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Observations on crossed products and invariants of Hopf algebras

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Introduction. Let $B = A \#_\sigma H$ denote a crossed product of the associative algebra A with the finite-dimensional Hopf algebra H. By studying the process of induction of modules from A to B in the case where H is pointed we show in Section 1 that the Jacobson radicals of B and A are related by

$$J(B)^{\dim_k H} \subseteqq J(A)\, B\,.$$

We then specialize to the situation where A is an H-module algebra, the cocycle σ is trivial (so B is a smash product), and the trace map from A to the algebra of H-invariants A^H is surjective. Making essential use of the well-known Morita context linking B with the algebra of H-invariants A^H we investigate the transfer of properties from A to A^H. In Section 3 we show, for example, that if A is right Noetherian (right Artinian) then so is A^H. In fact, A is Noetherian (Artinian) as right A^H-module in this case. Furthermore, if Kdim (A_A) exists then Kdim $(A^H_{A^H})$ exists as well and is bounded above by Kdim (A_A). The results concerning the Noetherian property have first been obtained by S. Montgomery ([7], Theorem 4.4.2) and have motivated much of our research in this section. Furthermore, we extend most of [5], Theorem 3.3 from group algebras to pointed Hopf algebras. For example, we show that if the right A-module W is Noetherian (Artinian) then W_{A^H}, the restriction of W to A^H, is likewise. Further, Kdim (W) exists if and only if Kdim (W_{A^H}) does and in this case both are equal. Finally, we prove the following estimate for the right global dimension of A^H:

$$\begin{aligned} \text{r.gldim}\, A^H &\leqq \text{r.gldim}\, B + \min\{\text{fdim}_{A^H} A,\ \text{pdim}\, A_{A^H}\} \\ &\leqq \text{r.gldim}\, A + \text{gldim}\, H + \min\{\text{fdim}_{A^H} A,\ \text{pdim}\, A_{A^H}\}\,. \end{aligned}$$

Here, fdim denotes flat dimension and pdim denotes projective dimension. Most of the results in Section 3 follow in a fairly straightforward manner from some general facts about Morita contexts which are established in Section 2. This is independent of Hopf algebras and may therefore be of interest in its own right.

It is a pleasure to thank Susan Montgomery for helpful discussions and for communicating Example 1.2 to us.

Notations. Our references for general material about Hopf algebras are the standard texts [1] and [10]. For crossed products in particular we follow the notes [7]. Throughout this article, we will keep the following notations.

k	denotes a commutative field;
H	will be a Hopf algebra over k, with comultiplication Δ and counit ε;
A	denotes an associative k-algebra so that there is a weak H-action on A, denoted $(h, a) \mapsto h \cdot a$ $(h \in H, a \in A)$;
$B = A \#_\sigma H$	will denote a crossed product, with cocycle $\sigma : H \times H \to A$.

Thus B is an associative algebra such that there is an isomorphism of left A-modules

$$A \otimes_k H \xrightarrow{\cong} B, \quad a \otimes h \mapsto a \# h\,.$$

The map $a \mapsto a \# 1$ identifies A with a subalgebra of B. Defining a k-linear map $\gamma : H \to B$ by

$$\gamma(h) = 1 \# h \quad (h \in H)\,,$$

we have $a\gamma(h) = a \# h$ for $a \in A$, $h \in A$. It is known (cf. [7], Chap. 7) that γ is convolution invertible and satisfies the following identity, for $h \in H$ and $a \in A$,

$$(*) \qquad \gamma(h)\, a = \sum (h_1 \cdot a)\, \gamma(h_2)\,.$$

Finally, $J(\cdot)$ always denotes the Jacobson radical, $\ell(\cdot)$ denotes composition length, and Kdim $(\cdot)$ denotes the Krull dimension in the sense of Gabriel and Rentschler.

1. The Jacobson radical

1.1. Free ring extensions. In this subsection, we collect a few general facts about ring extensions $R \subseteq S$ (same 1) so that S is free as left R-module. Much of this material is known and we give suitable references to the literature whenever available. Recall that the *Loewy length* of a module W is the smallest integer t so that $WJ^t = 0$, where J denotes the Jacobson radical, or ∞ if no such t exists.

Lemma. *Let $R \subseteq S$ be an extension of rings so that S is free as left R-module. Then:*
(a) *R is a direct summand of S as left R-module.*
(b) $J(S) \cap R \subseteq J(R)$.
(c) *For any right R-module V, the annihilator* $\operatorname{ann}_S(V \otimes_R S)$ *is the largest ideal of S that is contained in* $\operatorname{ann}_R(V)S$.
(d) *If there exists a finite upper bound d for the Loewy lengths of all S-modules $V \otimes_R S$, where V is a simple right R-module, then $J(S)^d \subseteq J(R)S$.*

P r o o f. (a) Let $\{s_i\}$ be a left R-module basis of S and write $1 = \sum r_i s_i$ $(r_i \in R)$. Putting $I = \sum r_i R$ we must have $I = R$, for otherwise $IS \neq S$, by freeness of S over R, contradicting the fact that $1 \in IS$. Thus we can write $1 = \sum r_i t_i$ with $t_i \in R$ and defining $\pi : {}_R S \to {}_R R$ by $\pi(s_i) = t_i$ we obtain the required projection which is the identitiy on R.

(b) With $U(\cdot)$ denoting unit groups, part (a) implies that $U(S) \cap R = U(R)$ which in turn yields (b) (cf. [8], Lemma 7.1.3).

(c) This is proved in [3], 10.4.

(d) By assumption and (c), $J(S)^d \subseteqq \operatorname{ann}_S(V \otimes_R S) \subseteqq \operatorname{ann}_R(V) S$ holds for all simple R-modules V. Consequently,

$$J(S)^d \subseteqq \bigcap_V (\operatorname{ann}_R(V)S) = (\bigcap_V \operatorname{ann}_R(V))\, S = J(R)S\,,$$

where V runs over the simple R-modules and where the first equality uses freeness of S over R. This completes the proof. □

1.2. Induced modules. We now return to the ring extension $A \subseteqq B = A \#_\sigma H$. Part (a) of the lemma below implies that if H is finite-dimensional and pointed then the hypothesis of Lemma 1 (d) is satisfied, with $d = \dim_k H$. (Note that the composition length clearly is an upper bound for the Loewy length.)

Lemma. *Assume that H is finite-dimensional and pointed. Let V be a right A-module. Then*:

(a) $\ell(V \otimes_A B_A) = \ell(V) \cdot \dim_k H$. *Consequently*, $\ell(V \otimes_A B_B) \leqq \ell(V) \cdot \dim_k H$.

(b) *V is Noetherian (Artinian) if and only if $V \otimes_A B_A$ is. In this case, $V \otimes_A B_B$ has the same property.*

(c) *The Loewy length of $V \otimes_A B_A$ is bounded above by the product of the Loewy length of V with the length of the coradical series of H.*

Proof. Let $\{H_n\}$ denote the coradical filtration of H and let $G = G(H)$ denote the set of group-like elements of H. So

$$H_{-1} = 0 \subseteqq H_0 = kG \subseteqq \ldots \subseteqq H_n \subseteqq H_{n+1} \subseteqq \ldots \subseteqq H_t = H$$

for some t. Using formula (*) and the fact that $\Delta H_n \subseteqq \sum_0^n H_i \otimes H_{n-i}$ we see that each $A\gamma(H_n)$ is an A-A-subbimodule of B which is a direct summand of B as left A-modules (since ${}_AB \cong A \otimes_k \gamma(H)$). Thus we can define A-submodules $W_n \subseteqq V \otimes_A B_A$ by

$$W_n = V \otimes_A A\gamma(H_n)\,.$$

Put $\overline{W_n} = W_n/W_{n-1}$ and $\overline{H_n} = H_n/H_{n-1}$ $(n \geqq 0)$. The lemma is a consequence of the following more precise assertion:

Each $\overline{W_n}$ is isomorphic to a direct sum of $\dim_k \overline{H_n}$ many G-conjugates of V.

Since each G-conjugate of V has the same (composition and Loewy) length as V, and is Noetherian (Artinian) if and only if V is, all claims in the lemma about $V \otimes_A B_A$ follow from this assertion. The statements about $V \otimes_A B_B$ are immediate.

For the proof we use the Taft-Wilson theorem (cf. [7], p. 64) which implies that every $h \in H_n$ can be written in the form $h = \sum_{x \in G} h_x$ with

$$(**) \qquad h_x \in H_n,\ \Delta h_x = x \otimes h_x \mod H_n \otimes H_{n-1}\,.$$

Fix such elements $h_x \in H_n$ (for various $x \in G$) so that their images in $\overline{H_n}$ form a k-basis of $\overline{H_n}$ and let V^x the image of $V \otimes_A A\gamma(h_x)$ in $\overline{W_n}$. Then, as k-space, $\overline{W_n}$ is the direct sum of the various V^x. Moreover, for $v \in V$, $a \in A$ one computes using (*) and (**)

$$(v \otimes \gamma(h_x))\, a = v \otimes (x \cdot a)\, \gamma(h_x) \mod W_{n-1}$$
$$= v(x \cdot a) \otimes \gamma(h_x) \mod W_{n-1}\,.$$

Thus, letting v^x denote the image of $v \otimes \gamma(h_x)$ in $\overline{W}_n$, we have a k-linear isomorphism $V \xrightarrow{\cong} V^x$, $v \mapsto v^x$, so that

$$v^x a = (v(x \cdot a))^x\,.$$

This shows that V^x is the conjugate of V corresponding to (the inverse of) the automorphism $a \mapsto x \cdot a$ of A, and our assertion is proved. □

The following example shows that part (a) of the above Lemma does not extend to the non-pointed case, even in the special case of smash products. The example is due to Susan Montgomery and Don Passman.

Example. Let G be a finite group and put $A = k[G]$, the group algebra of G, and $H = (k[G])^*$. Then A is an H-module algebra via $h \cdot g = h(g)g$ ($h \in H$, $g \in G$) and so we can form the smash product $B = A \# H$. Now let V be any right A-module and put $\overline{V} = V \otimes_A B_A$. Then

$$\overline{V} \cong V \otimes_k A_A\,,$$

with the diagonal action of G on the right. To see this, note that $\overline{V} = \bigoplus_{g \in G} V \otimes p_g$, where $p_g \in H$ is the projection, $p_g(x) = \delta_{g,x}$ for $g, x \in G$. The G-action is given by $(v \otimes p_g)\, x = vx \otimes p_{x^{-1}g}$. Thus the required isomorphism is given by $v \otimes p_g \mapsto v \otimes g^{-1}$. As is well-known, $V \otimes_k A_A \cong A_A^{(\dim_k V)}$, and hence

$$\overline{V} \cong A_A^{(\dim_k V)}\,,$$

the free right A-module of rank $\dim_k V$. Consequently,

$$\ell(\overline{V}) = \dim_k V \cdot \ell(A)\,.$$

Now assume that G is non-abelian and let k be a splitting field for G so that char k does not divide the order of G. Assume further that V is irreducible of maximal dimension. Then

$$\dim_k V \cdot \ell(A) = \dim_k V \cdot \sum_W \dim_k W > \sum_W (\dim_k W)^2 = \dim_k H\,,$$

where W runs over a full set of irreducible A-modules. Thus (a) of the Lemma does not hold here. – We also note that, by choosing k so that char k divides the order of G in the foregoing, one sees that $V \otimes_A B_A$ need not be completely reducible, even though V is irreducible and H is semisimple (cf. [7], Example 7.4.4).

1.3. Jacobson radicals. The following proposition is the main result of this section. Part (a) is immediate from Lemma 1(b) while part (b) is a consequence of Lemma 1(d) and Lemma 2.

Proposition. (a) $J(B) \cap A \subseteq J(A)$.
(b) *If H is finite-dimensional and pointed then* $J(B)^{\dim_k H} \subseteq J(A)B$.

2. Some results on Morita contexts.

2.1. Morita contexts. In this section, let $(R, S, {}_RP_S, {}_SQ_R, f, g)$ be a fixed Morita context. Here, R and S are arbitrary rings (with 1) and

$$f : P \otimes_S Q \to R \quad \text{and} \quad g : Q \otimes_R P \to S$$

are bimodule maps which satisfy the associativity conditions

$$f(p \otimes q)p' = pg(q \otimes p') \quad \text{and} \quad g(q \otimes p)q' = qf(p \otimes q')$$

for $p, p' \in P$ and $q, q' \in Q$. *Throughout this section we will assume that the map f is surjective.* We note that this assumption implies that ${}_RP$ and Q_R are generators, P_S and ${}_SQ$ are finitely generated projective, and f is an isomorphism (cf. [2], Theorem II.3.4).

2.2. Chain conditions. In the following lemma, we use $\mathfrak{L}(\cdot)$ to denote the lattice of submodules of the module in question.

Lemma. *For each S-module V_S, there exist order preserving maps*

$$\mathfrak{L}(V \otimes_S Q_R) \underset{\pi}{\overset{\mu}{\rightleftarrows}} \mathfrak{L}(V_S)$$

with $\pi \circ \mu = \mathrm{id}$. Moreover, π respects direct sums.

P r o o f. For any submodule U of $V \otimes_S Q_R$ define

$$\mu(U) = \mathrm{Im}\,(U \otimes_R P \xrightarrow{\mathrm{incl} \otimes \mathrm{id}_P} V \otimes_S Q \otimes_R P \xrightarrow{\mathrm{id}_V \otimes g} V \otimes_S S = V).$$

Then $\mu(U)$ is a submodule of V_S, being the image of an S-module homomorphism, and μ is clearly order preserving. Similarly, for any submodule W of V_S, define

$$\pi(W) = \mathrm{Im}\,(W \otimes_S Q_R \xrightarrow{\mathrm{incl} \otimes \mathrm{id}_Q} V \otimes_S Q_R).$$

Again, $\pi(W)$ is a submodule of $V \otimes_S Q_R$ and π is order preserving. Moreover, the map $\mathrm{incl} \otimes \mathrm{id}_Q$ is injective, since ${}_SQ$ is projective. Thus, if W_1 and W_2 are submodules of V_S with zero intersection, then

$$\pi(W_1 \oplus W_2) = \pi(W_1) \oplus \pi(W_2).$$

Finally,

$$\pi(\mu(U)) = U \cdot f(P \otimes_S Q).$$

To see this, note that $\pi(\mu(U))$ is the image of the map

$$U \otimes_R P \otimes_S Q \xrightarrow{\mathrm{incl} \otimes \mathrm{id}_P \otimes \mathrm{id}_Q} V \otimes_S Q \otimes_R P \otimes_S Q \xrightarrow{\mathrm{id}_V \otimes g \otimes \mathrm{id}_Q} V \otimes_S S \otimes_S Q = V \otimes_S Q.$$

Consider $u = \sum v_i \otimes q_i \in U$. Then the image of $u \otimes p \otimes q$ under this map is

$$\sum v_i g(q_i \otimes p) \otimes q = \sum v_i \otimes g(q_i \otimes p)q = \sum v_i \otimes q_i f(p \otimes q) = uf(p \otimes q),$$

which proves the above equality. Since $f(P \otimes_S Q) = R$, we get $\pi(\mu(U)) = U$, as required. □

Proposition. (a) *Let V_S be an S-module. Then:*

(1) *If V_S is Noetherian (Artinian) then $V \otimes_S Q_R$ is likewise.*

(2) $\ell(V \otimes_S Q_R) \leqq \ell(V_S)$.

(3) *If* Kdim (V_S) *exists then so does* Kdim $(V \otimes_S Q_R)$ *and* Kdim $(V \otimes_S Q_R) \leqq$ Kdim (V_S).

(4) *If V_S is completely reducible then so is $V \otimes_S Q_R$.*

(b) *If S is right Noetherian (right Artinian, semisimple) then so is R. Furthermore, if* Kdim (S_S) *exists then* Kdim (R_R) *exists too and is bounded above by* Kdim (S_S).

Proof. (a1)–(a3) are clear from the lemma. For (a4), assume that V_S is completely reducible. Then for any submodule U of $V \otimes_S Q_R$ there exists a submodule W of V with $\mu(U) \oplus W = V$. Applying π to this, we obtain $U \oplus \pi(W) = \pi(V) = V \otimes_S Q$. Therefore, $V \otimes_S Q_R$ is completely reducible.

For (b), take $V = S_S$ in (a) to deduce that Q_R is Noetherian (Artinian, completely reducible) if S_S has these properties. Moreover, if Kdim (S_S) exists then Kdim (Q_R) exists too and is bounded above by Kdim (S_S). Since Q_R is a generator, R_R inherits all these properties, with Kdim $(R_R) \leqq$ Kdim $(Q_R) \leqq$ Kdim (S_S) in case the latter exists. □

2.3. Homological dimension.

Lemma. *Let V_R and W_R be right R-modules. Then there are third quadrant spectral sequences*

$$E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Tor}_q^R(V, P), \mathrm{Hom}_R(Q, W)) \underset{p}{\Rightarrow} \mathrm{Ext}_R^n(V, W)$$

and

$$E_2^{p,q} = \mathrm{Ext}_S^p(V \otimes_R P, \mathrm{Ext}_R^q(Q, W)) \underset{p}{\Rightarrow} \mathrm{Ext}_R^n(V, W).$$

Proof. Let U_S be a given right S-module. Letting $\mathfrak{M}_S$ denote the category of right S-modules and similarly for other rings we define functors

$$G: \mathfrak{M}_R \to \mathfrak{M}_S, \quad G(X) = X \otimes_R P_S$$

and

$$F: \mathfrak{M}_S \to \mathfrak{M}_{\mathbb{Z}}, \quad F(Y) = \mathrm{Hom}_S(Y, U).$$

Then

$$FG(X) = \mathrm{Hom}_S(X \otimes_R P, U) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_S(P, U))$$

via the adjoint isomorphism ([9], Theorem 2.11). Also, since P_S is projective, $G(X)$ is right F-acyclic if X_R is projective, that is, $\mathrm{Ext}_S^i(G(X), U) = 0$ holds for $i > 0$. Grothendieck's theorem ([9], Theorem 11.40) now gives a third quadrant spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Tor}_q^R(V, P), U) \underset{p}{\Rightarrow} \mathrm{Ext}_R^n(V, \mathrm{Hom}_S(P, U)).$$

Now, for a given W_R, take $U = \mathrm{Hom}_R(Q, W)$, viewed as right S-module via $(fs)(q) = f(sq)$, as usual. Then the adjoint isomorphism and the fact that $P \otimes_S Q \cong R$ via f together imply that

$$\mathrm{Hom}_S(P, U) \cong \mathrm{Hom}_R(P \otimes_S Q, W) \cong \mathrm{Hom}_R(R, W) \cong W.$$

Thus the above spectral sequence yields the first spectral sequence of the lemma. The second spectral sequence is established similarly using the functors

$$G : \mathfrak{M}_R \to \mathfrak{M}_S, \quad G(X) = \operatorname{Hom}_R(Q, X)$$

and

$$F : \mathfrak{M}_S \to \mathfrak{M}_{\mathbb{Z}}, \quad F(Y) = \operatorname{Hom}_S(V \otimes_R P, Y). \quad \square$$

Corollary. r.gldim $R \leqq$ r.gldim $S + \min\{\operatorname{fdim}_R P, \operatorname{pdim} Q_R\}$.

Proof. If $n = p + q >$ r.gldim $S + \operatorname{fdim}_R P$ then $p >$ r.gldim S or $q > \operatorname{fdim}_R P$ and so $\operatorname{Ext}_S^p(\operatorname{Tor}_q^R(\cdot, P), \cdot) = 0$. The first spectral sequence in the lemma now gives $\operatorname{Ext}_R^n(\cdot, \cdot) = 0$ which proves that r.gldim $R \leqq$ r.gldim $S + \operatorname{fdim}_R P$. Similarly, the second spectral sequence implies that r.gldim $R \leqq$ r.gldim $S + \operatorname{pdim} Q_R$. $\square$

The above spectral sequences also yield estimates for the homological dimensions of modules. For example, if V_R is a right R-module then arguing as above using the second spectral sequence of the lemma one obtains

$$\operatorname{pdim} V_R \leqq \operatorname{pdim} V \otimes_R P_S + \operatorname{pdim} Q_R.$$

For further results on homological dimensions in Morita contexts we refer to [6].

3. Applications to smash products and invariants

3.1. Preliminaries. Throughout this section, A will be a left H-module algebra and H will be finite-dimensional. Fix a left integral $0 \neq t \in H$. Then there is a Morita context $(R, S, {}_R P_S, {}_S Q_R, f, g)$ with $S = A \# H$, the smash product of A and H, $R = A^H$, the algebra of H-invariants, and $P = Q = A$, with suitable bimodule actions. For details we refer to [7], Sect. 4.5. Letting

$$\hat{t} : A \to A^H, \quad \hat{t}(a) = t \cdot a$$

denote the trace map afforded by t, the maps f and g in the Morita context are given by

$$f = (\cdot, \cdot) : A \otimes_{A \# H} A \to A^H, \quad (a_1, a_2) = \hat{t}(a_1 a_2)$$

and

$$g = [\cdot, \cdot] : A \otimes_{A^H} A \to A \# H, \quad [a_1, a_2] = a_1 t a_2.$$

We will assume throughout this section that f is surjective or, equivalently, that the trace map is surjective. Note that this assumption is independent of the choice of t, since t is determined up to a scalar. Furthermore, the assumption holds, for example, if H is semisimple.

For any right $A \# H$-module V and any linear form $\phi \in H^*$, we put

$$V_\phi = \{v \in V \mid vh = \phi(h)v \quad \text{for all } h \in H\}.$$

Thus $V_\varepsilon = V^H$, the H-invariants in V. It is easily checked that each V_ϕ is in fact a module over A^H. Below, the case where $\phi = \alpha$ is the so-called *distinguished group-like element* will be important (cf. [7], 2.2.3). The element α satisfies

$$th = \alpha(h)t \quad (h \in H).$$

For unimodular H (e.g., for H semisimple), one has $\alpha = \varepsilon$.

3.2. Chain conditions. The following result is an application of Proposition 2.2. Similar conclusions could be drawn in the case where g is surjective (which holds, e.g., when $A \# H$ is a simple ring). The Noetherianness statements in part (b) below are due to S. Montgomery ([7], Theorem 4.4.2).

Theorem. (a) *Let V be a right $A \# H$-module Then:*

(1) *If V is Noetherian (Artinian) then V_α is likewise (as module over A^H).*

(2) $\ell(V_\alpha) \leqq \ell(V)$.

(3) *If* Kdim (V) *exists then so does* Kdim (V_α) *and* Kdim $(V_\alpha) \leqq$ Kdim (V).

(4) *If V is completely reducible then so is V_α.*

(b) *If A is right Noetherian (right Artinian) then so is A^H. In fact, A is Noetherian (Artinian) as right A^H-module in this case. Furthermore, if* Kdim (A_A) *exists then* Kdim $(A^H_{A^H})$ *exists too and is bounded above by* Kdim (A_A). *Finally, if A and H are semisimple then so is A^H.*

P r o o f. (a) In view of Proposition 2.2(a), it suffices to show that

$$V_\alpha \cong V \otimes_{A \# H} A$$

as right A^H-modules. For this, fix $a \in A$ with $\hat{t}(a) = 1$ and put $e = at \in A \# H$. Then $tat = t$ holds in $A \# H$, and hence $e^2 = e$, and ${}_{A \# H}A \cong A \# H t = A \# H e$ as $A \# H$-A^H-bimodules. Therefore, $V \otimes_{A \# H} A \cong Ve$ as A^H-modules. Finally, one checks that $Ve = V_\alpha$ which completes the proof of (a).

(b) This is an application of Proposition 2.2(b). We only have to make sure that the above properties for A entail the corresponding properties for $A \# H$. Inasmuch as $A \# H$ is a finitely generated right A-module, this is clear for right Noetherianness, right Artinianness, and existence of Krull dimension, with Kdim $(A \# H_{A \# H}) \leqq$ Kdim $(A \# H_A) \leqq$ Kdim (A_A). Finally, semisimplicity of $A \# H$ follows from semisimplicity of A and H by [7], Theorem 7.4.2. □

We now consider restriction of A-modules to A^H. The following results extends most of [5], Theorem 3.3 from group algebras to pointed Hopf algebras.

Proposition. *Assume that H is pointed. Let W be a right A-module. Then:*

(1) *If W is Noetherian (Artinian) then W_{A^H} is likewise.*

(2) $\ell(W_{A^H}) \leqq \ell(W) \cdot \dim_k H$.

(3) Kdim (W) *exists if and only if* Kdim (W_{A^H}) *does and in this case both are equal.*

P r o o f. The result is an application of part (a) of Proposition 2.2, with V the induced $A \# H$-module $V = W \otimes_A A \# H$. Note that $V \otimes_{A \# H} A_{A^H} \cong W_{A^H}$. Thus we have to keep track of the transfer of properties from W to V for which we use Lemma 1.2. This result immediately takes care of the Noetherian, Artinian, and length statements, thereby proving parts (1) and (2).

For Kdim, note that if Kdim (W_{A^H}) exists then certainly Kdim (W) does and is bounded above by Kdim (W_{A^H}). So assume that Kdim (W) exists. As we have shown in the proof of Lemma 1.2, V_A has a finite series of submodules with factors isomorphic to conjugates of W. This implies that Kdim (V_A) exists and is equal to Kdim (W). Consequently, Kdim $(V_{A\#H})$ also exists and is bounded above by Kdim (W). Proposition 2.2 (a 3) now implies that Kdim (W_{A^H}) exists and is bounded above by Kdim (W), which completes the proof of (3). □

3.3. Homological dimension.

Proposition.

$$\begin{aligned} \text{r.gldim}\, A^H &\leqq \text{r.gldim}\, A \# H + \min\{\text{fdim}_{A^H} A, \text{pdim}\, A_{A^H}\} \\ &\leqq \text{r.gldim}\, A + \text{gldim}\, H + \min\{\text{fdim}_{A^H} A, \text{pdim}\, A_{A^H}\}. \end{aligned}$$

Proof. The first inequality is an application of Corollary 2.3 and the second inequality follows from [4], Corollary 4. (Note that r.gldim H = l.gldim H, because the antipode of H is an antiautomorphism of H.) □

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