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Class Groups of Multiplicative Invariants

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Let G be a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$. Then G acts on the Laurent polynomial ring $k[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ over the field k via the natural G-action on the multiplicative group generated by the variables $X_1, \ldots, X_d$ ($\cong \mathbb{Z}^d$). We show that the class group of the ring of invariants of this action is isomorphic to $\mathrm{Hom}(G/N, k^*) \oplus \mathrm{H}^1(G/D, (\mathbb{Z}^d)^D)$, where N denotes the subgroup of G that is generated by all reflections in G and D the subgroup generated by the reflections that are diagonalizable over $\mathbb{Z}$.

INTRODUCTION

Let $S = kA$ denote the group algebra of a finitely generated free abelian group A over the field k. Multiplicative actions are actions of a subgroup G of $\mathrm{GL}(A)$ on S by means of the unique extension of the natural G-action on A. Identifying A with $\mathbb{Z}^d$ for some d, we can think of S as the Laurent polynomial ring $k[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ and G becomes a subgroup of $\mathrm{GL}_d(\mathbb{Z})$. In this note, we will always assume that G is finite. The ring theoretic properties of rings of invariants $R = S^G$ arising in this fashion ("multiplicative invariants") have been investigated by Farkas in a series of papers (e.g., [F, F2, F3]). In particular, Farkas raised the problem of determining when R is a unique factorization domain [F3, Section 3]. Our goal here is to answer this question by describing the class group $\mathrm{Cl}\, R$ of R.

THEOREM. $\mathrm{Cl}\, R \cong \mathrm{Hom}(G/N, k^*) \oplus \mathrm{H}^1(G/D, A^D)$, *where N denotes the subgroup of G that is generated by all reflections in G and D the subgroup generated by the reflections that are diagonalizable over $\mathbb{Z}$.*

This result makes [BrL, Theorem 5.2] precise. It also illustrates Farkas' comment [F, Introduction] that "even for reflection groups, it is rare that

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the fixed ring of the group algebra is a polynomial ring"—or even a unique factorization domain, in sharp contrast with the more classical cases of finite group actions on local unique factorization domains or polynomial algebras. In both instances, the class group of the ring of invariants is known to be isomorphic to $\mathrm{Hom}(G/N, k^*)$, where k is the residue field or the field of constants, respectively, and N is the subgroup generated by all pseudoreflections in G (cf. [Si, Be]). The proofs are all based on Samuel's work [S], and ours is no exception. The novelty in our approach, perhaps, consists in the systematic use of the functorial behavior of Samuel's exact sequence, specifically its behavior under change of the acting group. This is made explicit in Section 1 after reviewing Samuel's theory of Galois descent and some pertinent facts from general ramification theory. We also briefly comment on the aforementioned cases of group actions on local unique factorization domains and polynomial rings.

Section 2 is devoted specifically to multiplicative actions, working over a Krull domain k rather than a field, at no extra cost. After a discussion of reflections in $\mathrm{GL}_d(\mathbb{Z})$, in particular their invariant rings and the structure of the group D in the Theorem, the description of $\mathrm{Cl}\, R$ follows rather effortlessly from the material in Section 1. We illustrate the result by computing all class groups $\mathrm{Cl}\, R$ for multiplicative invariants in rank $d = 2$. This computation was partly motivated by the fact [BrL, 4.3] that, in rank 2 and over an algebraically closed field k whose characteristic does not divide the order of G, the Grothendieck group $G_0(R)$ of finitely generated R-modules has the form $G_0(R) \cong \mathbb{Z} \cdot [R] \oplus \mathrm{Cl}\, R$.

Notations and Conventions. All rings in this article are commutative with 1. For any ring S, we put

$$\mathrm{X}(S) = \{\mathfrak{P} : \mathfrak{P} \text{ is a prime ideal of } S \text{ of height } 1\}.$$

Group actions will be written exponentially, $s \mapsto s^g$, and $(\cdot)^G$ denotes G-invariants. Our notation concerning divisors, class groups, etc. follows [Fo].

1. SAMUEL'S THEORY OF GALOIS DESCENT

1.1. Ramification Indices and Maps between Class Groups

Let $R \subseteq S$ be an extension of Krull domains such that S is integral over R and let K and L denote the fields of fractions of R and S, respectively. Then $K \subseteq L$ and S is the integral closure of R in L. Sending $\mathfrak{P}$ to $\mathfrak{p} = \mathfrak{P} \cap R$, we obtain a surjective map $\mathrm{X}(S) \to \mathrm{X}(R)$. Furthermore, for each $\mathfrak{P} \in \mathrm{X}(S)$, the localization $S_{\mathfrak{P}}$ is a discrete valuation domain. In particular, there is a unique integer e such that $\mathfrak{p}S_{\mathfrak{P}} = \mathfrak{P}^e S_{\mathfrak{P}}$, the *ramification index* of $\mathfrak{p}$ in $\mathfrak{P}$. It will be denoted by $e(\mathfrak{P}/\mathfrak{p})$. We note the following obvious transitivity property: If T is a Krull domain with $R \subseteq T \subseteq S$ and $\mathfrak{q} = \mathfrak{P} \cap T$ then

$$e(\mathfrak{P}/\mathfrak{p}) = e(\mathfrak{P}/\mathfrak{q}) \cdot e(\mathfrak{q}/\mathfrak{p}). \tag{1}$$

Recall that $\operatorname{Div} S = \bigoplus_{\mathfrak{P} \in X(S)} \mathbb{Z} \operatorname{div}(\mathfrak{P}) \cong \mathbb{Z}^{(X(S))}$ is free abelian on $X(S)$. We have injective homomorphisms

$$j = j_R^S : \operatorname{Div} R \hookrightarrow \operatorname{Div} S, \operatorname{div} \mathfrak{p} \mapsto \sum_{\mathfrak{P} : \mathfrak{P} \cap R = \mathfrak{p}} e(\mathfrak{P}/\mathfrak{p}) \operatorname{div} \mathfrak{P} \qquad (\mathfrak{p} \in X(R))$$

$$k = j|_{\operatorname{Prin} R} : \operatorname{Prin} R \hookrightarrow \operatorname{Prin} S, \qquad \operatorname{div}(xR) \mapsto \operatorname{div}(xS) \qquad (x \in K^*).$$

Thus j passes down to a homomorphism of class groups

$$i = i_R^S : \operatorname{Cl} R \to \operatorname{Cl} S$$

([Fo, in particular, pp. 30–31]).

1.2. Finite Group Actions

Let S be a Krull domain and let G be a finite group acting faithfully by automorphisms on S. Putting $R = S^G$, the ring of G-invariants in S, we are in the situation of Section 1.1 (cf. [S, Chap. III, Section 1]). Moreover, the field extension $K = Q(R) \subseteq L = Q(S)$ is Galois, with Galois group G, and all primes $\mathfrak{P}$ of S lying over a given prime $\mathfrak{p}$ of R are conjugate under G (cf. [Bou, Chap. 5, pp. 34, 42]). In particular, the ramification index $e(\mathfrak{P}/\mathfrak{p})$ in Section 1.1 depends only on $\mathfrak{p} \in X(R)$, and so we can put

$$e(\mathfrak{p}) = e(\mathfrak{P}/\mathfrak{p}),$$

where $\mathfrak{P}$ is any prime of S lying over $\mathfrak{p}$. Let $G^T(\mathfrak{P}) = \{g \in G : s^g - s \in \mathfrak{P}$ for all $s \in S\}$ denote the *inertia group* of $\mathfrak{P}$ and put $T = S^{G^T(\mathfrak{P})}$, the ring of $G^T(\mathfrak{P})$-invariants in S. Then

$$e(\mathfrak{p}) = e(\mathfrak{P} \cap T). \tag{2}$$

This number divides $|G^T(\mathfrak{P})|$ and equality holds, for example, if $|G^T(\mathfrak{P})|$ is nonzero in $R/\mathfrak{p}$. In view of (1), it follows that $e(\mathfrak{p})$ is divisible by the order of any subquotient of $G^T(\mathfrak{P})$ whose order is nonzero in $R/\mathfrak{p}$ ([Se], pp. 20–22, with $A = R_{\mathfrak{p}} \subseteq B = S_{\mathfrak{p}}$).

1.3. Samuel's Exact Sequence

Keep the notations of Section 1.2. The group G operates on $\operatorname{Div} S$ via its obvious action on $X(S)$, and this action stabilizes $\operatorname{Prin} S$: $\operatorname{div}(xS)^g = \operatorname{div}(x^g S)$. Thus G also acts on $\operatorname{Cl} S$. Since the image of the map j in Section 1.1 clearly consists of G-invariants, j gives rise to maps

$$j' : \operatorname{Div} R \hookrightarrow (\operatorname{Div} S)^G, \quad \operatorname{div} \mathfrak{p} \mapsto e(\mathfrak{p}) \cdot \sum_{\mathfrak{P} : \mathfrak{P} \cap R = \mathfrak{p}} \operatorname{div} \mathfrak{P} \qquad (\mathfrak{p} \in X(R)),$$

$$k' = j'|_{\operatorname{Prin} R} : \operatorname{Prin} R \hookrightarrow (\operatorname{Prin} S)^G, \operatorname{div}(xR) \mapsto \operatorname{div}(xS) \qquad (x \in K^*),$$

$$i' : \operatorname{Cl} R \to (\operatorname{Cl} S)^G.$$

These maps are related by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Prin} R & \xrightarrow{\subseteq} & \operatorname{Div} R & \longrightarrow & \operatorname{Cl} R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle i'} & & \\ 0 & \longrightarrow & (\operatorname{Prin} S)^G & \xrightarrow{\subseteq} & (\operatorname{Div} S)^G & \longrightarrow & (\operatorname{Cl} S)^G & & \end{array} \tag{3}$$

The cokernels of j' and k' can be described as follows. The group $(\operatorname{Div} S)^G$ is free abelian on the set of G-orbits in $X(S)$ and lying over identifies this orbit set with $X(R)$. Hence we have an isomorphism $(\operatorname{Div} S)^G \xrightarrow{\cong} \mathbb{Z}^{(X(R))}$ sending the orbit sum $\sum_{\mathfrak{P} : \mathfrak{P} \cap R = \mathfrak{p}} \operatorname{div} \mathfrak{P}$ to the basis element corresponding to $\mathfrak{p}$. So

$$\operatorname{Coker} j' \cong \bigoplus_{\mathfrak{p} \in X(R)} \mathbb{Z}/e(\mathfrak{p})\mathbb{Z}.$$

The exact sequence $1 \to U(S) \xrightarrow{\subseteq} L^* \xrightarrow{\operatorname{div}} \operatorname{Prin} S \to 0$ consists of G-equivariant maps and, hence, it leads to a cohomology sequence

$$\begin{aligned} 1 &\to U(S)^G = U(R) \xrightarrow{\subseteq} (L^*)^G = K^* \\ &\to (\operatorname{Prin} S)^G \to H^1(G, U(S)) \to H^1(G, L^*) = 0 \to \dots . \end{aligned}$$

Here the last equality holds by Hilbert's theorem 90. Since $K^*/U(R) \cong \operatorname{Prin} R$, we obtain an exact sequence $0 \to \operatorname{Prin} R \xrightarrow{k'} (\operatorname{Prin} S)^G \to H^1(G, U(S)) \to 0$, whence

$$\operatorname{Coker} k' \cong H^1(G, U(S)).$$

Since j' is injective and $\operatorname{Ker} i' = \operatorname{Ker} i$, the Snake Lemma applied to (3) now yields Samuel's exact sequence

$$0 \to \operatorname{Ker} i \xrightarrow{\mu} \mathrm{H}^1(G, U(S)) \xrightarrow{\phi} \bigoplus_{\mathfrak{p} \in \mathrm{X}(R)} \mathbb{Z}/\mathrm{e}(\mathfrak{p})\mathbb{Z} \to \operatorname{Coker} i' \qquad (4)$$

([S, Chap. III, Section 1]; cf. also [Fo, pp. 82–83] or [BrL2, Section 2.5]).

1.4. Changing the Group

In the situation of Section 1.3, let H be a subgroup of G and put $T = S^H$. Our goal is to compare Samuel's exact sequences (4) for G and for H. For emphasis, we will write $(4)_G, i_H$, etc. to indicate the acting group. First, we have a commutative diagram $(3)_H$ analogous to $(3)_G$; replace G by H and R by T throughout. The inclusions $R \subseteq T \subseteq S$ imply that $j_G = j_H \circ j_R^T$, and similarly for the maps k and i. Therefore, connecting the first rows of $(3)_G$ and $(3)_H$ by means of the maps k_R^T, j_R^T, and i_R^T and the second rows via the obvious inclusion maps, we have a "morphism" from $(3)_G$ to $(3)_H$. By naturality of the Snake Lemma sequence (cf. [HS, p. 100]), this induces a morphism from $(4)_G$ to $(4)_H$. Explicitly, the map $\operatorname{Ker} i_G \to \operatorname{Ker} i_H$ is given by i_R^T, the map $\mathrm{H}^1(G, U(S)) \to \mathrm{H}^1(H, U(S))$ is the usual cohomology restriction map, and the map $\bigoplus_{\mathfrak{p} \in \mathrm{X}(R)} \mathbb{Z}/\mathrm{e}(\mathfrak{p})\mathbb{Z} \to \bigoplus_{\mathfrak{q} \in \mathrm{X}(T)} \mathbb{Z}/\mathrm{e}(\mathfrak{q})\mathbb{Z}$ has the following description: For each $\mathfrak{p} \in \mathrm{X}(R)$ let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ denote the primes of T lying over $\mathfrak{p}$ and let $(\rho_H^G)_{\mathfrak{p}}$: $\mathbb{Z}/\mathrm{e}(\mathfrak{p})\mathbb{Z} \to \bigoplus_{i=1}^t \mathbb{Z}/\mathrm{e}(\mathfrak{q}_i)\mathbb{Z}$ be the canonical "diagonal" map. (Recall that $\mathrm{e}(\mathfrak{p}) = \mathrm{e}(\mathfrak{q}_i) \cdot \mathrm{e}(\mathfrak{q}_i/\mathfrak{p})$ by Section 1.1, Eq. (1).) The required map is the direct sum of the $(\rho_H^G)_{\mathfrak{p}}$'s, viewing each $\bigoplus_{i=1}^t \mathbb{Z}/\mathrm{e}(\mathfrak{q}_i)\mathbb{Z}$ as canonically embedded in $\bigoplus_{\mathfrak{q} \in \mathrm{X}(T)} \mathbb{Z}/\mathrm{e}(\mathfrak{q})\mathbb{Z}$. Denoting this map by ρ_H^G, we thus have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{Ker} i_G & \xrightarrow{\mu_G} & \mathrm{H}^1(G, \mathrm{U}(S)) & \xrightarrow{\phi_G} & \bigoplus_{\mathfrak{p} \in \mathrm{X}(R)} \mathbb{Z}/\mathrm{e}(\mathfrak{p})\mathbb{Z} & \to & \operatorname{Coker} i'_G \\
 & & \downarrow i_R^T|_{\operatorname{Ker} i_G} & & \downarrow \operatorname{Res}_H^G & & \downarrow \rho_H^G & & \downarrow \\
0 & \to & \operatorname{Ker} i_H & \xrightarrow{\mu_H} & \mathrm{H}^1(H, \mathrm{U}(S)) & \xrightarrow{\phi_H} & \bigoplus_{\mathfrak{q} \in \mathrm{X}(T)} \mathbb{Z}/\mathrm{e}(\mathfrak{q})\mathbb{Z} & \to & \operatorname{Coker} i'_H
\end{array} \qquad (5)$$

The verification is straightforward from the description of the sequence (4) given in Section 1.3.

1.5. Controlling Ramification

In the setting of Section 1.2, we will say that a collection $\mathscr{X}$ of subgroups of G *controls ramification of R in S* if the following condition is satisfied:

For every $\mathfrak{p} \in X(R)$ with $e(\mathfrak{p}) \neq 1$ there exists $H \in \mathscr{X}$ and a prime ideal $\mathfrak{q}$ of S^H lying over $\mathfrak{p}$ with $e(\mathfrak{q}) = e(\mathfrak{p})$.

The following remarks are immediate consequences of (1) and (2), respectively:

• If $\mathscr{X}$ controls ramification of R in S and every member of $\mathscr{X}$ is contained, up to G-conjugacy, in some member of $\mathscr{Y}$ then $\mathscr{Y}$ controls ramification as well.

• The collection of all nonidentity inertia groups $G^T(\mathfrak{P})$ ($\mathfrak{P} \in X(S)$) controls ramification of R in S.

PROPOSITION. *If $\mathscr{X}$ is any collection of subgroups of G which controls ramification of R in S then μ_G yields an isomorphism* $\operatorname{Ker} i_G \cong \bigcap_{H \in \mathscr{X}} (\operatorname{Res}^G_H)^{-1}(\operatorname{Im} \mu_H)$.

Proof. The map $\rho = \{\rho^G_H\}_{H \in \mathscr{X}} : \bigoplus_{\mathfrak{p} \in X(R)} \mathbb{Z}/e(\mathfrak{p})\mathbb{Z} \to \prod_{H \in \mathscr{X}} \bigoplus_{\mathfrak{q} \in X(S^H)} \mathbb{Z}/e(\mathfrak{q})\mathbb{Z}$ is injective. For, if $0 \neq a = (a_{\mathfrak{p}}) \in \bigoplus_{\mathfrak{p} \in X(R)} \mathbb{Z}/e(\mathfrak{p})\mathbb{Z}$ is given with $a_{\mathfrak{p}_0} \neq 0$, say, then choose $H \in \mathscr{X}$ and $\mathfrak{q} \in X(S^H)$ such that $\mathfrak{q}$ lies over $\mathfrak{p}_0$ and $e(\mathfrak{q}) = e(\mathfrak{p}_0)$. Then $(\rho^G_H)_{\mathfrak{p}_0}(a_{\mathfrak{p}_0}) \neq 0$ and so $\rho^G_H(a) \neq 0$. Thus ρ is injective and we deduce from (5) above that

$$\mu_G(\operatorname{Ker} i_G) = \operatorname{Ker} \phi_G = \operatorname{Ker} \rho \circ \phi_G = \operatorname{Ker}\left(\{\phi_H \circ \operatorname{Res}^G_H\}_{H \in \mathscr{X}}\right)$$
$$= \bigcap_{H \in \mathscr{X}} \left(\operatorname{Res}^G_H\right)^{-1}(\operatorname{Im} \mu_H),$$

as claimed. ∎

COROLLARY. *If S and all S^H ($H \in \mathscr{X}$) are unique factorization domains then* $\operatorname{Cl} R \cong \bigcap_{H \in \mathscr{X}} \operatorname{Ker}(\operatorname{Res}^G_H)$.

1.6. Polynomial Rings (cf. [Be])

Let $S = S(V)$ denote the symmetric algebra of a finite-dimensional k-vector space V and let G be a finite subgroup of $\operatorname{GL}(V)$, acting on S via the unique extension of the canonical action on V. We put $\operatorname{char} k = p$ (≥ 0). The nonidentity inertia groups $E = G^T(\mathfrak{P})$ for $\mathfrak{P} \in X(S)$ are easily described: $\mathfrak{P}$ contains the nonzero subspace $W = \sum_{e \in E} V^{e-1}$ of V. Hence W must be one-dimensional, say $W = kv$, and $\mathfrak{P} = vS$. Also, E consists of pseudoreflections, since it acts trivially on V/W. Putting $E_p = \{e \in E : v^e = v\} = E \cap \operatorname{SL}(V)$, an elementary abelian p-group, we have $G^T(\mathfrak{P}) = E_p \rtimes H$, where $H \cong \det(E)$ is cyclic of order t, say. Since t is not divisible by p, Maschke's theorem implies that $V \cong W \oplus V/W$ as kH-modules. It follows that $S^H \cong S(V/W)[v^t]$ is a polynomial ring. Furthermore, $v^t \in \mathfrak{P} \cap S^E$ and so we conclude from Section 1.2 that $t = e(\mathfrak{P} \cap R) = e(\mathfrak{P} \cap S^H)$. This shows that the set $\mathscr{X}$ of cyclic subgroups H of G that are generated by pseudoreflections of order not divisible by p controls ramification of R in S. Finally, $H^1(G, U(S)) \cong \mathrm{Hom}(G, k^*)$, since $U(S) = k^*$, and $\mathrm{Ker}(\mathrm{Res}_H^G)$ is the set of homomorphisms that are trivial on H. The corollary thus implies that $\mathrm{Cl}\, R$ is isomorphic with the subgroup of $\mathrm{Hom}(G, k^*)$ consisting of those homomorphisms that are trivial on all pseudoreflections of p'-order, hence, on all pseudoreflections. In other words, letting N denote the subgroup of G that is generated by all pseudoreflections, we have $\mathrm{Cl}\, R \cong \mathrm{Hom}(G/N, k^*)$.

1.7. Local Unique Factorization Domains ([Si])

Let G be a finite group acting faithfully on the Noetherian local unique factorization domain S and suppose that the G-action on the residue field $k = S/\mathfrak{M}$ of S is trivial and that $|G|^{-1} \in k$. Then G acts on the k-vector space $V = \mathfrak{M}/\mathfrak{M}^2$ and the corresponding map $\epsilon : G \to \mathrm{GL}(V)$ is injective (e.g., [Si, Lemma 3]). Furthermore, $H^1(G, 1 + \mathfrak{M}) = 0$ [Si, Lemma 2] and so the canonical map $\pi : U(S) \twoheadrightarrow k^*$ yields an embedding $H^1(G, U(S)) \hookrightarrow \mathrm{Hom}(G, k^*)$. This embedding is an isomorphism if $U(S)$ contains an element of order equal to the exponent of G, or if π splits. Now let $\mathfrak{P} \in X(S)$ be given and put $H = G^T(\mathfrak{P})$. Since xS is prime for every $x \in \mathfrak{M}$, $x \notin \mathfrak{M}^2$, the subspace $W = \mathfrak{P} + \mathfrak{M}^2/\mathfrak{M}^2$ of V has dimension 0 or 1. Furthermore, H acts trivially on V/W. Thus, either $H = \langle 1 \rangle$ or $H = \langle h \rangle$ for some element h such that $\epsilon(h)$ is a pseudoreflection on V. In any event, S^H is a unique factorization domain: the map ϕ_H of (5) is identical with the embedding $H^1(H, U(S)) \hookrightarrow \mathrm{Hom}(H, k^*) \cong H \cong \mathbb{Z}/e(\mathfrak{P} \cap R)\mathbb{Z} \hookrightarrow \bigoplus_{\mathfrak{q} \in X(S^H)} \mathbb{Z}/e(\mathfrak{q})\mathbb{Z}$. Corollary 1.5, with $\mathscr{X}$ the set of all nonidentity inertia groups $G^T(\mathfrak{P})$ for $\mathfrak{P} \in X(S)$, thus implies that $\mathrm{Cl}\, R \hookrightarrow \mathrm{Hom}(G/N, k^*)$, where N denotes the subgroup of G that is generated by all $h \in G$ such that $\epsilon(h)$ is a pseudoreflection, and equality holds if $U(S)$ contains an element of order equal to the exponent of G, or if π splits.

2. MULTIPLICATIVE ACTIONS

2.1. Notations and Basic Facts

Throughout this section, $S = kA$ denotes the group algebra of a free abelian group $A \cong \mathbb{Z}^d$ over the commutative Krull domain k and G is a finite subgroup of $\mathrm{GL}(A) \cong \mathrm{GL}_d(\mathbb{Z})$. The action of G on A extends uniquely to an action on S. We continue to write $R = S^G$. Since S is isomorphic to the Laurent polynomial ring $k[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$, S is a Krull domain as well and the embedding $k \hookrightarrow S$ gives rise to an isomorphism $\mathrm{Cl}\, k \xrightarrow{\cong} \mathrm{Cl}\, S$ [Fo, pp. 7, 36]. Hence, in the notation of Section 1.4,

$$\mathrm{Cl}\, S^H = \mathrm{Ker}\, i_H \oplus \mathrm{Cl}\, k, \qquad \mathrm{Coker}\, i'_H = 0$$

hold for all subgroups H of G.

2.2. Reflections

A matrix $g \in \mathrm{GL}_d(\mathbb{Q})$ is called a *pseudoreflection* if $g - 1$ has rank 1. The pseudoreflections of finite order in $\mathrm{GL}_d(\mathbb{Q})$ are exactly the $\mathrm{GL}_d(\mathbb{Q})$-conjugates of the matrix

$$d = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

They are called *reflections*. Specializing to $g \in \mathrm{GL}(A) = \mathrm{GL}_d(\mathbb{Z})$, it is easy to see that g is a reflection if and only if g is conjugate in $\mathrm{GL}_d(\mathbb{Z})$ to either d or the matrix

$$e = \begin{pmatrix} -1 & 1 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

For brevity we will call the reflection g of "type d" or "type e" correspondingly. Note that $\mathrm{H}^1(\langle g \rangle, A) \cong \mathbb{Z}/2\mathbb{Z}$ for g of type d while $\mathrm{H}^1(\langle g \rangle, A) = 0$ for g of type e.

2.3. LEMMA. *The nonidentity inertia groups $G^T(\mathfrak{P})$, for $\mathfrak{P} \in \mathrm{X}(S)$, are exactly the subgroups H of G that are generated by a reflection. For each such H, the ring of invariants S^H has the form $S^H \cong k[X_1^{\pm 1}, \ldots, X_{d-1}^{\pm 1}, X]$. Consequently,* $\mathrm{Cl}\, S^H = \mathrm{Cl}\, k$ *and* $\mathrm{Ker}\, i_H = 0$.

Proof. If B denotes the subgroup of A that is generated by the elements $a^{-1}a^g = a^{g-1}$ for $g \in G^T(\mathfrak{P})$, $a \in A$, then $B \neq 1$ and the ideal $\omega B = \sum_{b \in B}(b-1)S$ of S is contained in $\mathfrak{P}$. Since $S/\omega B \cong k[A/B]$ and $\mathfrak{P}$ has height 1, it follows that $d - 1 = \dim kA/\mathfrak{P} \leq \dim k[A/B] = d - \mathrm{rk}\, B$, whence B is cyclic. Hence all elements $1 \neq g \in G^T(\mathfrak{P})$ are reflections. In particular, they have determinant -1. Thus, if $1 \neq g_1, g_2 \in G^T(\mathfrak{P})$ are given then $\det(g_1 g_2) = 1$, whence $g_1 g_2 = 1$. This shows that $G^T(\mathfrak{P})$ is generated by a reflection.

Conversely, if $g \in G$ is a reflection then $\sqrt{\mathrm{Im}_A(g-1)} = \{a \in A : a^n \in \mathrm{Im}_A(g-1)$ for some $n \in \mathbb{Z}\}$ is cyclic, say generated by a, and $\bar{A} = A/\langle a\rangle$ is torsion-free and g-trivial. Therefore, $\mathfrak{P} = (a-1)S$ has height 1 and $\mathfrak{P}$ is prime, since $S/\mathfrak{P} \cong k\bar{A}$. Further, $g \in G^T(\mathfrak{P})$ and so $G^T(\mathfrak{P}) = \langle g\rangle$, by the first paragraph of the proof.

Now let $H = \langle g\rangle$ for some reflection $g \in G$. Then $C = \mathrm{Ker}_A(g-1) = \{a \in A : a^g = a\}$ is a subgroup of A of corank 1 and A/C is torsion-free. Therefore, $A = C \times \langle a\rangle$ for some $a \in A$. Since $\det(g) = -1$, we have $a^g = a^{-1}c$ for some $c \in C$. Thus $\{1, a^n, (a^g)^n : n > 0\}$ is a transversal for C in A and so $kA = kC \oplus \bigoplus_{n>0} kCa^n \oplus \bigoplus_{n>0} kC(a^g)^n$ and $S^H = kA^{\langle g\rangle} = kC \oplus \bigoplus_{n>0} kC(a^n + (a^g)^n)$. Using the identity

$$a^n + (a^g)^n = \left(a^{n-1} + (a^g)^{n-1}\right)(a + a^g) - \left(a^{n-2} + (a^g)^{n-2}\right)c$$

we see that $S^H = kC[\alpha]$, where $\alpha = a + a^g$ is transcendental over kC. This proves the assertion about the structure of S^H, and the remaining assertions are now clear, by [Fo, p. 36]. ∎

2.4. LEMMA. *Let $d_1, \ldots, d_r$ be the distinct reflections of type d in G and let D denote the (normal) subgroup of G they generate. Then*:

(1) $A = \langle a_1\rangle \times \cdots \times \langle a_r\rangle \times C$, *where* $C = \bigcap_{i=1}^r \mathrm{Ker}_A(d_i - 1) = A^D$ *and* $a_i^{d_j} = a_i$ $(i \neq j)$, $a_i^{d_i} = a_i^{-1}$.

(2) *D is elementary abelian of rank r.*

(3) *The map* $\{\mathrm{res}^D_{\langle d_i\rangle}\} : H^1(D, A) \to \prod_{i=1}^r H^1(\langle d_i\rangle, A) \cong (\mathbb{Z}/2\mathbb{Z})^r$ *is an isomorphism.*

Proof. We prove (1) by induction on r. Assertions (2) and (3) will then be easy consequences. Writing $\langle a_i\rangle = \mathrm{Ker}_A(d_i + 1)$ and $C_i = \mathrm{Ker}_A(d_i - 1)$, we have $A = \langle a_i\rangle \times C_i$ for each i. Thus the case $r = 1$ of (1) is clear. Next, we consider the case of two distinct reflections d_1, d_2 of type d in G and claim the following.

Claim. $a_2 \in C_1$.

Proof of Claim. Let B denote the subgroup of A that is generated by a_1 and a_2. Then $B = \langle a_1\rangle \times (B \cap C_1)$ and so d_1 acts as a reflection of type d on B. Similarly for d_2. Put $x = d_1 d_2 \in G$. Then $x \neq 1$, because $d_1 \neq d_2$, and x acts trivially on A/B, because both d_1 and d_2 do. Hence x must act nontrivially on B, since x has finite order. In particular, B is not infinite cyclic (otherwise both d_i would act as -1), and so $\{a_1, a_2\}$ is a $\mathbb{Z}$-basis of B. The matrices of d_1 and d_2 on B_1 with respect to this basis have the form $d_1 = \left(\begin{smallmatrix} -1 & 0 \\ m & 1 \end{smallmatrix}\right)$ and $d_2 = \left(\begin{smallmatrix} 1 & n \\ 0 & -1 \end{smallmatrix}\right)$ for suitable integers m, n which must in fact be even, because the d_i are reflections of type d on B.

Note that the trace of x on B is $mn - 2$. On the other hand, since x has finite order $\neq 1$ on B, the trace has the form $\omega + \bar{\omega}$ for some complex root of unity $\omega \neq 1$. We conclude that the trace is in fact -2, and so $x = -1$ and $m = n = 0$. This proves the claim. ∎

We now finish the proof of (1). By induction, we know that $A = \langle a_1 \rangle \times \cdots \times \langle a_{r-1} \rangle \times C'$, where $C' = \bigcap_{i=1}^{r-1} C_i$ and $a_i^{d_j} = a_i$ $(i \neq j)$, $a_i^{d_i} = a_i^{-1}$ for $i, j = 1, \ldots, r-1$. The claim further implies that $a_r \in C'$ and $a_i \in C_r$ $(i = 1, \ldots, r-1)$. Therefore,

$$C_r = (\langle a_1 \rangle \times \cdots \times \langle a_{r-1} \rangle \times C') \cap C_r = \langle a_1 \rangle \times \cdots \times \langle a_{r-1} \rangle \times C,$$

since $C' \cap C_r = C$, and so

$$A = \langle a_r \rangle \times C_r = \langle a_1 \rangle \times \cdots \times \langle a_r \rangle \times C.$$

This proves (1).

For (2), just note that in view of (1) all d_i commute with each other.

As for (3), using the fact that each part in the above decomposition of A is D-stable and $\mathrm{H}^1(D, C) = \mathrm{Hom}(D, C) = 0$, we have

$$\mathrm{H}^1(D, A) \cong \prod_{i=1}^{r} \mathrm{H}^1(D, \langle a_i \rangle).$$

Similarly, $\mathrm{H}^1(\langle d_i \rangle, A) \cong \mathrm{H}^1(\langle d_i \rangle, \langle a_i \rangle)$. Using the 5-term sequence and noting that $\langle a_i \rangle^{\langle d_i \rangle} = 0$, we obtain an exact sequence

$$0 \to \mathrm{H}^1\big(D/\langle d_i \rangle, \langle a_i \rangle^{\langle d_i \rangle}\big) = 0 \to \mathrm{H}^1(D, \langle a_i \rangle) \stackrel{\mathrm{res}}{\to}$$
$$\mathrm{H}^1(\langle d_i \rangle, \langle a_i \rangle)^{D/\langle d_i \rangle} \to \mathrm{H}^1\big(D/\langle d_i \rangle, \langle a_i \rangle^{\langle d_i \rangle}\big) = 0.$$

Since $D/\langle d_i \rangle$ acts trivially on $\mathrm{H}^1(\langle d_i \rangle, \langle a_i \rangle) \cong \mathbb{Z}/2\mathbb{Z}$, we conclude that

$$\mathrm{res}^D_{\langle d_i \rangle} : \mathrm{H}^1(D, \langle a_i \rangle) \to \mathrm{H}^1(\langle d_i \rangle, \langle a_i \rangle)$$

is an isomorphism and, hence, so is the map

$$\mathrm{H}^1(D, A) \stackrel{\cong}{\to} \textstyle\prod_{i=1}^r \mathrm{H}^1(D, \langle a_i \rangle) \stackrel{\prod \mathrm{res}^D_{\langle d_i \rangle}}{\longrightarrow} \prod_{i=1}^r \mathrm{H}^1(\langle d_i \rangle, \langle a_i \rangle)$$
$$\stackrel{\cong}{\to} \textstyle\prod_{i=1}^r \mathrm{H}^1(\langle d_i \rangle, A).$$

Since the latter map equals $\{\mathrm{res}^D_{\langle d_i \rangle}\}$, the lemma is proved. ∎

2.5. THEOREM (Notations 2.1). *Let N denote the (normal) subgroup of G that is generated by all reflections in G and let D denote the (normal) subgroup generated by the reflections of type d (cf. Lemma 2.4). Then*

$$\mathrm{Cl}\, R \cong \mathrm{Cl}\, k \oplus \mathrm{Hom}(G/N, \mathrm{U}(k)) \oplus \mathrm{H}^1(G/D, A^D).$$

Proof. In view of Section 2.1, it suffices to show that $\operatorname{Ker} i_G \cong \operatorname{Hom}(G/N, \mathrm{U}(k)) \oplus \mathrm{H}^1(G/D, A^D)$. For this, we will use Proposition 1.5 with $\mathscr{X}$ the set of all subgroups H of G that are generated by a reflection. Since $\operatorname{Im} \mu_H = 0$ for all such H, by Lemma 2.3, we obtain that $\operatorname{Ker} i_G \cong \bigcap_H \operatorname{Ker}(\operatorname{Res}_H^G)$, where $\operatorname{Res}_H^G : \mathrm{H}^1(G, \mathrm{U}(S)) \to \mathrm{H}^1(H, \mathrm{U}(S))$. But $\mathrm{U}(S) = \mathrm{U}(k) \times A$ and so $\mathrm{H}^1(H, \mathrm{U}(S)) \cong \operatorname{Hom}(H, \mathrm{U}(k)) \oplus \mathrm{H}^1(H, A)$ and similarly for G. The map Res_H^G becomes the direct sum of the restriction maps $r_H^G : \operatorname{Hom}(G, \mathrm{U}(k)) \to \operatorname{Hom}(H, \mathrm{U}(k))$ and $\operatorname{res}_H^G : \mathrm{H}^1(G, A) \to \mathrm{H}^1(H, A)$. Thus $\bigcap_H \operatorname{Ker}(\operatorname{Res}_H^G) = \bigcap_H \operatorname{Ker}(r_H^G) \oplus \bigcap_H \operatorname{Ker}(\operatorname{res}_H^G)$. Clearly, $\bigcap_H \operatorname{Ker}(r_H^G) \cong \operatorname{Hom}(G/N, \mathrm{U}(k))$. In $\bigcap_H \operatorname{Ker}(\operatorname{res}_H^G)$, it suffices to let H run over the subgroups that are generated by a reflection of type d, since $\mathrm{H}^1(H, A) = 0$ if H is generated by a reflection of type e (see Section 2.2). Since all these H are contained in D and $\bigcap_H \operatorname{Ker}(\operatorname{res}_H^D) = 0$, by Lemma 2.4, we conclude that

$$\bigcap_H \operatorname{Ker}(\operatorname{res}_H^G) = \operatorname{Ker}(\operatorname{res}_D^G).$$

Finally, $\operatorname{Ker}(\operatorname{res}_D^G) \cong \mathrm{H}^1(G/D, A^D)$, and so the theorem is proved. ∎

We remark that $\mathrm{H}^1(G/D, A^D) \cong \mathrm{H}^1(G/\mathbb{C}_G(A^D), A^D)$, where $\mathbb{C}_G(A^D) = \{g \in G : a^g = a \text{ for all } a \in A^D\}$. This follows from the fact that $\mathrm{H}^1(\mathbb{C}_G(A^D)/D, A^D) \cong \operatorname{Hom}(\mathbb{C}_G(A^D)/D, A^D) = 0$, because $\mathbb{C}_G(A^D)/D$ is finite and A^D is torsion-free. Similarly, $\mathrm{H}^1(G/D, A^D) \cong \mathrm{H}^1(G, A^D)$.

2.6. EXAMPLE (The inversion action; (cf. [BrL2, Section 5.3]). Let $G = \langle g \rangle$ be the group of order 2 acting on A by $a^g = a^{-1}$ and assume that $d > 1$. (For $d = 1$, R is a polynomial ring over k and so $\operatorname{Cl} R = \operatorname{Cl} k$.) Then G contains no reflections and so $\operatorname{Cl} R \cong \operatorname{Cl} k \oplus \operatorname{Hom}(G, \mathrm{U}(k)) \oplus \mathrm{H}^1(G, A) = \operatorname{Cl} k \oplus (\mathbb{Z}/2\mathbb{Z})^\epsilon \oplus (\mathbb{Z}/2\mathbb{Z})^d$, where $\epsilon = 0$ if $\operatorname{char} Q(k) = 2$ and $\epsilon = 1$ otherwise.

2.7. EXAMPLE (The rank-2 cases). Let $d = 2$. We will describe $\operatorname{Cl} R$ for the rings of invariants $R = S^G$ for each nonidentity finite subgroup G of $\mathrm{GL}_2(\mathbb{Z})$. The classification of these groups, up to conjugation in $\mathrm{GL}_2(\mathbb{Z})$ is, of course, standard, but it is briefly sketched here in a form suitable for our purposes.

First assume that $G \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then G contains no reflections. Specifically, since $\mathrm{SL}_2(\mathbb{Z}) \cong C_6 *_{C_2} C_4$ with C_6 generated by $x_1 = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and C_4 by $x_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, G is conjugate (in $\mathrm{SL}_2(\mathbb{Z})$) to a subgroup of either $\langle x_1 \rangle$ or $\langle x_2 \rangle$. This leads to the following possibilities:

(1) $G = \langle -\mathrm{Id} \rangle \cong C_2 : \operatorname{Cl} R \cong \operatorname{Cl} k \oplus \operatorname{Hom}(C_2, \mathrm{U}(k)) \oplus (\mathbb{Z}/2\mathbb{Z})^2$ (cf. Section 2.6),

(2) $G = \langle x_2 \rangle \cong C_4 : \operatorname{Cl} R \cong \operatorname{Cl} k \oplus \operatorname{Hom}(C_4, \mathrm{U}(k)) \oplus \mathbb{Z}/2\mathbb{Z}$,

(3) $G = \langle -x_1 \rangle \cong C_3 : \mathrm{Cl}\, R \cong \mathrm{Cl}\, k \oplus \mathrm{Hom}(C_3, \mathrm{U}(k)) \oplus \mathbb{Z}/3\mathbb{Z}$,

(4) $G = \langle x_1 \rangle \cong C_6 : \mathrm{Cl}\, R \cong \mathrm{Cl}\, k \oplus \mathrm{Hom}(C_6, \mathrm{U}(k))$.

If G is not contained in $\mathrm{SL}_2(\mathbb{Z})$ then G contains a reflection. In fact, G is generated by reflections, because the reflections in G are precisely the elements of G with determinant -1. Thus $\mathrm{Cl}\, R \cong \mathrm{Cl}\, k \oplus \mathrm{H}^1(G/D, A^D)$. First suppose that G contains a reflection of type d. Replacing G by a suitable conjugate in $\mathrm{GL}_2(\mathbb{Z})$, we may assume that $d = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G$. If d is the only reflection of type d in G then it must be central. So $G \subseteq \mathbb{C}_{\mathrm{GL}_2(\mathbb{Z})}(d) = \langle d, -d \rangle$ and, hence, $G = \langle d \rangle$. If G contains another reflection of type d then (cf. Lemma 2.4) $D = \langle d, -d \rangle$ must be normal in G. Since $\mathbb{N}_{\mathrm{GL}_2(\mathbb{Z})}(D) = D \cup Dy$, where $y = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we conclude that either $G = D$ or $G = D \rtimes \langle y \rangle = \langle d, y \rangle$. In all cases, $G = D$ or $A^D = 0$ and, consequently, $\mathrm{H}^1(G/D, A^D) = 0$. Thus, summarizing:

$$\left.\begin{array}{ll} (5) & G = \langle d \rangle \cong C_2 \\ (6) & G = \langle d, -d \rangle \cong C_2 \times C_2 \\ (7) & G = \langle d, y \rangle \cong D_8 \end{array}\right\} \quad \mathrm{Cl}\, R \cong \mathrm{Cl}\, k.$$

The remaining cases have no reflections of type d. They are easily classified by considering the subgroup $G_1 = G \cap \mathrm{SL}_2(\mathbb{Z})$ which must be either trivial or conjugate to one of the groups (1)–(4). Also, if G_1 is nontrivial then $A^{G_1} = 0$ and so the 5-term sequence implies that $\mathrm{H}^1(G, A) \cong \mathrm{H}^1(G_1, A)^{G/G_1}$ which is easily computed. Here are the results:

(8) $G = \langle y \rangle \cong C_2 : \mathrm{Cl}\, R \cong \mathrm{Cl}\, k$ (cf. Lemma 2.3),

(9) $G = \langle y, -y \rangle \cong C_2 \times C_2 : \mathrm{Cl}\, R \cong \mathrm{Cl}\, k \oplus \mathbb{Z}/2\mathbb{Z}$,

(10) $G = \langle -x_1, y \rangle \cong D_6 : \mathrm{Cl}\, R \cong \mathrm{Cl}\, k \oplus \mathbb{Z}/3\mathbb{Z}$,

(11) $G = \langle -x_1, -y \rangle \cong D_6 : \mathrm{Cl}\, R \cong \mathrm{Cl}\, k$,

(12) $G = \langle x_1, y \rangle \cong D_{12} : \mathrm{Cl}\, R \cong \mathrm{Cl}\, k$.

Note that $\mathrm{Cl}\, R$ is nontrivial in cases (9) and (10), even though the groups G are generated by reflections.

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