

ON THE CLASS EQUATION FOR HOPF ALGEBRAS

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ABSTRACT. We give a simple proof of the Kac-Zhu class equation for semisimple Hopf algebras over an algebraically closed field of characteristic 0.

1. INTRODUCTION

The class equation ([K], Theorem 2, [Z], Theorem 1) expresses a fundamental arithmetic property of semisimple Hopf algebras over an algebraically closed field of characteristic 0. It has featured prominently in many classification results for semisimple Hopf algebras of small or otherwise restricted dimension, notably in the work of Masuoka (cf. [M] and the references therein). All proofs heretofore ([K], [S]) ultimately depend on the finiteness of the class number of algebraic number fields. The purpose of this short note is to give a simpler, more representation theoretic proof whose main ingredient is the known explicit form of the central primitive idempotents in split semisimple algebras.

In §2, we gather the requisite background material and give the statement of the class equation. The reader is referred to [L] or [S] for complete details. §3 then presents our proof of the class equation.

Throughout, H denotes a finite dimensional Hopf algebra over a field k of characteristic $p \geq 0$, with augmentation ε and antipode \mathcal{S} . All H -modules are assumed to be finite dimensional left modules.

2. THE CLASS EQUATION

2.1. Character algebra and Grothendieck ring. The *character algebra* $R(H)$ of H is the k -subalgebra of the dual Hopf algebra H^* that is generated by the characters χ_V of H -modules V . A \mathbb{Z} -form of $R(H)$ is provided by the *Grothendieck ring* $G_0(H)$, the abelian group generated by the isomorphism classes $[V]$ of H -modules V modulo the relations $[V] = [U] + [W]$ for each short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$. Multiplication in $G_0(H)$ is given by $[V] \cdot [W] = [V \otimes_k W]$. The *character map* $\chi : G_0(H) \rightarrow R(H)$, $[V] \mapsto \chi_V$, induces an isomorphism of k -algebras

$$R(H) \cong G_0(H) \otimes_{\mathbb{Z}} k .$$

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Putting $[V]^* = [V^*]$, where $V^* = \text{Hom}_k(V, k)$, one obtains an anti-automorphism $*$ of the ring $G_0(H)$. Similarly, the antipode \mathcal{S}^* of H^* stabilizes $R(H)$ and thus yields an anti-automorphism of $R(H)$. The character map respects these anti-automorphisms:

$$\chi_{V^*} = \mathcal{S}^*(\chi_V) .$$

We will fix a complete set of simple H -modules V_1, \dots, V_t , with $V_1 = k_\varepsilon$ the “trivial” H -module, and we put $\chi_i = \chi_{V_i}$, the irreducible characters of H . Then $\{[V_1], \dots, [V_t]\}$ is a \mathbb{Z} -basis of $G_0(H)$ and $\{\chi_1, \dots, \chi_t\}$ is a k -basis of $R(H)$, with $[V_1] = [k_\varepsilon]$ and $\chi_1 = \varepsilon$ the respective identity elements of $G_0(H)$ and $R(H)$.

Examples. (1) $H = kG$, the group algebra of a finite group G over a splitting field k . The dual Hopf algebra H^* is the algebra of k -valued functions on G , $H^* = k^G$, with pointwise algebra operations, and $R(H)$ is the subalgebra of all k -valued functions f on G that are constant on the conjugacy classes of G and satisfy $f(g) = f(g_{p'})$ for all $g \in G$. Here, $g_{p'}$ denotes the p -regular part of g , that is, the component of g whose order is not divisible by $p = \text{char } k$. So

$$R(H) \cong k^{T(G_{p'})} ,$$

where $T(G_{p'})$ denotes the set of G -conjugacy classes of p -regular elements of G . Of course, if p does not divide the order of G then $T(G_{p'})$ is the set of all conjugacy classes of G . Furthermore, via Brauer characters, $G_0(H) \subseteq G_0(H) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{T(G_{p'})}$.

(2) $H = (kG)^*$, the dual Hopf algebra of kG . Here, $R(H) = H^* = kG$ and $G_0(H) = \mathbb{Z}G$.

2.2. The semisimple case. Now assume H split semisimple. Then $G_0(H)$ is a semiprime ring and $*$ is an involution (cf. [L], Proposition 3.2 and Remark 2.5(2)). The form $\langle \cdot, \cdot \rangle : G_0(H) \times G_0(H) \rightarrow \mathbb{Z}$ that is defined by

$$\langle [V], [W] \rangle = \dim_k \text{Hom}_H(V, W^*)$$

is bilinear, associative, symmetric, and $*$ -invariant ([L], §3.1). A pair of dual \mathbb{Z} -bases of $G_0(H)$ with respect to $\langle \cdot, \cdot \rangle$ is given by $\{[V_1], \dots, [V_t]\}$ and $\{[V_1]^*, \dots, [V_t]^*\}$: $\langle [V_i], [V_j]^* \rangle = \delta_{i,j}$.

By k -linear extension to $R(H) \cong G_0(H) \otimes_{\mathbb{Z}} k$, one obtains a form $\beta : R(H) \times R(H) \rightarrow k$ with analogous properties. In particular, $R(H)$ is a *symmetric algebra*, and $\{\chi_1, \dots, \chi_t\}, \{\mathcal{S}^*(\chi_1), \dots, \mathcal{S}^*(\chi_t)\}$ is a pair of dual bases for β . If $p = \text{char } k = 0$, then $R(H)$ is semisimple, as a consequence of semiprimeness of $G_0(H)$.

2.3. The Class Equation (Kac-Zhu). Assume H semisimple and k algebraically closed of characteristic $p = 0$. Then, for every primitive idempotent $e = e^2 \in R(H)$, $\dim_k eH^*$ divides $\dim_k H$.

Examples. For $H = kG$, the class equation reduces to the fact that the size of each conjugacy class of G divides $|G|$. For $H = (kG)^*$, it says that the dimension of each simple kG -module divides $|G|$ (Frobenius’ Theorem).

3. THE PROOF

Letting A denote the ring of algebraic integers in k , it suffices to show that

$$d := \frac{\dim_k H}{\dim_k eH^*} \in A .$$

To this end, we first express d in terms of the central primitive idempotent $\widehat{e} \in \mathcal{Z}(R(H))$, the center of $R(H)$, with $e = e\widehat{e}$. In other words, $\widehat{e}R(H)$ is the simple component of the split semisimple k -algebra $R(H)$ containing e . Thus, if m denotes the dimension of the irreducible $R(H)$ -module $eR(H)$, then $\widehat{e}R(H) \cong (eR(H))^m$. This implies $\widehat{e}H^* \cong (eH^*)^m$, and hence

$$d = \frac{m \dim_k H}{\dim_k \widehat{e}H^*} .$$

Central primitive idempotents can be expressed in terms of dual bases (cf. [CR], Proposition (9.17)). In our situation, in view of §2.2, this expression takes the following form:

$$\widehat{e} = \omega(\widetilde{e})^{-1}\widetilde{e} \quad \text{with} \quad \widetilde{e} = \sum_i \mu(\mathcal{S}^*(\chi_i))\chi_i ,$$

where $\omega : \mathcal{Z}(R(H)) \rightarrow k$ denotes the central character of the irreducible $R(H)$ -module $eR(H)$ and μ its (ordinary) character; so $m = \mu(\varepsilon)$ and $\omega(z) = m^{-1}\mu(z)$. We claim that

$$d = \omega(\widetilde{e}) .$$

Indeed, defining $L_f \in \text{End}_k(H^*)$ by $L_f(g) = fg$ for $f, g \in H^*$, the element $x = (f \mapsto \text{trace}(L_f)) \in H^{**} = H$ is an integral of H ([LS], Proposition 9; recall that H is involutory, by [LR]). Since \widehat{e} is idempotent, we have $\dim_k \widehat{e}H^* = \text{trace}(L_{\widehat{e}})$, and so

$$\dim_k \widehat{e}H^* = \widehat{e}(x) = \omega(\widetilde{e})^{-1}\widetilde{e}(x) .$$

Finally, $\chi_1(x) = x(\varepsilon) = \dim_k H^* = \dim_k H$ and $\chi_i(x) = 0$ for $i > 1$, since xV_i consists of H -invariants in V_i and hence must vanish. Thus,

$$\widetilde{e}(x) = \mu(\varepsilon) \dim_k H = m \dim_k H$$

and our claim follows.

To complete the proof, we remark that if ϕ is any character of $R(H)$, then, identifying $G_0(H)$ with its image in $R(H)$ under the character map, we have $\phi(G_0(H)) \subseteq A$. For, any $\alpha \in G_0(H)$ satisfies a monic polynomial over \mathbb{Z} , and hence so does the image $\Phi(\alpha)$ under the representation Φ belonging to ϕ , and all eigenvalues of $\Phi(\alpha)$ as well. Thus, $\phi(\alpha) = \text{trace}(\Phi(\alpha)) \in A$. In particular, $\widetilde{e} = \sum_i \mu(\mathcal{S}^*(\chi_i))\chi_i$ belongs to $G_0(H) \otimes_{\mathbb{Z}} A$. Hence \widetilde{e} is integral over \mathbb{Z} and, therefore, so is $d = \omega(\widetilde{e})$.

REFERENCES

[CR] C. W. Curtis and I. Reiner, *Methods of Representation Theory, Vol. 1*, Wiley-Interscience, New York, 1981. MR **90k**:20001
 [K] G. I. Kac, *Certain arithmetic properties of ring groups*, Functional Anal. Appl. **6** (1972), 158–160. MR **46**:3687
 [LR] R. G. Larson and D. E. Radford, *Semisimple cosemisimple Hopf algebras*, Amer. J. Math. **109** (1987), 187–195. MR **89a**:16011
 [LS] R. G. Larson and M. E. Sweedler, *An associative orthogonal bilinear form for Hopf algebras*, Amer. J. Math. **91** (1969), 75–94. MR **39**:1523
 [L] M. Lorenz, *Representations of finite dimensional Hopf algebras*, J. Algebra **188** (1997), 476–505. CMP 97:08
 [M] A. Masuoka, *Some further classification results of semisimple Hopf algebras*, Commun. Algebra **24** (1996), 307–329. MR **96k**:16070

- [S] H.-J. Schneider, *Lectures on Hopf Algebras*, Lecture Notes, Universidad Nacional de Córdoba, 1995.
- [Z] Y. Zhu, *Hopf algebras of prime dimension*, *Internat. Math. Res. Notices* **1** (1994), 53–59. MR **94j**:16072

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