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K_0 of Invariant Rings and Nonabelian H^1

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We give a description of the kernel of the induction map $K_0(R) \to K_0(S)$, where S is a commutative ring and $R = S^G$ is the ring of invariants of the action of a finite group G on S. The description is in terms of $H^1(G, \mathrm{GL}(S))$. © 1999 Academic Press

Key Words: ring of invariants; group action; nonabelian 1-cohomology; Grothendieck group; induction map; Galois action; Hilbert's Theorem 90.

INTRODUCTION

This article is concerned with the relationship between $K_0(S)$ and $K_0(R)$, where S is a commutative ring and $R = S^G$ denotes the subring of invariants under the action of a finite group G on S.

Specifically, working under the assumption that the trace $\mathrm{tr}: S \to R$, $s \mapsto \sum_{g \in G} s^g$, is surjective, we shall study the kernel of the induction map

$$\mathrm{Ind}_R^S = K_0(f) : K_0(R) \to K_0(S)$$

that is associated with the inclusion $f: R \hookrightarrow S$. We will describe an embedding of $\mathrm{Ker}(\mathrm{Ind}_R^S)$ into the cohomology set $H^1(G, \mathrm{GL}(S))$. Moreover, we will endow $H^1(G, \mathrm{GL}(S))$ with a natural commutative monoid structure, essentially coming from the "block diagonal" maps $\mathrm{GL}_n \times \mathrm{GL}_m \to \mathrm{GL}_{n+m} \hookrightarrow \mathrm{GL}$, such that our embedding identifies $\mathrm{Ker}(\mathrm{Ind}_R^S)$ with the group of units $\mathrm{U}(H^1(G, \mathrm{GL}(S)))$. To further describe this unit group, we define S_H, for any subgroup H of G, to be the factor of S modulo the intersection of all maximal ideals of S whose inertia group contains H.

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Letting $\rho_H : H^1(G, \mathrm{GL}(S)) \to H^1(H, \mathrm{GL}(S_H))$ denote the map that is given by restriction from G to H and the canonical map $S \twoheadrightarrow S_H$, our main result reads as follows.

THEOREM. $\mathrm{Ker}(\mathrm{Ind}_R^S) \cong \mathrm{U}(H^1(G, \mathrm{GL}(S))) = \bigcap_H \mathrm{Ker}\,\rho_H$, *where* H *ranges over all cyclic (or, equivalently, all) subgroups of* G.

The theorem follows via direct limits from a corresponding result, with $\mathrm{GL}_n(.)$ in place of $\mathrm{GL}(.)$, for the kernel of the maps $\mathbf{P}_n(f) : \mathbf{P}_n(R) \to \mathbf{P}_n(S)$, where $\mathbf{P}_n(.)$ denotes the set of isomorphism classes of f.g. projective modules of constant rank n.

As applications, we present a version of Hilbert's Theorem 90 for Galois actions on commutative rings and quickly derive the (known) structure of the Picard groups of linear and multiplicative invariants [K, L]. Some open problems are also discussed.

Notations and Conventions. Throughout this note,

S will be a ring, assumed commutative from Subsection 2.3 onwards,

G will be a finite group acting by automorphisms on S;

the action will be written $s \mapsto s^g$; and

$R = S^G$ will denote the ring of G-invariants in S.

We make the standing hypothesis that the trace map $\mathrm{tr} : S \to R$, $s \mapsto \sum_{g \in G} s^g$ is surjective or, equivalently,

$$\text{There exists an element } x \in S \text{ with } \mathrm{tr}(x) = 1. \qquad (*)$$

1. NONABELIAN H^1

1.1. DEFINITION. We recall the definition of nonabelian H^1 following [Se, Sect. I.5] (or [BS; Se2, p. 123ff]), except that our group actions are on the right.

Let X be a G-group, with G-action written as $x \mapsto x^g$. A (1-)*cocycle* is a map $d : G \to X$ satisfying

$$d(gg') = d(g)^{g'} d(g') \qquad (g, g' \in G).$$

The set of l-cocycles of G in X will be denoted by $Z^1(G, X)$. Two cocycles $d, e \in Z^1(G, X)$ are called *cohomologous* if there exists an element $x \in X$ satisfying $d(g) = x^g e(g) x^{-1}$ for all $g \in G$. This defines an equivalence relation on $Z^1(G, X)$. The set of equivalence classes is

$$H^1(G, X),$$

a pointed set with distinguished element the class of the *unit cocycle* $\mathbf{1}(g) = 1$ for all $g \in G$.

Remark. $H^1(G, X)$ parametrizes the conjugacy classes of complements of X in the split extension $X \rtimes G$, exactly as in the familiar special case where X is abelian (cf. [Ro, 11.1.2, 11.1.3]).

1.2. EXAMPLES. (1) Suppose G acts trivially on X. Then $Z^1(G, X) = \mathrm{Hom}(G, X)$ and

$$H^1(G, X) = \mathrm{Hom}(G, X)/X,$$

with X acting by conjugation on $\mathrm{Hom}(G, X)$. The distinguished class consists of $\mathbf{1}$ alone.

(2) If $G = \langle g \rangle$ is cyclic of order m, then each cocycle $d \in Z^1(G, X)$ is determined by the element $x = d(g) \in X$, and the eligible elements of X are precisely those satisfying the condition $x^{g^{m-1}} x^{g^{m-2}} \cdots x^g x = 1$. Moreover, if $d, e \in Z^1(G, X)$ correspond in this manner to $x, y \in X$, respectively, then d and e are cohomologous precisely if there exists $z \in X$ with $x = z^g y z^{-1}$. Thus, writing $\sim$ for the equivalence relation on X determined by this condition, we have

$$H^1(G, X) \cong \{x \in X : x^{g^{m-1}} x^{g^{m-2}} \cdots x^g x = 1\} / \sim .$$

(3) If the order of X is finite and coprime to $|G|$ then $H^1(G, X)$ is trivial, by the uniqueness part of the Schur–Zassenhaus Theorem (cf. [Ro, 9.1.2]).

(4) If X is a linear algebraic group over an algebraically closed field whose characteristic does not divide $|G|$ and G acts algebraically on X then $H^1(G, X)$ is finite. This is essentially due to A. Weil who explicitly dealt with the case of a trivial G-action on X ([We], cf. also [Sl]). The general case is an easy consequence.

1.3. *Functoriality and Direct Limits.* Suppose that we are given a homomorphism of groups $\alpha : G \to G'$ and a G'-group X'. Then X' can be viewed as a G-group via α. Any map of G-groups (i.e., any group homomorphism compatible with the G-actions) $f : X' \to X$ gives rise to a map $Z^1(G', X') \to Z^1(G, X)$, $d \mapsto f \circ d \circ \alpha$, and this map passes down to a map of pointed sets

$$(\alpha, f)^1_* : H^1(G', X') \to H^1(G, X).$$

In particular, we have the *restriction maps* $H^1(G, X) \to H^1(H, X)$ for subgroups H of G and the *inflation maps* $H^1(G/N, X^N) \to H^1(G, X)$ for normal subgroups N. Here, X^N denotes the N-invariants in X. We will write f^1_* for $(\mathrm{Id}_G, f)^1_*$.

The following easy lemma is surely well known, but I am not aware of a reference. (For commutative cohomology, see [Br, Proposition (4.6), p. 195].)

LEMMA. *Let* (X_n, f_{mn}) *be a direct system of* G*-groups and let* $X = \varinjlim X_n$ *be the direct limit with its induced* G*-action* (*cf*. [Bou3, *p*. A I.117]). *If all* $f_{mn}: X_n \to X_m$ $(m \geq n)$ *are injective then* $H^1(G, X) \cong \varinjlim H^1(G, X_n)$ (*as pointed sets*).

Proof. Put $\varphi_{mn} = (f_{mn})^1_*: H^1(G, X_n) \to H^1(G, X_m)$ and $\varphi_n = (f_n)^1_*$, where $f_n: X_n \to X$ is the canonical map. Then the relations $f_m \circ f_{mn} = f_n$ entail $\varphi_m \circ \varphi_{mn} = \varphi_n$, and so there is a unique map $\varphi: \varinjlim H^1(G, X_n) \to H^1(G, X)$ with $\varphi \circ \psi_n = \varphi_n$, where $\psi_n: H^1(G, X_n) \to \varinjlim H^1(G, X_n)$ is the canonical map. We show that φ is bijective; the fact that φ respects distinguished elements is clear.

For surjectivity, let $d \in Z^1(G, X)$ be given. Since G is finite, there is an n with $d(G) \subseteq f_n(X_n)$, and so the class of d in $H^1(G, X)$ belongs to $\operatorname{Im} \varphi_n \subseteq \operatorname{Im} \varphi$.

As to injectivity, we must show that if $a, b \in H^1(G, X_n)$ satisfy $\varphi_n(a) = \varphi_n(b)$ then there exists $m \geq n$ with $\varphi_{mn}(a) = \varphi_{mn}(b)$ (cf. [Bou2, Proposition 6, p. E III.62]). Say a and b are the classes of $d, e \in Z^1(G, X_n)$, respectively. Then $f_n \circ d$ and $f_n \circ e$ are cohomologous in $Z^1(G, X)$, i.e., there exists $x \in X$ with $(f_n \circ d)(g) = x^g (f_n \circ e)(g) x^{-1}$ for all $g \in G$. Now $x \in f_m(X_m)$ for some $m \geq n$, say $x = f_m(y)$. Then

$$(f_m \circ f_{mn} \circ d)(g) = f_m(y^g)(f_m \circ f_{mn} \circ e)(g) f_m(y^{-1}).$$

Since all f_{mn} are injective, so are the maps f_n [Bou2, Remarque 1, p. E III.63]. Hence, $(f_{mn} \circ d)(g) = y^g (f_{mn} \circ e)(g) y^{-1}$ holds for all $g \in G$ which shows that $f_{mn} \circ d$ and $f_{mn} \circ e$ are cohomologous in $Z^1(G, X_m)$, as required. ∎

1.4. *The Monoid Structure of* $H^1(G, \mathrm{GL}(S))$. Lemma 1.3 impies in particular that in $H^1(G, \mathrm{GL}(S)) \cong \varinjlim H^1(G, \mathrm{GL}_n(S))$, where G acts on $\mathrm{GL}(S)$ and on $\mathrm{GL}_n(S)$ via its action on S. Our goal here is to endow $H^1(G, \mathrm{GL}(S))$ with the structure of a commutative monoid.

More generally, let X be any G-stable subgroup of $\mathrm{GL}(S)$ containing the matrices

$$s_{m,n} \underset{\text{def}}{=} (-1)^{(m+1)n} \begin{pmatrix} 0_{n\times m} & -1_{n\times n} & & \\ 1_{m\times m} & 0_{m\times n} & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}.$$

Note that $\det s_{m,n} = 1$. (In fact, it is not hard to show that the subgroup of $\mathrm{GL}(S)$ that is generated by the matrices $s_{m,n}$ consists of all monomial matrices with entries ± 1 and with determinant 1.) Hence $s_{m,n} \in \mathrm{E}(S)$, the group generated by the elementary matrices [Ba, Proposition 1.6, p. 226]. Furthermore,

$$s_{m,n}^{-1} \begin{pmatrix} a_{n\times n} & & & \\ & b_{m\times m} & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} s_{m,n} = \begin{pmatrix} b_{m\times m} & & & \\ & a_{n\times n} & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}.$$

Thus, if

$$x = \begin{pmatrix} x_{n\times n} & & \\ & 1 & \\ & & \ddots \end{pmatrix} \in X_n = X \cap \mathrm{GL}_n(S)$$

then, for any m,

$$x[m] \underset{\mathrm{def}}{=} \begin{pmatrix} 1_{m\times m} & & & \\ & x_{n\times n} & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} = s_{m,n}^{-1} x s_{m,n} \in X_{m+n}.$$

For $a, b \in H^1(G, X)$, define $a + b \in H^1(G, X)$ as follows. Choose cocycles $d, e \in Z^1(G, X)$ representing a and b, respectively. Since G is finite, we have $d \in Z^1(G, X_m)$ and $e \in Z^1(G, X_n)$ for suitable m and n. For $g \in G$, put

$$\begin{aligned}(d \oplus_m e)(g) &\underset{\mathrm{def}}{=} d(g) \cdot e(g)[m] \\ &= \begin{pmatrix} d(g)_{m\times m} & & & \\ & e(g)_{n\times n} & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \in X.\end{aligned}$$

It is trivial to verify that $d \oplus_m e$ is a cocycle, and we define $a + b$ to be its class in $H^1(G, X)$. To show that this is well-defined, we first note that the class of $d \oplus_m e$ is independent of m, as long as $d(G) \subseteq X_m$. Indeed, if $t \geq 0$ then

$$(d \oplus_{m+t} e)(g) = (s_{t,n}[m])^{-1} (d \oplus_m e)(g) s_{t,n}[m],$$

and $s_{t,n}[m] = s_{t,n}[m]^g \in X$. Thus, $d \oplus_{m+t} e$ and $d \oplus_m e$ are cohomologous. Now suppose that a and b are also represented by the cocycles $d' \in Z^1(G, X)$ and $e' \in Z^1(G, X)$, respectively. So there are matrices $x, y \in X$ with

$$d'(g) = x^g d(g) x^{-1} \qquad \text{and} \qquad e'(g) = y^g e(g) y^{-1}.$$

Fix r so that $x, y, d(G), e(G)$ are all contained in X_r, and hence so are $d'(G)$ and $e'(G)$. By the foregoing, it suffices to show that $d \oplus_r e$ and $d' \oplus_r e'$ are cohomologous. But, putting $z = x \cdot y[r] \in X$, we have

$$(d' \oplus_r e')(g) = z^g (d \oplus_r e)(g) z^{-1}.$$

Thus, $a + b \in H^1(G, X)$ is indeed well-defined. Commutativity and associativity of $+$ follow along similar lines, thereby making $H^1(G, X)$ a commutative monoid with neutral element the class of the unit cocycle. For a different description of the monoid structure of $H^1(G, X)$ for $X = \mathrm{GL}(S)$, see Lemma 2.7.

1.5. *Monoid Maps*. Let S' be another ring that is acted on by a group G' and suppose that we are given a group homomorphism $\alpha : G \to G'$ and a G-equivariant ring map $\phi : S' \to S$, where G acts on S' via α. Then we obtain a map of G-groups $\mathrm{GL}(\phi) : \mathrm{GL}(S') \to \mathrm{GL}(S)$. Thus, if $X' \subseteq \mathrm{GL}(S')$ is a G'-stable subgroup containing the matrices $s_{m,n} \in \mathrm{GL}(S')$ of Subsection 1.4 and if $X \subseteq \mathrm{GL}(S)$ is G-stable containing the image of X' then we have a map of G-groups $f : X' \to X$. The map induced on cocycles $Z^1(G', X') \to Z^1(G, X)$ (cf. Subsection 1.3) is easily seen to respect the operations $\oplus_m$ of Subsection 1.4, and hence the map $(\alpha, f)^1_* : H^1(G', X') \to H^1(G, X)$ of Subsection 1.3 is actually a monoid map. This applies in particular to restriction and induction maps (with $\phi = \mathrm{Id}_S$). Moreover, we have the following

LEMMA. *Let $X \subseteq \mathrm{GL}(S)$ be a G-stable subgroup with $X \supseteq \mathrm{E}(S)$ and let $f : X \to X^{\mathrm{ab}} = X/[X, X]$ denote the canonical map. Then $f^1_* : H^1(G, X) \to H^1(G, X^{\mathrm{ab}})$ is a monoid map, where $H^1(G, X^{\mathrm{ab}})$ has its usual group structure.*

Proof. Consider $d, e \in Z^1(G, X)$, say $d, e \in Z^1(G, X_m)$. Then, by [Ba, Proposition (1.7), p. 226],

$$\begin{aligned}(d \oplus_m e)(g) &\equiv d(g) \cdot e(g) \\ &= \begin{pmatrix} d(g)_{m \times m} e(g)_{m \times m} & & \\ & 1 & \\ & & \ddots \end{pmatrix} \mod \mathrm{E}(S).\end{aligned}$$

Inasmuch as $X^{\mathrm{ab}} = X/\mathrm{E}(S)$ (cf. [Ba, Theorem (2.1), p. 228]), our assertion follows. ∎

1.6. *Units*. The kernel of $\mathrm{Ind}_R^S : K_0(R) \to K_0(S)$ will turn out to be isomorphic to the group of units $\mathrm{U}(H^1(G, \mathrm{GL}(S)))$ of the monoid $H^1(G, \mathrm{GL}(S))$ (see Subsection 2.8). Here, we make some preliminary observations on the unit group

$$\mathrm{U}\big(H^1(G, X)\big),$$

where X is any G-invariant subgroup of $\mathrm{GL}(S)$ containing the matrices $s_{m,n}$, as in Subsection 1.4.

LEMMA. *Let* $N = \mathrm{Ker}_G(X)$ *denote the kernel of the action of* G *on* X. *Then* $\mathrm{U}(H^1(G, X)) \cong \mathrm{U}(H^1(G/N, X))$ *via inflation. In particular, if* G *acts trivially on* X, *then* $\mathrm{U}(H^1(G, X))$ *contains only the unit class.*

Proof. Use the inflation-restriction sequence $H^1(G/N, X) \to H^1(G, X) \to H^1(N, X)$ [Se, Sect. I.5.8]. This sequence is exact, the first map (inflation) is injective, and both maps are monoid maps. Thus it induces an exact sequence of groups

$$1 \to \mathrm{U}\big(H^1(G/N, X)\big) \to \mathrm{U}\big(H^1(G, X)\big) \to \mathrm{U}\big(H^1(N, X)\big).$$

Part (ii) therefore reduces to the claim that $\mathrm{U}(H^1(N, X))$ is trivial. To verify this, recall that $H^1(N, X) = \mathrm{Hom}(N, X)/X$, with X acting by conjugation on $\mathrm{Hom}(N, X)$, and the unit class consists of the unit map $\mathbf{1}$ alone (cf. Subsection 1.2). Thus, letting $\langle . \rangle$ denote X-conjugacy classes, the equation $\langle d \rangle + \langle e \rangle = \langle \mathbf{1} \rangle$ for $d, e \in \mathrm{Hom}(N, X)$ is equivalent with $(d \oplus_m e)(g) = 1$ for all $g \in N$, where m is chosen as above. But the latter condition says that $d = e = \mathbf{1}$. ∎

2. THE KERNEL OF INDUCTION

2.1. *The Skew Group Ring*. We will let

$$T = S * G$$

denote the *skew group ring* that is associated with the given G-action on S. Thus T is an associative ring containing S as a subring and G is a subgroup of $\mathrm{U}(T)$, the group of units of T. The elements of G form a free basis of T as a right S-module. Multiplication in T is based on the rule $ga \cdot hb = gha^h b$ for $a, b \in S$, $g, h \in G$. The ring S is an R-T-bimodule with action

$$r \cdot a \cdot gb = ra^g b \qquad (r \in R, a, b \in S, g \in G).$$

Hypothesis (*) entails that $txt = t$, where we have put $t = \sum_{g \in G} g \in T$. So $e = tx$ is an idempotent element of T with $eT = tT = tS \cong S_T$. In particular, S_T is projective and the ideal $I = TeT$ of T satisfies $I^2 = I$ and $S_T \cdot I = S_T$.

2.2. *Some Module Categories*. Let $\operatorname{proj} R$ denote the category of finitely generated projective (right) R-modules, and similarly for T, and let $\operatorname{add} S_T$ denote the full subcategory of $\operatorname{proj} T$ whose objects are the direct summands of the modules S_T^n for $n \geq 0$. The following lemma is well known but we include the proof for the reader's convenience.

LEMMA. (i) *The functors* $E : \operatorname{proj} R \to \operatorname{add} S_T$, $Q \mapsto Q \otimes_R S_T$ *and* $F : \operatorname{add} S_T \to \operatorname{proj} R$, $P \mapsto P^G$ *yield an equivalence of categories* $\operatorname{proj} R \approx \operatorname{add} S_T$.

(ii) *A module P in* $\operatorname{proj} T$ *belongs to* $\operatorname{add} S_T$ *precisely if $PI = P$*.

Proof. (i) For Q in $\operatorname{proj} R$, let $\varphi_Q : Q \to (F \circ E)(Q) = (Q \otimes_R S_T)^G$ denote the R-linear map given by $\varphi_Q(q) = q \otimes 1$. Then $\varphi_R : R \to (R \otimes_R S_T)^G \cong (S_T)^G = R$ is an isomorphism, and hence so is φ_{R^n} for every n and φ_Q for every Q. Thus φ is a natural equivalence of functors $\mathrm{Id}_{\operatorname{proj} R} \cong F \circ E$. Similarly, defining $\psi_P : (E \circ F)(P) = P^G \otimes_R S_T \to P$ for P in $\operatorname{add} S_T$ by $\psi_P(p \otimes s) = ps$, we obtain a natural equivalence of functors $E \circ F \cong \mathrm{Id}_{\operatorname{add} S_T}$.

(ii) All modules P in $\operatorname{add} S_T$ satisfy $PI = P$, because $S \cdot I = S$. Conversely, if P in $\operatorname{proj} T$ satisfies $P = PI = PeT$ then, for some n, $(eT)^n \cong S_T^n$ maps onto P, and so P is a direct summand of S_T^n. ∎

2.3. *Another Description of* $\operatorname{add} S_T$. *From now on, S is assumed commutative*. We let $\operatorname{Max} S$ denote the set of maximal ideals of S. For each $\mathfrak{M} \in \operatorname{Max} S$, we put $G^Z(\mathfrak{M}) = \{g \in G : \mathfrak{M}^g = \mathfrak{M}\}$, the *decomposition group* of $\mathfrak{M}$, and

$$T(\mathfrak{M}) = (S/\mathfrak{M}) * G^Z(\mathfrak{M}),$$

the skew group ring that is associated with the action of $G^Z(\mathfrak{M})$ on $S/\mathfrak{M}$. As in Subsection 2.1, $S/\mathfrak{M}$ is a right module over $T(\mathfrak{M})$; this module structure can be viewed as coming from S_T by restriction to $S * G^Z(\mathfrak{M})$ and reduction mod $\mathfrak{M}$. The following description of $\operatorname{add}(S_T)$ is gleaned from [Kr, Proposition 3].

LEMMA. *A module P in* $\operatorname{proj} T$ *belongs to* $\operatorname{add} S_T$ *if and only if, for all* $\mathfrak{M} \in \operatorname{Max} S$, *there is an isomorphism of $T(\mathfrak{M})$-modules* $P/P\mathfrak{M} \cong (S/\mathfrak{M})^r$ ($r = \operatorname{rank} P_{\mathfrak{M}}$).

Proof. The condition is necessary, because if P is a direct summand of S_T^n, then $P/P\mathfrak{M}$ is a direct summand of the homogeneous $T(\mathfrak{M})$-module $(S/\mathfrak{M})^n$.

For the converse, consider some $\mathfrak{M} \in \operatorname{Max} S$ and put $\mathfrak{M}^0 = \bigcap_{g \in G} \mathfrak{M}^g$, a G-stable ideal of S. Then

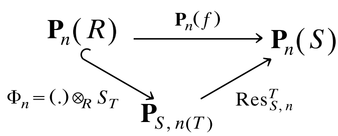
$$S/\mathfrak{M}^0 \cong \bigoplus_{g \in G^Z(\mathfrak{M}) \setminus G} S/\mathfrak{M}^g \cong (S/\mathfrak{M}) \otimes_{T(\mathfrak{M})} T$$

as T-modules. Similarly, $P/P\mathfrak{M}^0 \cong P \otimes_S S/\mathfrak{M}^0 \cong (P/P\mathfrak{M}) \otimes_{T(\mathfrak{M})} T$ as T-modules, with G acting "diagonally" on $P \otimes_S S/\mathfrak{M}^0$: $(p \otimes \bar{s})g = pg \otimes \bar{s}^g$. By hypothesis, $P/P\mathfrak{M}$ is isomorphic to $(S/\mathfrak{M})^r$ as $T(\mathfrak{M})$-modules, and so

$$P/P\mathfrak{M}^0 \cong \left(S/\mathfrak{M}^0\right)^r$$

as T-modules. Since $S = S \cdot I$ (cf. Subsection 2.1), this isomorphism implies $P = PI + P\mathfrak{M}^0$, and since $\mathfrak{M}$ was arbitrary, we further conclude that $P = PI$ (cf. [Bou, Proposition 11, p. 113]). In view of Lemma 2.2(ii), this shows that P belongs to add S_T. ∎

2.4. *The Induction Triangle*. For each $n \geq 0$, we let $\mathbf{P}_n(R)$ denote the set of isomorphism classes of f.g. projective R-modules of constant rank n, and similarly for $\mathbf{P}_n(S)$. These are pointed sets with distinguished elements $\langle R^n \rangle$ and $\langle S^n \rangle$, respectively, where $\langle . \rangle$ denotes isomorphism classes. Furthermore, $\mathbf{P}_{S,n}(T)$ will denote the set of isomorphism classes of f.g. projective right T-modules having constant rank n as S-modules, with distinguished element $\langle S_T^n \rangle$. We have a commutative diagram of pointed sets (cf. [Ba, Proposition (7.3), p. 130])



By Lemma 2.2(i), Φ_n is injective. The kernels of the other two maps will be described in Subsections 2.5 and 2.6 below. Recall that the *kernel* of a map of pointed sets is defined to be the preimage of the distinguished element of the target set.

2.5. *The Kernel of* $\operatorname{Res}^T_{S,n}$. We now consider the restriction map $\operatorname{Res}^T_{S,n} : \mathbf{P}_{S,n}(T) \to \mathbf{P}_n(S)$, as in Subsection 2.4.

LEMMA. $\operatorname{Ker}(\operatorname{Res}^T_{S,n}) \cong H^1(G, \operatorname{GL}_n(S))$ *as pointed sets*.

Proof. Each cocycle $d \in Z^1(G, \operatorname{GL}_n(S))$ gives rise to a T-module structure $(S^n)_d$ on S^n via

$$x \cdot gs = x^g d(g) s \qquad (x \in S^n,\ g \in G,\ s \in S).$$

This action extends the regular S-module structure on S^n. Conversely, if $\cdot$ is any right T-module operation on S^n extending the regular S-module structure then write, for $g \in G$,

$$e_i \cdot g = \sum_{j=1}^{n} e_j d_{i,j}(g),$$

where $e_i \in S^n$ is the basis element with 1 in the ith position, 0's elsewhere, and $d_{i,j}(g) \in S$. A routine verification shows that $d = (d_{i,j})$ is a cocycle of G in $\mathrm{GL}_n(S)$ and the given T-module structure on S^n is identical with $(S^n)_d$.

Since $(S^n)_d$ for the unit cocycle $d = 1$ is just S^n_T, we obtain a surjective map of pointed sets

$$Z^1(G, \mathrm{GL}_n(S)) \twoheadrightarrow \mathrm{Ker}\big(\mathrm{Res}^T_{S,n}\big), \qquad d \mapsto \langle (S^n)_d \rangle.$$

Finally, for $d, e \in Z^1(G, \mathrm{GL}_n(S))$, we have $(S^n)_d \cong (S^n)_e$ as T-modules precisely if there is an S-module isomorphism $(S^n)_d \xrightarrow{\cong} (S^n)_e$ that commutes with the G-actions, that is, for some matrix $a \in \mathrm{GL}_n(S)$, $(x \cdot g)a = (xa) \cdot g$ holds for all $x \in S^n$, $g \in G$. The latter condition is equivalent with $x^g d(g) = x^g a^g e(g) a^{-1}$ which in turn just says that d and e are cohomologous. This completes the proof of the lemma. ∎

Remark. With $T(\mathfrak{M})$ as in Subsection 2.3, we have a map of pointed sets $\mathbf{P}_{S,n}(T) \to \mathbf{P}_{S/\mathfrak{M},n}(T(\mathfrak{M}))$, $P \mapsto P/P\mathfrak{M}$, which restricts to a map $\mathrm{Ker}(\mathrm{Res}^T_{S,n}) \to \mathrm{Ker}(\mathrm{Res}^{T(\mathfrak{M})}_{S/\mathfrak{M},n})$. In terms of the identification provided by the above lemma, the latter becomes the map

$$\rho'_{\mathfrak{M},n} : H^1(G, \mathrm{GL}_n(S)) \to H^1\big(G^Z(\mathfrak{M}), \mathrm{GL}_n(S/\mathfrak{M})\big)$$

that is given by restriction from G to $G^Z(\mathfrak{M})$ and reduction modulo $\mathfrak{M}$.

2.6. *The Kernel of* $\mathbf{P}_n(f)$. For each subgroup $H \leq G$, we put

$$J(H) = \bigcap_{\substack{\mathfrak{M} \in \mathrm{Max}\, S \\ H \subseteq G^T(\mathfrak{M})}} \mathfrak{M} \qquad \text{and} \qquad S_H = S/J(H),$$

where $G^T(\mathfrak{M}) = \{g \in G : s^g - s \in \mathfrak{M} \text{ for all } s \in S\}$ is the *inertia group* of $\mathfrak{M}$. In addition to the maps $\rho'_{\mathfrak{M},n}$ introduced in Remark 2.5, we will consider the analogous restriction-reduction maps

$$\rho_{\mathfrak{M},n} : H^1(G, \mathrm{GL}_n(S)) \to H^1\big(G^T(\mathfrak{M}), \mathrm{GL}_n(S/\mathfrak{M})\big)$$

and

$$\rho_{H,n}: H^1(G, \mathrm{GL}_n(S)) \to H^1(H, \mathrm{GL}_n(S_H)).$$

Since the actions of $G^T(\mathfrak{M})$ on $\mathrm{GL}_n(S/\mathfrak{M})$ and of H on $\mathrm{GL}_n(S_H)$ are trivial, we have

$$H^1(G^T(\mathfrak{M}), \mathrm{GL}_n(S/\mathfrak{M})) = \mathrm{Hom}(G^T(\mathfrak{M}), \mathrm{GL}_n(S/\mathfrak{M}))/\mathrm{GL}_n(S/\mathfrak{M})$$

and similarly for $H^1(H, \mathrm{GL}_n(S_H))$ (cf. Subsection 1.2). We let $\mathscr{C}$ denote the set of cyclic subgroups of G.

PROPOSITION. *As pointed sets,*

$$\mathrm{Ker}\,\mathbf{P}_n(f) \cong \bigcap_{\mathfrak{M} \in \mathrm{Max}\, S} \mathrm{Ker}\,\rho'_{\mathfrak{M},n} = \bigcap_{\mathfrak{M} \in \mathrm{Max}\, S} \mathrm{Ker}\,\rho_{\mathfrak{M},n}$$
$$= \bigcap_{C \in \mathscr{C}} \mathrm{Ker}\,\rho_{C,n} = \bigcap_{H \leq G} \mathrm{Ker}\,\rho_{H,n}.$$

Proof. In view of the induction triangle in Subsection 2.4, $\mathrm{Ker}\,\mathbf{P}_n(f) \cong \mathrm{Im}\,\Phi_n \cap \mathrm{Ker}(\mathrm{Res}^T_{S,n})$. Furthermore, by virtue of Lemmas 2.2 and 2.3, if $\langle P \rangle \in \mathbf{P}_{S,n}(T)$ then $\langle P \rangle \in \mathrm{Im}\,\Phi_n$ iff $\langle P/P\mathfrak{M} \rangle$ is the distinguished element of $\mathbf{P}_{S/\mathfrak{M},n}(T(\mathfrak{M}))$ for all $\mathfrak{M} \in \mathrm{Max}\, S$. Therefore, by Remark 2.5,

$$\mathrm{Im}\,\Phi_n \cap \mathrm{Ker}(\mathrm{Res}^T_{S,n}) \cong \bigcap_{\mathfrak{M} \in \mathrm{Max}\, S} \mathrm{Ker}\,\rho'_{\mathfrak{M},n},$$

which establishes the isomorphism in the proposition. Now $\rho_{\mathfrak{M},n} = \mathrm{Res}^{G^Z(\mathfrak{M})}_{G^T(\mathfrak{M})} \circ \rho'_{\mathfrak{M},n}$, where

$$\mathrm{Res}^{G^Z(\mathfrak{M})}_{G^T(\mathfrak{M})}: H^1(G^Z(\mathfrak{M}), \mathrm{GL}_n(S/\mathfrak{M})) \to H^1(G^T(\mathfrak{M}), \mathrm{GL}_n(S/\mathfrak{M}))$$

is the restriction map. Since this map has trivial kernel, by the generalized "Theorem 90" [Se, Lemme 1, p. 129, Sect. 5.8(a)], we conclude that $\mathrm{Ker}\,\rho'_{\mathfrak{M},n} = \mathrm{Ker}\,\rho_{\mathfrak{M},n}$ which proves the first equality.

Consider $a \in H^1(G, \mathrm{GL}_n(S))$, say a is the class of $d \in Z^1(G, \mathrm{GL}_n(S))$. Then, in view of Subsection 1.2(1),

$$a \in \bigcap_{\mathfrak{M} \in \mathrm{Max}\, S} \mathrm{Ker}\,\rho_{\mathfrak{M},n}$$
$$\Leftrightarrow \forall \mathfrak{M} \in \mathrm{Max}\, S\ \forall g \in G^T(\mathfrak{M}): d(g) \equiv 1_{n\times n} \bmod \mathfrak{M}$$
$$\Leftrightarrow \forall g \in G\ \forall \mathfrak{M} \text{ with } \mathfrak{M} \supseteq J(\langle g \rangle): d(g) \equiv 1_{n\times n} \bmod \mathfrak{M}$$
$$\Leftrightarrow \forall g \in G: d(g) \equiv 1_{n\times n} \bmod J(\langle g \rangle)$$
$$\Leftrightarrow a \in \bigcap_{C \in \mathscr{C}} \mathrm{Ker}\,\rho_{C,n},$$

proving the second equality. Finally, the proof of $\bigcap_{\mathfrak{M}} \operatorname{Ker} \rho_{\mathfrak{M},n} = \bigcap_H \operatorname{Ker} \rho_{H,n}$ is completely analogous, based on the observation that $H \subseteq G^T(\mathfrak{M})$ if and only if $\mathfrak{M} \supseteq J(H)$. This completes the proof of the proposition. ∎

2.7. *Stabilization.* For $m \geq n$, we now consider the *stabilization maps* $\mathbf{P}_n(R) \to \mathbf{P}_m(R)$, $\langle P \rangle \mapsto \langle P \oplus R^{m-n} \rangle$, the analogous map for S, and the map $\mathbf{P}_{S,n}(T) \to \mathbf{P}_{S,m}(T)$, $\langle Q \rangle \mapsto \langle Q \oplus S_T^{m-n} \rangle$. These are maps of pointed sets which are compatible with the maps Φ_n, $\operatorname{Res}^T_{S,n}$, and $\mathbf{P}_n(f)$ in the induction triangle (Subsection 2.4). Thus we obtain a commutative diagram

$$\begin{array}{ccc} \varinjlim \mathbf{P}_n(R) & \xrightarrow{\varinjlim \mathbf{P}_n(f)} & \varinjlim \mathbf{P}_n(S) \\ & {\scriptstyle \varphi = \varinjlim \Phi_n} \searrow \quad \nearrow {\scriptstyle r = \varinjlim \operatorname{Res}^T_{S,n}} & \\ & \varinjlim \mathbf{P}_{S,n}(T) & \end{array}$$

with φ injective, by [Bou2, Proposition 7, p. E III.64]. Explicitly (cf. [Bou2, p. E III.61]),

$$\varinjlim \mathbf{P}_n(R) = \biguplus_{n \geq 0} \mathbf{P}_n(R) / \sim ,$$

where $\biguplus$ denotes the disjoint union and $x \sim y$ for $x = \langle P \rangle \in \mathbf{P}_n(R)$, $y = \langle Q \rangle \in \mathbf{P}_m(R)$ iff $P \oplus R^{t-n} \cong Q \oplus R^{t-m}$ for some $t \geq \max(m, n)$. In other words, $\langle P \rangle \sim \langle Q \rangle$ iff P and Q are stably isomorphic. Now, $\biguplus_{n \geq 0} \mathbf{P}_n(R)$ is a commutative monoid under $\oplus$, and the equivalence relation $\sim$ respects this structure. Thus, $\varinjlim \mathbf{P}_n(R)$ becomes a commutative monoid with identity element the stable isomorphism class of $\{0\}$, that is, the f.g. free modules. Actually, $\varinjlim \mathbf{P}_n(R)$ is a group: If $\langle P \rangle \in \mathbf{P}_n(R)$ is given then $P \oplus Q \cong R^r$ for suitable Q and r, and hence $\langle P \rangle \oplus \langle Q \rangle = \langle R^r \rangle \sim \langle 0 \rangle$. In fact, letting $\widetilde{K_0}(R)$ denote the kernel of

$$\operatorname{rank} : K_0(R) \to H_0(R) = \{\text{continuous maps } \operatorname{Spec} R \to \mathbb{Z}\},$$

as usual (cf. [Ba, p. 459]), the map sending $\langle P \rangle \in \mathbf{P}_n(R)$ to $[P] - [R^n] \in \widetilde{K_0}(R)$ passes down to a homomorphism of groups

$$\varinjlim \mathbf{P}_n(R) \to \widetilde{K_0}(R)$$

which is easily seen to be an isomorphism (cf. [W, Chap. II, Lemma 2.3.1]).

The foregoing is valid for *any* commutative ring R, and so also applies to $\varinjlim \mathbf{P}_n(S)$. Under the identification $\varinjlim \mathbf{P}_n \cong \widetilde{K_0}$, the top map $\varinjlim \mathbf{P}_n(f)$ of the above triangle becomes

$$\widetilde{K_0}(f) = \operatorname{Ind}_R^S|_{\widetilde{K_0}(R)} : \widetilde{K_0}(R) \to \widetilde{K_0}(S).$$

Things are similar for $\varinjlim \mathbf{P}_{S,n}(T)$: Isomorphism classes $\langle X \rangle \in \mathbf{P}_{S,n}(T)$ and $\langle Y \rangle \in \mathbf{P}_{S,m}(T)$ become identified precisely if $X \oplus S_T^{t-n} \cong Y \oplus S_T^{t-m}$ for some $t \geq \max(m, n)$. However, since the submonoid of $(\biguplus_{n \geq 0} \mathbf{P}_{S,n}(T), \oplus)$ that is generated by $\langle S_T \rangle$ need no longer be cofinal, $\varinjlim \mathbf{P}_{S,n}(T)$ is merely a commutative monoid. The maps φ and r of the above triangle are monoid maps and φ is mono, as was pointed out earlier.

Thus, summarizing, we have the following stabilized induction triangle of commutative monoids and groups

$$\begin{array}{ccc} \widetilde{K_0}(R) & \xrightarrow{\widetilde{K_0}(f)} & \widetilde{K_0}(S) \\ & {\scriptstyle \varphi = (.)\otimes_R S_T} \searrow \quad \nearrow {\scriptstyle r} & \\ & \varinjlim \mathbf{P}_{S,n}(T) & \end{array}$$

LEMMA. $\operatorname{Ker} r \cong H^1(G, \mathrm{GL}(S))$ *as commutative monoids*.

Proof. Since direct limits commute with kernels [Bou2, Corollary (ii), p. E III.65]),

$$\operatorname{Ker} r = \varinjlim \left(\operatorname{Ker} \operatorname{Res}_{S,n}^T\right).$$

Next, we infer from Lemma 2.5 (and its proof) that the stabilization map $\operatorname{Ker} \operatorname{Res}_{S,n}^T \to \operatorname{Ker} \operatorname{Res}_{S,n}^T$ $(m \geq n)$ translates into the map $H^1(G, \mathrm{GL}_n(S)) \to H^1(G, \mathrm{GL}_m(S))$ that is induced by $\mathrm{GL}_n(S) \to \mathrm{GL}_m(S)$, $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, Lemma 1.3 implies that $\operatorname{Ker} r \cong H^1(G, \mathrm{GL}(S))$, at least as pointed sets. The fact that this isomorphism does respect the additive structures is a consequence of the obvious isomorphism (using the notations of Subsections 1.4, 2.5)

$$(S^m)_d \oplus (S^n)_e \cong (S^{m+n})_{d \oplus_m e}$$

for $d \in Z^1(G, \mathrm{GL}_m(S))$ and $e \in Z^1(G, \mathrm{GL}_n(S))$. ∎

2.8. *Proof of the Main Result*. For each subgroup $H \leq G$, let

$$\rho_H : H^1(G, \mathrm{GL}(S)) \to H^1(H, \mathrm{GL}(S_H)) = \operatorname{Hom}(H, \mathrm{GL}(S_H))/\mathrm{GL}(S_H)$$

be given by restriction from G to H and reduction modulo $J(H)$; so $\rho_H = \varinjlim \rho_{H,n}$ (cf. Subsection 2.6). Similarly, for $\mathfrak{M} \in \operatorname{Max} S$, put

$$\rho_{\mathfrak{M}} = \varinjlim \rho_{\mathfrak{M},n} : H^1(G, \mathrm{GL}(S)) \to H^1\big(G^T(\mathfrak{M}), \mathrm{GL}(S/\mathfrak{M})\big).$$

By Subsection 1.5, these maps are monoid maps, and hence their kernels are submonoids of $H^1(G, \mathrm{GL}(S))$. It is now a simple matter to prove the Theorem stated in the Introduction. Recall that $\mathscr{C}$ denotes the set of cyclic subgroups of G.

THEOREM. $\operatorname{Ker} K_0(f) \cong \mathrm{U}(H^1(G, \mathrm{GL}(S)))$, *an isomorphism of groups, and*

$$\mathrm{U}\big(H^1(G, \mathrm{GL}(S))\big) = \bigcap_{C \in \mathscr{C}} \operatorname{Ker} \rho_C = \bigcap_{H \le G} \operatorname{Ker} \rho_H = \bigcap_{\mathfrak{M} \in \operatorname{Max} S} \operatorname{Ker} \rho_{\mathfrak{M}}.$$

If S is Noetherian of Krull dimension d and $n > d$ then $\operatorname{Ker} K_0(f) \cong \operatorname{Ker} \mathbf{P}_n(f)$.

Proof. Recall that $K_0(.)$ decomposes naturally as $K_0(.) = H_0(.) \oplus \widetilde{K_0}(.)$, with $\widetilde{K_0}(.) \cong \varinjlim \mathbf{P}_n(.)$. If the ring in question is Noetherian of Krull dimension d, then $\varinjlim \mathbf{P}_n(.) \cong \mathbf{P}_m(.)$ holds for any $m > d$ [Ba2].

Applying this to the inclusion $f : R \hookrightarrow S$, we obtain that $K_0(f) = H_0(f) \oplus \widetilde{K_0}(f)$. Here, $H_0(f)$ is injective (cf. [Ba, Lemma (3.1), p. 459]), and hence

$$\operatorname{Ker} K_0(f) = \operatorname{Ker} \widetilde{K_0}(f) \cong \varinjlim \operatorname{Ker} \mathbf{P}_n(f)$$

(again using the fact that direct limits commute with kernels). Moreover, if S is Noetherian of Krull dimension d (and hence so is R, by virtue of hypothesis (*)), then $\operatorname{Ker} K_0(f) \cong \operatorname{Ker} \mathbf{P}_n(f)$ for $n > d$ which establishes the last assertion of the theorem.

Writing the formula for $\operatorname{Ker} \mathbf{P}_n(f)$ in Proposition 2.6 as

$$\operatorname{Ker} \mathbf{P}_n(f) \cong \operatorname{Ker} \rho_n,$$

where $\rho_n = \{\rho_{C,n}\} : H^1(G, \mathrm{GL}_n(S)) \to \prod_{C \in \mathscr{C}} H^1(C, \mathrm{GL}_n(S_C))$, the stabilization map $\operatorname{Ker} \mathbf{P}_n(f) \to \operatorname{Ker} \mathbf{P}_m(f)$ $(m \ge n)$ becomes the map on H^1 that is induced by the maps $\mathrm{GL}_n \to \mathrm{GL}_m$, $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for S and S_C. Thus, using Lemma 1.3 and the fact that direct limits commute with kernels and with finite direct products (cf. [Bou2, *loc. cit.* and Proposition 10/Corollary, p. E III.67/8]), we deduce that

$$\operatorname{Ker} K_0(f) \cong \operatorname{Ker} \rho,$$

where $\rho = \{\rho_C\} : H^1(G, \mathrm{GL}(S)) \to \prod_{C \in \mathscr{C}} H^1(C, \mathrm{GL}(S_C))$. The isomorphism is additive, by Lemma 2.7. Since $\operatorname{Ker} K_0(f)$ is a group, its image in $H^1(G, \mathrm{GL}(S))$ must be contained in the unit group $\mathrm{U}(H^1(G, \mathrm{GL}(S)))$. On the other hand, we infer from Lemma 1.6 that $\mathrm{U}(H^1(G, \mathrm{GL}(S)))$ is contained in $\bigcap_{H \leq G} \operatorname{Ker} \rho_H$, which in turn is clearly contained in $\operatorname{Ker} \rho = \bigcap_{C \in \mathscr{C}} \operatorname{Ker} \rho_C$. Hence equality must hold throughout, that is,

$$\bigcap_{C \in \mathscr{C}} \operatorname{Ker} \rho_C = \bigcap_{H \leq G} \operatorname{Ker} \rho_H = \mathrm{U}\big(H^1(G, \mathrm{GL}(S))\big).$$

Finally, the equality $\bigcap_{H \leq G} \operatorname{Ker} \rho_H = \bigcap_{\mathfrak{M} \in \operatorname{Max} S} \operatorname{Ker} \rho_{\mathfrak{M}}$ is proved exactly as in the proof of Proposition 2.6, thereby completing the proof of the theorem. ∎

3. APPLICATIONS AND PROBLEMS

3.1. *Galois Actions*. The G-action on S is *Galois*, in the sense of Auslander and Goldman [AG], if and only if $G^T(\mathfrak{M})$ is trivial for all $\mathfrak{M} \in \operatorname{Max} S$ (cf. [CHR, Theorem 1.3, p. 4]). In this case, hypothesis (*) is satisfied [CHR, Lemma 1.6, p. 7]. Thus, by Proposition 2.6 and Theorem 2.8,

$$\operatorname{Ker} \mathbf{P}_n(f) \cong H^1(G, \mathrm{GL}_n(S)) \qquad \text{and} \qquad \operatorname{Ker} K_0(f) \cong H^1(G, \mathrm{GL}(S)).$$

In particular, $H^1(G, \mathrm{GL}(S))$ is a group with the operation of Subsection 1.4. In fact:

PROPOSITION. *If the action of G on S is Galois then* $\operatorname{Ker} K_0(f) \cong H^1(G, \mathrm{GL}(S))$ *is annihilated by a power of* $|G|$. *For S Noetherian of Krull dimension d, $|G|^d$ will do.*

Proof. By [CHR, Lemma 4.1, p. 13], S is f.g. projective of constant rank equal to $|G|$ as an R-module. Therefore,

$$[S_R] - |G|[R] = [S_R] - |G| 1_{K_0(R)} \in \widetilde{K_0}(R)$$

and, moreover, the restriction map $\operatorname{Res}_R^S : K_0(S) \to K_0(R)$ is defined. The composite $\operatorname{Res}_R^S \circ K_0(f) : K_0(R) \to K_0(R)$ is clearly multiplication with $[S_R]$. Hence,

$$\operatorname{Ker} K_0(f) \subseteq \operatorname{ann}_{K_0(R)}([S_R]).$$

Recall that $\widetilde{K_0}(R)$ is a nil ideal of $K_0(R)$, and if S (or, equivalently, R) is Noetherian of Krull dimension d, then $\widetilde{K_0}(R)^{d+1} = \{0\}$ [Ba, pp. 477, 473]. Furthermore, $\operatorname{Ker} K_0(f) \subseteq \widetilde{K_0}(R)$ (cf. the proof of Theorem 2.8). Hence, $\operatorname{Ker} K_0(f) \cdot ([S_R] - |G|)^t = \{0\}$ for some t, with $t = d$ a possible choice for S Noetherian of Krull dimension d. Consequently, $\operatorname{Ker} K_0(f) \cdot |G|^t = \{0\}$. ∎

The proposition can be viewed as an extension of Hilbert's "Theorem 90" (cf. [Se, Lemma 1, p. 129]) to commutative rings.

3.2. I don't know if Proposition 3.1 generalizes to arbitrary G-actions:

QUESTION. *Is* $\operatorname{Ker} K_0(f) \cong \mathrm{U}(H^1(G, \mathrm{GL}(S)))$ *always annihilated by* $|G|^d$ *if S is Noetherian of Krull dimension d*?

This is indeed so for $d = 0$, in which case $K_0(R) = H_0(R)$ and $K_0(f) = H_0(f)$ is mono, and for $d = 1$, where $K_0(R) = H_0(R) \oplus \operatorname{Pic}(R)$ and $\operatorname{Ker} K_0(f) = \operatorname{Ker} \operatorname{Pic}(f)$ is isomorphic to a subgroup of $H^1(G, \mathrm{U}(S))$ (cf. Subsection 3.3 below). By a routine direct limit argument, a positive answer to the above question would imply that the kernel of the induction map $K_0(f)\colon K_0(R) \to K_0(S)$ is always $|G|$-primary (i.e., every element is annihilated by a power of $|G|$), for any commutative ring S. Finally, I note that the dual statement for G_0 is known to hold [BrL1, BrL2]: The cokernel of the restriction map $G_0(S) \to G_0(R)$ is annihilated by $|G|^{d+1}$. The proof given in [BrL1] results from an analysis of the so-called *coniveau filtration* of G_0. The key to the above problem might very well be the *Grothendieck γ-filtration* [SGA6, FL]

$$K_0(R) = F_\gamma^0 K_0 \supseteq \widetilde{K_0}(R) = F_\gamma^1 K_0 \supseteq \cdots \supseteq F_\gamma^{d+1} K_0 = 0.$$

The first two slices are $F_\gamma^0/F_\gamma^1 = H_0(R)$ and $F_\gamma^1/F_\gamma^2 = \operatorname{Pic}(R)$. Not much appears to be known about the higher slices.

3.3. *Picard Groups*. For any commutative ring $\mathscr{R}$, the set $\mathbf{P}_1(\mathscr{R})$ of isomorphism classes of f.g. projective $\mathscr{R}$-modules of constant rank 1 forms a group under $\otimes_{\mathscr{R}}$, with identity element the distinguished element $\langle \mathscr{R} \rangle$. This group is the *Picard group* of $\mathscr{R}$, usually denoted $\operatorname{Pic}(\mathscr{R})$ (cf. [Ba, p. 131ff]).

Specializing Proposition 2.6 to the case $n = 1$ and letting $\mathrm{U}(.) = \mathrm{GL}_1(.)$ denote groups of units, we obtain the following result (cf. [Kr, DMV, L]):

PROPOSITION. *There is an isomorphism of groups*

$$\begin{aligned}\operatorname{Ker}\operatorname{Pic}(f) &\cong \bigcap_{C\in\mathscr{C}} \operatorname{Ker}\bigl(H^1(G,\mathrm{U}(S)) \to \operatorname{Hom}(C,\mathrm{U}(S_C))\bigr)\\ &= \bigcap_{H\le G} \operatorname{Ker}\bigl(H^1(G,\mathrm{U}(S)) \to \operatorname{Hom}(H,\mathrm{U}(S_H))\bigr)\end{aligned}$$

and an exact sequence of commutative monoids

$$1 \to \operatorname{Ker}\sigma \to \operatorname{Ker} K_0(f) \to \operatorname{Ker}\operatorname{Pic}(f) \to 1,$$

where $\sigma : H^1(G,\mathrm{SL}(S)) \to \prod_{C\in\mathscr{C}} H^1(C,\mathrm{SL}(S_C))$.

Proof. The fact that the isomorphism of Proposition 2.6 is an isomorphism of groups, not just of pointed sets, for $n = 1$ is a consequence of the identity of T-modules $S_d \otimes_S S_e \cong S_{de}$ for $d, e \in Z^1(G,\mathrm{U}(S))$. The exact sequence is a consequence of the exact sequence $1 \to \mathrm{SL}(.) \to \mathrm{GL}(.) \xrightarrow{\det} \mathrm{U}(.) \to 1$ which is split by the canonical embedding $\mathrm{U}(.) = \mathrm{GL}_1(.) \hookrightarrow \mathrm{GL}(.)$. Indeed, this sequence leads to a commutative diagram of pointed sets [Se, Sects. 5.4, 5.5]

$$\begin{array}{ccc}
1 & & 1\\
\downarrow & & \downarrow\\
H^1(G,SL(S)) & \xrightarrow{\sigma} & \prod_{C\in\mathscr{C}} H^1(C,SL(S_C))\\
\downarrow & & \downarrow\\
H^1(G,GL(S)) & \xrightarrow{\rho} & \prod_{C\in\mathscr{C}} H^1(C,GL(S_C))\\
\mu_1 \uparrow\downarrow \pi_1 & & \prod\pi_{1,C} \downarrow\uparrow \prod\mu_{1,C}\\
H^1(G,U(S)) & \xrightarrow{\rho_1} & \prod_{C\in\mathscr{C}} H^1(C,U(S_C))\\
\downarrow & & \downarrow\\
1 & & 1
\end{array}$$

with $\pi_1 \circ \mu_1 = \mathrm{Id}$ and $\pi_{1,C} \circ \mu_{1,C} = \mathrm{Id}$. Here, ρ and ρ_1 are the usual restriction-reduction maps, as in the proof of Theorem 2.8. The diagram yields the exact sequence

$$1 \to \operatorname{Ker}\sigma \to \operatorname{Ker}\rho \to \operatorname{Ker}\rho_1 \to 1$$

which is in fact a sequence of commutative monoids, by Subsection 1.5. Finally, $\operatorname{Ker} K_0(f) \cong \operatorname{Ker}\rho$ and $\operatorname{Ker}\operatorname{Pic}(f) \cong \operatorname{Ker}\rho_1$. ∎

3.4. *Linear Actions*. Here, $S = S(V)$ is the symmetric algebra of a finite dimensional k-vector space V and G is a subgroup of $\mathrm{GL}(V) = \mathrm{Aut}_k(V)$. The G-action on V extends uniquely to an action of G on S by k-algebra automorphisms. Hypothesis (*) amounts to the requirement that $|G|^{-1} \in k$, which will be assumed, and linear actions are never Galois. Both assertions follow from the existence of an augmentation $\varepsilon : S \to k$ which is G-invariant (i.e., $\varepsilon(s^g) = \varepsilon(s)$ holds for all $s \in S$, $g \in G$); it is given by $\varepsilon(V) = \{0\}$.

3.4.1. *The factors S_H*. We now describe the factors $S_H = S/J(H)$ of S that were introduced in Subsection 2.6. Fix a subgroup H of G and let $V(H)$ denote the subspace of V that is generated by the elements $v - v^h$ ($v \in V$, $h \in H$). Then $H \subseteq G^T(\mathfrak{M})$ iff $V(H) \subseteq \mathfrak{M}$. Thus, $J(H)$ is the intersection of all $\mathfrak{M} \in \operatorname{Max} S$ with $V(H) \subseteq \mathfrak{M}$. Now $V(H)S$ is a prime ideal of S; in fact, $S/V(H)S \cong S(V_H)$, where $V_H = V/V(H)$ is the vector space of H-coinvariants of V. Moreover, since kH is semisimple, we have $V = V^H \oplus V(H)$. Therefore, $J(H) = V(H)S$ and

$$S_H \cong S(V_H) \cong S(V^H).$$

The canonical map $S \twoheadrightarrow S_H$ is H-equivariantly split by $S_H \cong S(V^H) \hookrightarrow S$.

3.4.2. *Picard group* [K]. Here, $\operatorname{Pic}(S) = 1$ and so $\operatorname{Ker}\operatorname{Pic}(f) = \operatorname{Pic}(R)$. Also, $\mathrm{U}(S) = k^*$ and all $\mathrm{U}(S_H) = k^*$, and so the intersection in Proposition 3.3 reduces to the intersection of the kernels of the restriction maps $\operatorname{Hom}(G, k^*) \to \operatorname{Hom}(H, k^*)$ which is obviously trivial. Thus,

$$\operatorname{Pic}(R) = 1.$$

3.4.3. *A problem of Kraft*. Recall that $\operatorname{Pic}(R)$ is an image of $\widetilde{K_0}(R)$ (cf. Subsection 3.2). It is an open question, raised by Kraft [Kr, Problem 5.1], whether in fact $\widetilde{K_0}(R)$ is trivial or, equivalently, $\operatorname{Ker} K_0(f) = \{0\}$. A positive answer to Question 3.2 would easily entail this in characteristics > 0, and for fixed-point-free actions in characteristic 0. In the latter case, Kraft's problem has already been settled by a different method by Holland [Ho], at least for algebraically closed base fields k. The following proposition gives a cohomological reformulation of Holland's result and of a result of Gubeladze [Gu]. We put $S_+ = VS$ and, as usual, $\mathrm{GL}(S, S_+)$ denotes the kernel of the reduction map $\mathrm{GL}(S) \to \mathrm{GL}(S/S_+) = \mathrm{GL}(k)$, and similarly for $\mathrm{GL}_n(S, S_+)$.

PROPOSITION. *Assume that k is algebraically closed of characteristic* 0. *If G acts fixed-point-freely on V then $H^1(G, \mathrm{GL}(S, S_+))$ is trivial. If, in addition, G is abelian* (*and hence actually cyclic*) *then $H^1(G, \mathrm{GL}_n(S, S_+))$ is trivial for all n.*

Proof. By fixed-point-freeness, $V(H) = V$ holds for all non-identity subgroups H of G, and hence $J(H) = S_+$. Thus, in the notation of

Theorem 2.8, $\operatorname{Ker}\rho_G \subseteq \operatorname{Ker}\rho_H$ and so $\mathrm{U}(H^1(G, \mathrm{GL}(S))) = \operatorname{Ker}\rho_G$. Finally, by [Se, Proposition 38, p. 49], the split exact sequence of G-groups $1 \to \mathrm{GL}(S, S_+) \to \mathrm{GL}(S) \to \mathrm{GL}(S/S_+) = \mathrm{GL}(k) \to 1$ gives rise to an exact sequence of pointed sets

$$1 \to H^1(G, \mathrm{GL}(S, S_+)) \to H^1(G, \mathrm{GL}(S)) \overset{\rho_G}{\to} H^1(G, \mathrm{GL}(S/S_+)) \to 1.$$

Inasmuch as $\operatorname{Ker}\rho_G \cong \operatorname{Ker} K_0(f)$ is trivial, by [Ho], triviality of $H^1(G, \mathrm{GL}(S, S_+))$ follows (and conversely).

If G is abelian then R is an affine normal semigroup algebra (cf. [H]), and hence all f.g. projectives over R are free, by Gubeladze's Theorem [Gu]. In other words, all maps $\mathbf{P}_n(f)$ have trivial kernel which in turn says that $H^1(G, \mathrm{GL}_n(S, S_+))$ is trivial, exactly as above, using Proposition 2.6 instead of Theorem 2.8. ▮

3.5. *Multiplicative Actions*. In this case, $S = kA$ is the group algebra of a f.g. free abelian group A and G is a subgroup of $\mathrm{GL}(A) = \mathrm{Aut}_{\mathbb{Z}}(A)$ acting on S by means of the unique extension of the natural $\mathrm{GL}(A)$-action on A. Again, hypothesis (*) is equivalent to $|G|^{-1} \in k$ and multiplicative actions are never Galois, because of the G-invariant augmentation $\varepsilon : S \to k$ given by $\varepsilon(A) = \{1\}$.

3.5.1. *The factors* S_H. Fix a subgroup H of G, let $[A, H]$ denote the subgroup of A that is generated by the elements $a^{-1}a^h$ ($a \in A$, $h \in H$), and let $\omega([A, H])S$ denote the ideal of S that is generated by the elements $a - 1$ with $a \in [A, H]$. So $\omega([A, H])S$ is the kernel of the canonical map of group algebras $S = kA \twoheadrightarrow kA_H$, where $A_H = A/[A, H]$ denotes the H-coinvariants of A. Clearly, $H \subseteq G^T(\mathfrak{M})$ iff $\omega([A, H])S \subseteq \mathfrak{M}$. We claim that $\omega([A, H])S$ is a semiprime ideal of S. Since $|H|$ is nonzero in k, this will follow if we can show that the torsion group of A_H is annihilated by a power of $|H|$. To this end, write $(.)' = (.) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|H|]$ and view A' as a module over the group ring $\mathbb{Z}'H$. Putting $e = |H|^{-1}\sum_{h \in H} h \in \mathbb{Z}'H$, an idempotent of $\mathbb{Z}'H$, we have $A \subseteq A' = (A')^e \oplus (A')^{1-e} = (A')^H \oplus [A', H]$. Thus $A'/[A', H]$ is torsion-free, and hence so is $A/(A \cap [A', H])$. Since every element of $[A', H]/[A, H]$ is annihilated by a power of $|H|$, our claim follows. We conclude that $J(H) = \omega([A, H])S$ and

$$S_H \cong kA_H.$$

The canonical map $S \twoheadrightarrow S_H$ is not split, in general, as S_H need not be a domain.

3.5.2. *Picard group* [L]. Again, $\mathrm{Pic}(S) = 1$ and so $\mathrm{Pic}(R)$ can be determined from Proposition 3.3. Here, $\mathrm{U}(S) = k^* \times A$ and $\mathrm{U}(S_C) = k^* \times U_1$ with $A_C \subseteq U_1$, the group of normalized (augmentation 1) units of

$S_C = kA_C$ (a strict inclusion, in general). The maps $H^1(G, \mathrm{U}(S)) = \mathrm{Hom}(G, k^*) \times H^1(G, A) \to \mathrm{Hom}(C, \mathrm{U}(S_C)) = \mathrm{Hom}(C, k^*) \times \mathrm{Hom}(C, U_1)$ decompose as the direct product of the restriction maps $\mathrm{Hom}(G, k^*) \to \mathrm{Hom}(C, k^*)$ times the maps $H^1(G, A) \to H^1(C, A) \to H^1(C, A_C) = \mathrm{Hom}(C, A_C) \hookrightarrow \mathrm{Hom}(C, U_1)$. The first factor contributes nothing to the kernel, as for linear actions. Since the map $H^1(C, A) \to H^1(C, A_C)$ is mono for cyclic C (e.g., [B, p. 79]), the contribution from the second factor is the kernel of the restriction map $H^1(G, A) \to H^1(C, A)$. Therefore,

$$\mathrm{Pic}(R) = \bigcap_{C \in \mathscr{C}} \mathrm{Ker}\big(\mathrm{Res}^G_C : H^1(G, A) \to H^1(C, A)\big).$$

This group need not be trivial; the results of a systematic computer aided investigation of all cases with rank $A \leq 4$ are reported in [L].

3.6. *Moding out the Radical.* Returning to the general situation where S is an arbitrary commutative ring satisfying (*), we briefly consider the reduction maps $p : \mathrm{GL}(S) \to \mathrm{GL}(S/\mathrm{rad}\, S)$ and $p_n : \mathrm{GL}_n(S) \to \mathrm{GL}_n(S/\mathrm{rad}\, S)$. Here, $\mathrm{rad}\, S$ denotes the *Jacobson radical* of S. The following lemma is an application of Lemma 2.5.

LEMMA. *The maps* $(p_n)^1_* : H^1(G, \mathrm{GL}_n(S)) \to H^1(G, \mathrm{GL}_n(S/\mathrm{rad}\, S))$ *have trivial kernel, and so does* $p^1_* : H^1(G, \mathrm{GL}(S)) \to H^1(G, \mathrm{GL}(S/\mathrm{rad}\, S))$.

Proof. It suffices to consider the maps $(p_n)^1_*$; the case of p^1_* then follows by taking $\varinjlim$.

Using the identification $\mathrm{Ker}(\mathrm{Res}^T_{S,n}) \cong H^1(G, \mathrm{GL}_n(S))$ of Lemma 2.5 and writing $\overline{S} = S/\mathrm{rad}\, S$ and $\overline{T} = \overline{S} * G$ (cf. Subsection 2.1), the map $(p_n)^1_*$ becomes the map $\mathrm{Ker}(\mathrm{Res}^T_{S,n}) \to \mathrm{Ker}(\mathrm{Res}^{\overline{T}}_{\overline{S},n})$ that is given by $\langle P \rangle \mapsto \langle \overline{P} = P/P\,\mathrm{rad}\, S \rangle$. Say $\langle P \rangle$ belongs to the kernel of this map, that is, $\overline{P} \cong \overline{S}^n_{\overline{T}}$ or, equivalently, $P/P\,\mathrm{rad}\, S \cong S^n_T / S^n_T \mathrm{rad}\, S$. Since $\mathrm{rad}\, S \subseteq \mathrm{rad}\, T$ (cf. [P, Theorem 7.2.5]), the Nakayama Lemma implies that $P \cong S^n_T$ (cf. [Ba, Proposition (2.12), p. 90]). Thus $\langle P \rangle$ is the distinguished element of $\mathrm{Ker}(\mathrm{Res}^T_{S,n})$, as required. ∎

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