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On Galois descent for Hochschild and cyclic homology

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Abstract. Let G be a finite group acting by automorphisms on an algebra S over some commutative ring k. We show that if the action of G restricted to the center of S is Galois in the sense of [C-H-R], then $HH_*(S^G) \cong HH_*(S)^G$. An analogous result holds for cyclic homology, provided the order of G is invertible in k.

Introduction

Let G be a finite group acting by automorphisms on an algebra S over some commutative ring k. Then G acts on the Hochschild homology $HH_*(S)$ and on the cyclic homology $HC_*(S)$ of S. The relationship between the invariants of this action on the one hand and the cyclic or Hochschild homology of the algebra of invariants S^G on the other is rather opaque. In the special situation where the action of G on S is Galois in the sense of [C-H-R] and the order of G is invertible in k, the obvious "induction" map $HH_*(S^G) \to HH_*(S)^G$ is at least surjective (see §4 below for a marginally more general formulation). It need however *not* be injective as the explicit computations of $HH_0(A_1(\mathbb{C})^G)$ for certain Galois actions on the Weyl algebra $S = A_1(\mathbb{C})$ in [A-H-V] show. In these examples, $HH_0(A_1(\mathbb{C})^G)$ is nonzero while $HH_0(A_1(\mathbb{C})) = 0$. Our goal in this article is to prove that, if the action of G restricted to the *center* of S is Galois (in which case the action will be called centrally Galois), then induction from S^G to S does in fact yield an isomorphism

$$HH_*(S^G) \cong HH_*(S)^G.$$

This is achieved in §6, and a corresponding result for cyclic homology quickly follows by the usual application of the 5-lemma to the Connes-Gysin sequence,

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provided the order of G is invertible in k. For commutative S, both isomorphisms have been obtained in [W-G] as a consequence of a general result on étale extensions of commutative algebras. Our approach, instead, is to analyze the homology $HH_*(T)$ of the skew group ring $T = S * G$ that is associated with the given action of G on S. Using a description of $HH_*(T)$ in terms of certain hyperhomology groups ([Lo]), we show that restriction from T to S yields an isomorphism $HH_*(T) \cong HH_*(S)^G$. The above isomorphism then follows by means of a Morita isomorphism between $HH_*(S^G)$ and $HH_*(T)$.

Notations and conventions

Our general reference concerning Hochschild and cyclic homology is [L] whose notation we will follow here. All algebras considered in this article are over some commutative base ring k and $\otimes$ denotes $\otimes_k$. Bimodules are understood to have identical k-operations on both sides. In addition, we will keep the following notations throughout this article.

S	will be a unital k-algebra;
G	denotes a finite group acting by k-algebra automorphisms on S; this action will be denoted $s \mapsto s^g$ ($s \in S, g \in G$);
$R = S^G$	is the subalgebra of G-invariants in S;
$T = S * G$	will denote the skew group ring of G over S.

Thus T is an associative algebra which is additively isomorphic to the ordinary group ring $S[G]$ but whose multiplication is determined by the rule $sg = gs^g$ ($s \in S, g \in G$). As S-S-bimodule, T is the direct sum of the subbimodules Sg for $g \in G$. Finally, S can be viewed as R-T-bimodule via $r \cdot s \cdot s'g = (rss')^g$ ($r \in R$, $s, s' \in S, g \in G$). Similarly, S can be made into a T-R-bimodule.

Proofs

1. *Maps on Hochschild homology*

Let A and B be k-algebras and let ${}_AP_B$ be an A-B-bimodule such that P_B is finitely generated and projective. Then there is a k-linear map on Hochschild homology

$$HH_*^P : HH_*(A) \to HH_*(B)$$

which is obtained as follows (see [Lo], §§1.2 and 1.4). Choose dual bases for P, that is, elements $p_i \in P$, $q_i \in P_B^* = Hom_B(P_B, B)$ $(i = 1, 2, \ldots, r)$ with $p = \Sigma_{i=1}^r p_i q_i(p)$ for all $p \in P$. Then the map

$$\Phi^P : C(A) \to C(B)$$

which on $C_n(A) = A^{\otimes(n+1)}$ is defined by

$$\Phi_n^P(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{(i_0, \ldots, i_n)} q_{i_0}(a_0 p_{i_1}) \otimes q_{i_1}(a_1 p_{i_2}) \otimes \cdots \otimes q_{i_n}(a_n p_{i_0})$$

is a chain map whose homotopy type is independent of the choice of the dual bases $\{p_i\}$, $\{q_i\}$ for P. Thus the induced map on homology, $HH_*^P = H_*(\Phi^P)$, is well-defined and only depends on the isomorphism type of the A-B-bimodule P. Furthermore, if C is another k-algebra and ${}_BQ_C$ is a B-C bimodule which is finitely generated and projective over C then

$$HH_*^{P \otimes_B Q} = HH_*^Q \circ HH_*^P.$$

2. *Special cases*

The following special cases will be of particular interest for our puposes.

(*a*) *Action of G on $HH_*(S)$*. Taking $A = B = S$ in §**1** and $P = Sg \subseteq T$ for $g \in G$, we obtain maps $HH_*^{Sg} : HH_*(S) \to HH_*(S)$ which yield a right action of G on $HH_*(S)$. The map HH_*^{Sg} is afforded by the chain map

$$\Phi_g = \Phi^{S_g} : C(S) \to C(S), \qquad s_0 \otimes s_1 \otimes \cdots \otimes s_n \mapsto s_0^g \otimes s_1^g \otimes \cdots \otimes s_n^g.$$

(*b*) *Induction from R to S and from* S to T. Using $A = R$, $B = S$ and $P = {}_RS_S$ we obtain an induction map

$$\mathrm{Ind}_R^S = H_*^{{}_RS_S} : HH_*(R) \to HH_*(S).$$

The canonical corresponding chain map $\Phi^{{}_RS_S} : C(R) \to C(S)$ simply comes from the inclusion $R \hookrightarrow S$, which makes it clear that

$$\mathrm{Im}\,(\mathrm{Ind}_R^S) \subseteq HH_*(S)^G,$$

where $HH_*(S)^G$ denotes the G-invariants in $HH_*(S)$. (Alternatively, this follows from the fact that ${}_RS_S \otimes_S Sg \cong {}_RS_S$ as R-S-bimodules.) Similarly, the embedding $S \hookrightarrow T$ yields an induction map $\mathrm{Ind}_S^T : HH_*(S) \to HH_*(T)$ which is easily seen to factor through the canonical epimorphism of $HH_*(S)$ onto the G-coinvariants $HH_*(S)_G$ (cf. [Lo], §2.3). Thus we obtain a map

$$\overline{\mathrm{Ind}_S^T} : HH_*(S)_G \to HH_*(T).$$

(c) *Restriction from T to S.* With $A = T$, $B = S$, and $P = {}_TT_S$ we obtain a restriction map

$$\mathrm{Res}_S^T = H_*^{{}_TT_S} : HH_*(T) \to HH_*(S).$$

Since the multiplication of T gives a T-S-isomorphism $T \otimes_S Sg \cong Tg = T$, *we deduce that* $HH_*^{Sg} \circ \mathrm{Res}_S^T = \mathrm{Res}_S^T$. Thus

$$\mathrm{Im}\,(\mathrm{Res}_S^T) \subseteq HH_*(S)^G.$$

By [Lo], Lemma 2.3(a), one has

$$\mathrm{Res}_S^T \circ \overline{\mathrm{Ind}_S^T} = \overline{\mathrm{tr}} : HH_*(S)_G \to HH_*(S)^G,$$

where $\overline{\mathrm{tr}}$ is the G-trace map on $HH_*(S)$ (see the Appendix).

We remark that the above inclusion $\mathrm{Im}\,(\mathrm{Res}_S^T) \subseteq HH_*(S)^G$ can be sharpened to

$$\mathrm{Im}\,(\mathrm{Res}_S^T) = \mathrm{tr}\,(HH_*(S)).$$

In fact, using the dual bases $p_g = g$ and $q_g(\Sigma\, xs_x) = s_g$ $(g \in G)$ of T_S one computes that the chain map $\Phi^{{}_TT_S}$ maps the element $g_0s_0 \otimes g_1s_1 \otimes \cdots \otimes g_ns_n \in C(T)$ to 0 if $g_0 \cdots g_n \neq e$ (so Res_S^T vanishes on the components $HH_*(T)_{[g]}$ with $g \neq e$; cf. [Lo], §2.2) and to

$$\mathrm{tr}\,(s_0^{g_0^{-1}} \otimes s_1^{g_1^{-1}g_0^{-1}} \otimes \cdots \otimes s_n^{g_n^{-1}\cdots g_0^{-1}})$$

otherwise. This fact will however not be needed in the proofs of our main results.

3. *Galois actions*

The action of G on S is called *Galois* if $T = TtT$, where $t = \Sigma_{g \in G}\, g \in T$. The latter condition is equivalent with the existence of elements $x_i, y_i \in S$ $(i = 1, \ldots, n)$

such that

$$\sum_{i=1}^{n} x_i y_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} x_i y_i^g = 0 \quad \text{for all } e \neq g \in G. \tag{*}$$

In this case, S is finitely generated and projective as R-module (on either side; e.g., [P], §29, Exercises 3 and 4). Thus from §1 we infer the existence of a map

$$H_*^{{}_TS_R} : HH_*(T) \to HH_*(R).$$

LEMMA. *Suppose that the action of G on S is Galois.*
(a) $\mathrm{Ind}_R^S \circ H_*^{{}_TS_R} = \mathrm{Res}_S^T$.
(b) *If there exists an* $z \in S$ *with* $\mathrm{tr}(z) = 1$ *then* $H_*^{{}_TS_R}$ *is an isomorphism.*

Proof. (a) The map $S \otimes_R S \to T, s \otimes s' \mapsto sts'$ is a T-T-bimodule isomorphism (see [Co]). Thus the left hand side in (a) is equal to $H_*^{{}_RS_S} \circ H_*^{{}_TS_R} = H_*^{{}_TS \otimes_R S_S} = H_*^{{}_TT_S}$, which proves (a).

(b) In this case, the bimodules ${}_TS_R$ and ${}_RS_T$ yield a Morita equivalence between R and T. Specifically, the map $S \otimes_T S \to R, s \otimes s' \mapsto \mathrm{tr}(ss')$ is an R-R-bimodule isomorphism (see [Co]). Therefore, $H_*^{{}_RS_T}$ is inverse to $H_*^{{}_TS_R}$. □

In view of §2, part (a) of the lemma implies the following inclusions for Galois actions:

$$\mathrm{tr}(HH_*(S)) = \mathrm{Im}(\mathrm{Res}_S^T) \subseteq \mathrm{Im}(\mathrm{Ind}_R^S) \subseteq HH_*(S)^G.$$

4. *Module structures*

Let M be an S-S-bimodule. Then the Hochschild homology of S with coefficients in M, $H(S, M)$, becomes a module over the center $Z(S)$ of S by means of the action of $Z(S)$ on the chain complex $C(S, M)$ which, for a given $z \in Z(S)$ is defined by (cf. [L], 1.1.5)

$$\lambda_z(m \otimes s_1 \otimes \cdots \otimes s_n) = (zm) \otimes s_1 \otimes \cdots \otimes s_n.$$

This yields the structure map

$$\phi : Z(S) \to \mathrm{End}_k(H_*(S, M)), \quad \phi(z) = H_*(\lambda_z).$$

Similarly, one can consider the right action of $Z(S)$ on $C(S, M)$ that is given by

$$\rho_z(m \otimes s_1 \otimes \cdots \otimes s_n) = (mz) \otimes s_1 \otimes \cdots \otimes s_n.$$

However, by [L], E.1.1.2, λ_z and ρ_z are homotopic and, consequently, they yield the same map on homology:

$$\phi(z) = H_*(\rho_z).$$

In the special case where $M = S$, the actions of G (as in §2(a)) and $Z(S)$ on $HH_*(S)$ combine to give a right $Z(S) * G$-module structure on $HH_*(S)$. Indeed, the chain maps ρ_z and Φ_g satisfy $\rho_{z^g} = \Phi_g \circ \rho_z \circ \Phi_{g^{-1}}$ for all $g \in G, z \in Z(S)$. Therefore, the Lemma in the Appendix has the following immediate consequence.

LEMMA. *Assume that there exists* $z \in Z(S)$ *with* $\mathrm{tr}(z) = 1$. *Then the trace map* $\overline{\mathrm{tr}} : HH_*(S)_G \to HH_*(S)^G$ *is an isomorphism and* $H_n(G, HH_*(S)) = 0$ *holds for all* $n > 0$.

We remark that the Lemma implies in particular that, if the action of G on S is Galois and there exists $z \in Z(S)$ with $\mathrm{tr}(z) = 1$, then all inclusions at the end of §3 are equalities.

5. *Centrally Galois actions*

We will call the action of G on S *centrally Galois* if the restricted action on the center $Z(S)$ of S is Galois or, equivalently, if the elements x_i, y_i in §3 can be chosen to belong to $Z(S)$. In this case, by [C-H-R], Lemma 1.6, there also exists an element $z \in Z(S)$ with $\mathrm{tr}(z) = 1$. In particular, the Lemmas in §§3 and 4 apply. Furthermore, we have the following vanishing result for the Hochschild homology of S with coefficients in the bimodules $Sg \subseteq T$.

LEMMA. *Suppose that the action of* G *on* S *is centrally Galois. Then* $H_*(S, Sg) = 0$ holds for all $e \neq g \in G$.

Proof. We use the maps λ_z and ρ_z of §4 in the special case where $M = Sg$. It follows from $sgz^g = zsg$ that $\rho_{z^g} = \lambda_z$, and hence the structure map $\phi : Z(S) \to \mathrm{End}_k(H_*(S, Sg))$ satisfies $\phi(z) = \phi(z^g)$ for all $z \in Z(S)$. Applying ϕ to the equations (*) in §3, we deduce that $1 = 0$ holds in $\mathrm{End}_k(H_*(S, Sg))$ if $g \neq e$ which proves the lemma. □

6. THEOREM. *Suppose that the action of G on S is centrally Galois. Then the maps* $\mathrm{Res}_S^T : HH_*(T) \to HH_*(S)^G$ *and* $\mathrm{Ind}_R^S : HH_*(R) \to HH_*(S)^G$ *are isomorphisms.*

Proof. In view of the Lemma in §3, it suffices to prove the assertion for Res_S^T. To this end, we use the following description of $HH_*(T)$ (cf. [Lo], §2.6):

$$HH_*(T) \cong \bigoplus_g H_*(\mathbf{C}_G(g), C(S, Sg)),$$

where g runs over a complete representative set of the conjugacy classes of G and $H_*(\mathbf{C}_G(g), C(S, Sg))$ denotes the hyperhomology of the centralizer $\mathbf{C}_G(g)$ of g in G with coefficients in the complex $C(S, Sg)$. By [B], (5.10) on p. 169, there exists a spectral sequence

$$E^2_{p,q} = H_0(\mathbf{C}_G(g), H_q(S, Sg)) \underset{p}{\Rightarrow} H_{p+q}(\mathbf{C}_G(g), C(S, Sg)).$$

Therefore, the Lemma in §5 implies that $H_*(\mathbf{C}_G(g), C(S, Sg)) = 0$ holds for $g \neq e$, and hence $HH_*(T)$ is isomorphic with the $(g = e)$-component of the above direct sum. For $g = e$, the spectral sequence becomes

$$E^2_{p,q} = H_p(G, HH_q(S)) \underset{p}{\Rightarrow} H_{p+q}(G, C(S)).$$

The Lemma in §4 implies that $E^2_{p,q} = 0$ holds for all $p > 0$ and, consequently, the edge homomorphism $E^2_{0,*} = H_0(G, HH_*(S)) \to H_*(G, C(S))$ is an isomorphism. The composite of this edge map with the isomorphism $H_*(G, C(S)) \cong HH_*(T)$ is just the map $\overline{\mathrm{Ind}_S^T}$ of §2(b). Thus we conclude that $\overline{\mathrm{Ind}_S^T}$ yields an isomorphism

$$\overline{\mathrm{Ind}_S^T} : HH_*(S)_G \xrightarrow{\cong} HH_*(T).$$

Finally, by §2(c) and the Lemma in §4, the composite

$$\mathrm{Res}_S^T \circ \overline{\mathrm{Ind}_S^T} = \overline{\mathrm{tr}} : HH_*(S)_G \to HH_*(S)^G$$

is an isomorphism, whence Res_S^T is an isomorphism as well, and the theorem is proved. □

7. *Cyclic homology*

In the situation of §1, there is an analogous map for cyclic homology

$$HC_*^P : HC_*(A) \to HC_*(B).$$

In particular, one has a G-action and restriction and induction maps for cyclic homology as in §2. Furthermore, the maps H_*^p and HC_*^p yield a commutative diagram of Connes-Gysin sequences (see [Lo], §1.3)

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & HH_n(A) & \longrightarrow & HC_n(A) & \longrightarrow & HC_{n-2}(A) & \longrightarrow & HH_{n-1}(A) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle H_n^p} & & \downarrow{\scriptstyle HC_n^p} & & \downarrow{\scriptstyle HC_{n-2}^p} & & \downarrow{\scriptstyle H_{n-1}^p} & \\ \cdots \longrightarrow & HH_n(B) & \longrightarrow & HC_n(B) & \longrightarrow & HC_{n-2}(B) & \longrightarrow & HH_{n-1}(B) & \longrightarrow \cdots \end{array}$$

Thus the above Theorem has the following consequence.

COROLLARY. *Suppose that the action of G on S is centrally Galois and that $|G|^{-1} \in k$. Then the maps* $\mathrm{Res}_S^T : HC_*(T) \to HC_*(S)^G$ *and* $\mathrm{Ind}_R^S : HC_*(R) \to HC_*(S)^G$ *are isomorphisms.*

Proof. We concentrate on Ind_R^S. By assumption on $|G|$, the G-fixed point functor is exact on k-modules and so the above commutative diagram yields the following commutative diagram with exact rows and with all vertical maps equal to Ind_R^S.

$$\begin{array}{ccccccccc} HC_{n-1}(R) & \longrightarrow & HH_n(R) & \longrightarrow & HC_n(R) & \longrightarrow & HC_{n-2}(R) & \longrightarrow & HH_{n-1}(R) \\ \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} \\ HC_{n-1}(S)^G & \longrightarrow & HH_n(S)^G & \longrightarrow & HC_n(S)^G & \longrightarrow & HC_{n-2}(S)^G & \longrightarrow & HH_{n-1}(S)^G \end{array}$$

The assertion thus follows from the 5-lemma by induction on n. (Note that all homologies under consideration are zero in negative degrees.) □

Appendix: The G-trace map

In this Appendix we collect some well-known facts concerning the G-trace map. Let A be any k-algebra on which G acts by automorphisms. Then, for any right module V over the skew group ring $A * G$, we can consider $V^G = H^0(G, V)$, the G-invariants in V. The G-trace map is defined by

$$\mathrm{tr} : V \to V^G, \qquad v \mapsto \sum_{g \in G} vg = vt,$$

where we have put $t = \Sigma_{g \in G}\, g \in T$. In particular, one has the usual G-trace $\mathrm{tr} : A \to A^G$. Letting $V_G = H_0(G, V) = V/(v(g-1) \mid v \in V, g \in G)$ denote the G-

coinvariants of V, one observes that the G-trace factors through the canonical epimorphism $V \to V_G$. Thus we obtain a map

$$\overline{\mathrm{tr}} : V_G \to V^G.$$

LEMMA. *Suppose that* $\mathrm{tr}\,(z) = 1$ *for some* $z \in A$. *Then, for every right* $A * G$-*module* V, *the trace map* $\overline{\mathrm{tr}} : V_G \to V^G$ *is an isomorphism and* $H_n(G, V) = 0$ *holds for all* $n > 0$.

Proof. Recall that the Tate cohomology $\hat{H}^*(G, V)$ satisfies $\hat{H}^0(G, V) = \mathrm{Coker}\,(\overline{\mathrm{tr}})$, $\hat{H}^{-1}(G, V) = \mathrm{Ker}\,(\overline{\mathrm{tr}})$, and $\hat{H}^{-n-1}(G, V) = H_n(G, V)$ for all $n > 0$ (see [B], §V1.4). Thus it suffices to show that $\hat{H}^*(G, V)$ vanishes. To this end, let $\phi : A * G \to \mathrm{End}_k(V)^{op}$ denote the structure map of the $A * G$-module V. Then, in $\mathrm{End}_k(V)^{op}$, we have $\mathrm{Id} = \mathrm{tr}\,(\phi(z))$. Since $\hat{H}^*(G, \mathrm{Id})$ is the identity on $\hat{H}^*(G, V)$ and $\hat{H}^*(G, \mathrm{tr}\,(\phi(z))$ is zero (cf. [Ba], 15.3), we conclude that $\hat{H}^*(G, V) = 0$, as required. □

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