G}))^{r+s}$; so $2 = (r+s)(|G| - 1)$. When $n = 3$, this leads to either $r = 1, s = 0$ or $r = 0, s = 1$. In the former case, rank $A = 2$ and so R^G is surely Cohen–Macaulay, being a normal domain of dimension 2. If $r = 0, s = 1$ then A is a G -permutation lattice of rank 3. Hence, $R = k[A]$ is a localization of the symmetric algebra $S(A \otimes k)$, and likewise for the subalgebras of invariants. Since linear invariants of dimension ≤ 3 are known to be Cohen–Macaulay (e.g., [Ke₂]), R^G is Cohen–Macaulay in this case as well. For $n = 2$, there are three cases to consider, one of which ($r = 2, s = 0$) leads to an invariant algebra of dimension 2 which is clearly Cohen–Macaulay. Thus, we are left with the possibilities $r = 1, s = 1$, and $r = 0, s = 2$. Explicitly, after an obvious choice of basis, G acts as one of the following groups on A :

Case 1.

$$G_1 = \left\langle g_1 = \left(\begin{array}{c|cc} -1 & & \\ \hline & 0 & 1 \\ & 1 & 0 \end{array} \right) \right\rangle.$$

Case 2.

$$G_2 = \left\langle \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & & 1 & 0 \end{array} \right) \right\rangle.$$

For G_2 , $A \cong \mathbb{Z}^4$ is a permutation lattice. Hence, as above, it suffices to check that the linear invariant algebra $S(V)^G$ for $V = A \otimes k$ is Cohen–Macaulay which is indeed the case, by [ES], since $\dim V/V^G = 2$. For G_1 , one can proceed as follows: Embed G_1 into $\Gamma = \langle g_1, \text{diag}(-1, 1, 1) \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and denote the corresponding basis of $A \cong \mathbb{Z}^3$ by $\{x, y, z\}$; so $g_1(x) = x^{-1}$, $g_1(y) = z$, and $g_1(z) = y$. One easily checks that $R^\Gamma = k[\xi, \sigma_1, \sigma_2^{\pm 1}]$, where $\xi = x + x^{-1}$, $\sigma_1 = y + z$, and $\sigma_2 = yz$. Furthermore, $R = k[A] = R^\Gamma \oplus xR^\Gamma \oplus yR^\Gamma \oplus xyR^\Gamma$. With this, the invariant subalgebra R^{G_1} is easily determined; the result (for $\text{char } k = 2$) is $R^{G_1} = R^\Gamma \oplus (xy + x^{-1}z)R^\Gamma$ which is indeed Cohen–Macaulay. This completes the proof for $G \cong \mathbb{Z}/2$ or $\cong \mathbb{Z}/3$.

We now sketch the remaining case, $G = \langle g \rangle \cong \mathbb{Z}/4$. The action on $\bar{A} = A/A^G$ can then be described by $G|_{\bar{A}} = \langle (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}) \rangle$; so $\bar{A} \cong \mathbb{Z}[G]/(g^2 + 1)$. With this, one calculates $\text{Ext}_G(\bar{A}, \mathbb{Z}) \cong \mathbb{Z}/2$. Thus, there is exactly one (up to isomorphism) non-split extension of G -modules $0 \rightarrow \mathbb{Z} \rightarrow U \rightarrow \bar{A} \rightarrow 0$. A suitable module U is $U = \mathbb{Z}[G]/(g - 1)(g^2 + 1)$. Furthermore, one calculates $\text{Ext}_G(U, \mathbb{Z}) = 0$. Consequently, either $A \cong A^G \oplus \bar{A}$ or $A \cong \mathbb{Z}^m \oplus U$, and hence either $R^G \cong k[\bar{A}]^G[A^G]$ which is Cohen–Macaulay because $k[\bar{A}]^G$ has dimension 2, or $R^G \cong k[U]^G[X_1^{\pm 1}, \dots, X_r^{\pm 1}]$ which is Cohen–Macaulay precisely if $k[U]^G$ is. This reduces the problem to the case where $A = U$ which can be handled by direct calculation, taking advantage of the fact that a conjugate of group G_1 is contained in G . We leave the details to the reader. ■

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REFERENCES

- [AM] A. Adem and R. J. Milgram, “Cohomology of Finite Groups,” Springer-Verlag, Berlin, 1994.
- [AG] M. Auslander and O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.* **97** (1959), 749–765.
- [Ba] A. Babakhanian, “Cohomological Methods in Group Theory,” Dekker, New York, 1972.
- [Be] D. J. Benson, “Representations and Cohomology,” Vol. I, Cambridge Univ. Press, Cambridge, UK, 1995.
- [Bou] N. Bourbaki, “Algèbre commutative,” Chaps. 5, 6, Hermann, Paris, 1964.
- [BS] M. P. Brodmann and R. Y. Sharp, “Local Cohomology,” Cambridge Univ. Press, Cambridge, UK, 1998.

- [Br] K. S. Brown, "Cohomology of Groups," Springer-Verlag, New York, 1982.
- [BH] W. Bruns and J. Herzog, "Cohen–Macaulay Rings," Cambridge Univ. Press, Cambridge, UK, 1993.
- [CHR] S. U. Chase, D. K. Harrison, and A. Rosenberg, Galois theory and cohomology of commutative rings, *Mem. Amer. Math. Soc.* **52** (1965).
- [CoR] L. Corry and S. Rosset, The splitting data of cohomology classes, *Arch. Math.* **44** (1985), 418–423.
- [CR] C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Wiley, New York, 1962.
- [ES] G. Ellingsrud and T. Skjelbred, Profondeur d'anneaux d'invariants en caractéristique p , *Compositio Math.* **41** (1980), 233–244.
- [HE] M. Hochster and J. A. Eagon, Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci, *Amer. J. Math.* **93** (1971), 1020–1058.
- [Ke₁] G. Kemper, On the Cohen–Macaulay property of modular invariant rings, *J. Algebra* **215** (1999), 330–351.
- [Ke₂] G. Kemper, "Die Cohen–Macaulay Eigenschaft in der modularen Invariantentheorie," Habilitationsschrift, Universität Heidelberg, 1999.
- [Ke₃] G. Kemper, The depth of invariant rings and cohomology (with an appendix by Kay Magaard), IWR preprint, 99-25, Heidelberg, 1999.
- [Lo₁] M. Lorenz, Multiplicative invariants and semigroup algebras, *Algebras Representation Theory* **4** (2001), 293–304.
- [Lo₂] M. Lorenz, Class groups of multiplicative invariants, *J. Algebra* **177** (1995), 242–254.
- [Nak] H. Nakajima, Invariants of finite abelian groups generated by transvections, *Tokyo J. Math.* **3** (1980), 201–204.
- [Ro] J. J. Rotman, "An Introduction to Homological Algebra," Academic Press, Orlando, 1979.