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Grothendieck groups of invariant rings: linear actions of finite groups

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0 Introduction

Let G be a finite group and let k be a field with $|G|^{-1} \in k$. Let V be a finite dimensional vector space over k, with basis $\{X_1, \ldots, X_n\}$, and suppose that G acts as automorphisms of V. This action extends to give an action of G as a group of k–algebra automorphisms of the symmetric algebra $S = S(V) = k[X_1, \ldots, X_n]$. Denote the fixed ring S^G under this action by R. The purpose of this article is to describe the Grothendieck group $G_0(R)$ of finitely generated R–modules.

We shall present $G_0(R)$ as a factor group of the Grothendieck ring $G_0(kG)$ of finitely generated modules over the group algebra kG. (Recall that the ring structure of $G_0(kG)$ is afforded by $\otimes_k$.) To do so we need the following notation. For each subgroup H of G let $V(H)$ be the subspace of V spanned by $\{v - v^h : v \in V, h \in H\}$. This is a $kN_G(H)$–module, and hence so is $\wedge^i V(H)$, where $\wedge^i$ denotes the i–th exterior power. Thus we may define

$$\alpha_H := \sum_{i \geqq 0} (-1)^i [\wedge^i V(H)] ,$$

an element of $G_0(kN_G(H))$. Further, define a subgroup of $G_0(kN_G(H))$,

$$G^*(H) := \langle [X] : X \text{ an irreducible } kN_G(H)\text{–module with } X^H = 0 \rangle ,$$

where X^H denotes the space of H–invariants in X. For a subgroup N of G let Ind_{kN}^{kG} denote the group homomorphism from $G_0(kN)$ to $G_0(kG)$ afforded by the induction functor $(.) \otimes_{kN} kG$. Define a subgroup $\mathcal{U}$ of $G_0(kG)$ by

$$\mathcal{U} := \sum_{1 \neq H \subseteq G} \mathrm{Ind}_{kN_G(H)}^{kG}(\alpha_H G^*(H)) .$$

Our main result can now be stated as

Theorem. $G_0(R) \cong G_0(kG)/\mathcal{U}$.

The strategy of the proof is as follows. Let $T = S * G$, the skew group ring corresponding to the given action, let $t = \sum_{g \in G} g \in T$ and $I = TtT$. The fixed point functor $(.)^G$ yields an exact sequence ([Br-L], Corollary 1.3(i))

$$G_0(T/I) \xrightarrow{\mathrm{Infl}_{T/I}^T} G_0(T) \xrightarrow{(.)^G} G_0(R) \to 0 \tag{1}$$

where $\mathrm{Infl}_{T/I}^T$ is the inflation map (viewing T/I–modules as T–modules). By Quillen's induction theorem ([Q], Theorem 7), there is an isomorphism

$$\mathrm{Ind}_{kG}^T : G_0(kG) \xrightarrow{\cong} G_0(T) .$$

Thus $G_0(R)$ is isomorphic to $G_0(kG)$ modulo the image of the map $(\mathrm{Ind}_{kG}^T)^{-1} \circ \mathrm{Infl}_{T/I}^T$. The issue is to describe the image.

The layout of the paper is as follows. Some preliminary results and proofs are given in Sect. 1. These show, first, that the subgroup $\mathscr{U}$ of $G_0(kG)$ sought for the main theorem is generated by subgroups $\mathscr{K}_H$, for $1 \neq H \leqq G$, (Prop. 1.2); and then that, by means of an inductive argument, we may concentrate on those subgoups H of G which are normal (Prop. 1.6). The contribution of the normal subgroups of G is analysed in Sect. 2, and the proof of the theorem is completed in Sect. 3. (We also point out in **3.4** that it is not in general necessary to consider *all* the non-identity subgroups H of G in calculating $\mathscr{U}$.)

Partial results on the structure of $G_0(R)$ have been obtained previously by various authors ([A-R2], [H-S], [H-M-W] and [M] for example). Most of this work has used the formal power series ring $\widehat{S} = k[[X_1,\ldots,X_n]]$ rather than the polynomial algebra $S(V)$, although, with minor amendments, methods and results can be switched as desired between these two settings. We explain briefly in **4.5** that our results and proofs deal with this case also, and indicate in **4.2** and **4.7**(1) how some special cases obtained by the above authors can be deduced from the main theorem. In particular, we point out that the main result of [H-M-W] is the case of the above theorem for G abelian. Also included in Sect. 4 are various worked examples (**4.4, 4.6, 4.7**(2) and **4.9**), and a discussion (**4.4**) of the question: When is $G_0(R) \cong \mathbb{Z}$? As we show by examples, the classical sufficient condition stemming from the Sheppard-Todd-Chevalley theorem is not necessary.

In **4.8** we observe that the Sheppard-Todd-Chevalley theorem permits one first to factor out the subgroup of pseudoreflections from G before applying the Main Theorem; in particular this leads to improved bounds for the exponent of the torsion subgroup of $G_0(R)$.

Notations and conventions

We shall retain throughout the paper the notation defined in the introduction. Note in particular that exponential notation ($s \mapsto s^g$) is used to denote the action of $g \in G$ on $s \in S$ (and on $v \in V$). Thus the skew group ring $T = S * G$ is the free left S-module with the elements of G as a basis and with multiplication given by the rule $sg = gs^g (s \in S, g \in G)$. For a subset of X of G, write $t_X = \sum_{x \in X} x \in T$; when $X = G$ we usually omit the suffix. The ideal of a ring A generated by a collection of subsets $B, \ldots$ and elements $b, \ldots$ is denoted by $(B, \ldots, b, \ldots)$; the corresponding notation for subgroups of a group will use the brackets $\langle \ldots \rangle$.

1 Some reductions

1.1 For each subgroup H of G, define an ideal of S,

$$I(H) := (s - s^h : s \in S, h \in H) = (v - v^h : v \in V, h \in H) .$$

The following easily checked facts will be used freely:

- $H \subseteq D \Rightarrow I(H) \subseteq I(D)$;
- $I(H) + I(D) = I(\langle H, D \rangle)$;
- $I(H)^g = I(H^g)$, for all $g \in G$.

Recall that $I := TtT = StS$. In view of the exact sequence (1) of the Introduction, this ideal and the ramification locus $I \cap S$ are fundamental to our analysis. They are closely approximated by

Lemma. ([*Br-L*], *Lemma 2.2*). $\prod_{1 \neq H \subseteq G} I(H) \subseteq I \cap S \subseteq \bigcap_{1 \neq H \subseteq G} I(H)$.

We also define

$$I^0(H) := \bigcap_{g \in G} I(H)^g \quad \text{and} \quad S^0(H) := S/I^0(H) ;$$

$$J^0(H) := I^0(H)T \quad \text{and} \quad T^0(H) := T/J^0(H) \cong S^0(H) * G ;$$

$$J(H) := J^0(H) + I \quad \text{and} \quad T(H) := T/J(H) \cong T^0(H)/(t) .$$

The above lemma implies that $\prod_{1 \neq H \subseteq G} J(H) \subseteq I$. Hence

$$G_0(T/I) = \sum_{1 \neq H \subseteq G} \mathrm{Infl}^T_{T(H)}(G_0(T(H))) .$$

1.2 Note that $G_0(T)$ is a (left) $G_0(kG)$–module. Thus, if X_{kG} and W_T are finitely generated modules then $X \otimes_k W$ is a finitely generated right T–module under the action $(x \otimes w)sg = xg \otimes wsg$ for $x \in X$, $w \in W$, $s \in S$ and $g \in G$. Exactness of $\otimes_k$ ensures that this defines a module action on $G_0(T)$. Note also that kG is a T–kG–bimodule, with T acting via the ring epimorphism $T \to T/VT \cong kG$, and T is a kG–free kG–T–bimodule via the embedding of rings $kG \hookrightarrow T$. At the level of Grothendieck groups these maps yield inverse isomorphisms:

Lemma. (i) (*Quillen* [*Q*], [*H*], *Theorem 1.2*)

$$\mathrm{Ind}^T_{kG} : G_0(kG) \to G_0(T) : [M] \mapsto [M \underset{kG}{\otimes} T]$$

is an isomorphism of $G_0(kG)$*–modules.*

(ii) ([*Br-L2*], *Lemma 2.2*) *The inverse isomorphism to* Ind_{kG}^{T} *is*

$$\pi : G_0(T) \to G_0(kG) : [W] \mapsto \sum_{i\geqq 0}(-1)^i[\mathrm{Tor}_i^T(W,kG)]\,.$$

(iii) ([*Br-L2*], *Lemma 2.3*) *The endomorphism*

$$\pi \circ \mathrm{Infl}_{T/VT}^{T} : G_0(kG) \to G_0(kG)$$

is given by multiplication by $\alpha_V := \sum_{i\geqq 0}(-1)^i[\wedge^i V] \in G_0(kG)$.

For a subgroup H of G the isomorphism from $G_0(S * H)$ to $G_0(kH)$ given by (ii) of the above lemma will be denoted by π_H.

We now write

$$\mathscr{I} := \mathrm{Infl}_{T/I}^{T}(G_0(T/I)) \subseteq G_0(T) \quad \text{and} \quad \mathscr{K} := \pi(\mathscr{I}) \subseteq G_0(kG)\,;$$

$$\mathscr{I}_H := \mathrm{Infl}_{T(H)}^{T}(G_0(T(H))) \subseteq G_0(T) \quad \text{and} \quad \mathscr{K}_H := \pi(\mathscr{I}_H) \subseteq G_0(KG)\,.$$

With this notation, the exact sequence (1) of the introduction coupled with the isomorphism π of Lemma 1.2 (ii) yields (i) of the following result. The first equality in (ii) has already been noted in **1.1** and the second equality follows by applying π.

Proposition. (i) $G_0(R) \cong G_0(T)/\mathscr{I} \cong G_0(kG)/\mathscr{K}$.
(ii) $\mathscr{I} = \sum_{1 \neq H \subseteq G} \mathscr{I}_H$ *and* $\mathscr{K} = \sum_{1 \neq H \subseteq G} \mathscr{K}_H$.

In view of the above proposition, our goal will be to describe $\mathscr{I}_H$ and $\mathscr{K}_H$ for a fixed subgroup $H \subseteq G$.

1.3 For a fixed subgroup $H \subseteq G$, put

$$S^*(H) := \prod_{g \in \mathscr{T}_H} S/I(H^g)\,,$$

where $\mathscr{T}_H$ is a set of right coset representatives of $N_G(H)$ in G with $1 \in \mathscr{T}_H$.

Lemma. *The canonical embedding* $\varphi : S^0(H) \hookrightarrow S^*(H)$ *makes* $S^*(H)$ *into a finitely generated* $S^0(H)$*–module. The factor* coker φ *is annihilated by* Y^{h-1}, *where* $h = [G : N_G(H)]$ *and*

$$Y = \bigcap_{1 \neq x \in \mathscr{T}_H} I^0(\langle H, H^x\rangle)\,.$$

Furthermore, there is a canonical action of G *on* $S^*(H)$ *which makes* φ *a* G*-equivariant map.*

Proof. The idempotents corresponding to the components $S/I(H^g)$ of $S^*(H)$ generate $S^*(H)$ as $S^0(H)$–module. For $x \in G$ and $\sigma = \left(s_g + I(H^g)\right)_{g \in \mathscr{T}_H} \in S^*(H)$ put

$$\sigma^x = \left(s_g^x + I\left(H^{gx}\right)\right)_{g \in \mathscr{T}_H} \in S^*(H)\,.$$

This defines an action of G on $S^*(H)$ so that φ is G-equivariant.

Note that

$$Y = \bigcap_{g \in G} \left(\bigcap_{1 \neq x \in \mathcal{T}_H} I(\langle H, H^x \rangle) \right)^g = \bigcap_{\substack{x,y \in \mathcal{T}_H \\ x \neq y}} (I(H^x) + I(H^y)) \, .$$

Thus the assertion about coker φ is a direct consequence of the Chinese Remainder Theorem below.

Chinese Remainder Theorem. *Suppose $P_1, \ldots, P_h$ are ideals of the ring A. Put $Q = \bigcap_{i \neq j}(P_i + P_j)$. Then the cokernel of the canonical homomorphism $\varphi : A \to B := \prod_{i=1}^h A/P_i$ is annihilated by Q^{h-1}.*

Proof. Letting $e_j \in B$ denote the idempotent corresponding to the component A/P_j of B, it suffices to show that $e_j Q^{h-1} \in \varphi A$ holds for all j. By symmetry, we may concentrate on $j = 1$. Let $q_1, q_2, \ldots, q_{h-1} \in Q$ be given and write

$$q_l = p_{1,l} + p_l \qquad (l = 1, \ldots, h-1) \, ,$$

where $p_{1,l} \in P_1$, $p_l \in P_{l+1}$. Then

$$q_1 q_2 \cdots q_{h-1} = a + p \, ,$$

with $a \in P_1$ and $p = p_1 p_2 \cdots p_{h-1} \in \bigcap_{i=2}^h P_i$. Therefore,

$$\begin{aligned} e_1 \cdot q_1 q_2 \cdots q_{h-1} &= (a + p + P_1, 0, \ldots, 0) \\ &= (p + P_1, p + P_2, \ldots, p + P_h) \\ &= \varphi(p) \, . \end{aligned}$$

Thus $e_1 Q^{h-1} \in \varphi(A)$, as required. □

1.4 Using the notation of **1.3** we can define, for $H \subseteq G$,

$$T^*(H) = S^*(H) * G \, ,$$

the skew group ring of G over $S^*(H)$. The embedding $\varphi : S^0(H) \hookrightarrow S^*(H)$ extends to an embedding

$$\varphi : T^0(H) = S^0(H) * G \hookrightarrow T^*(H)$$

which makes $T^*(H)$ a finitely generated module (left or right) over $T^0(H)$. Furthermore, since the ideal Y of **1.3** is G–invariant, YT is a twosided ideal of T and Lemma 1.3 implies that

$$(\operatorname{coker} \varphi) \cdot (YT)^{h-1} = (0) \, .$$

Lemma. (i) *There is a ring homomorphism*

$$\psi : T(H) = T^0(H)/(t) \longrightarrow T^*(H)/(t)$$

which makes $T^*(H)/(t)$ a finitely generated module (left or right) over $T(H)$. Furthermore,

$$(\operatorname{coker}\psi)\cdot(YT)^{h-1}=(0)$$

and

$$(YT)^{h-1}\cdot(\ker\psi)\cdot(YT)^{h-1}=(0)\,.$$

(ii) *Let*

$$\rho : G_0(T^*(H)/(t)) \longrightarrow G_0(T(H))$$

denote restriction along ψ. Then the image of the map $\mathrm{Infl}^{T}_{T(H)}\circ\rho$ in $G_0(T)$ coincides with the image of the map

$$G_0(S*N/(I(H),t_N)) \xrightarrow{\mathrm{Infl}} G_0(S*N) \xrightarrow{\mathrm{Ind}^{T}_{S*N}} G_0(T)\,,$$

where $N=N_G(H)$ and $t_N=\sum_{g\in N} g\in S*N\subseteq T$.

Proof. (i) The map ψ is obtained from φ by factoring out the ideals generated by $t=\sum_{g\in G} g$. Thus the claim about $\operatorname{coker}\psi$ is clear from the corresponding fact for φ. As to $\ker\psi$, note that

$$\ker\psi = S^*tS^*\cap T^0/S^0tS^0,$$

where we have written S^* for $S^*(H)$, T^0 for $T^0(H)$, etc. Since

$$Y^{h-1}S^*tS^*Y^{h-1}\subseteq S^0tS^0,$$

the statement about $\ker\psi$ follows.

(ii) To prove the assertions about Grothendieck groups, we first establish the following

Claim. *$T^*(H)\cong M_h((S/I(H))*N)$, and under this isomorphism $t=\sum_{g\in G} g$ gets mapped to the matrix with t_N in every component.*

Proof of claim. Let $e_g\in S^*$ denote the idempotent corresponding to the component $S/I(H^g)$ of S^* and note that

$$e_g=e_1^g=g^{-1}e_1g\quad(g\in\mathcal{T}_H)\,.$$

Therefore,

$$e_1T^*e_1=e_1(S^**G)e_1=e_1S^*e_1*N\cong(S/I(H))*N\,.$$

Also, as right T^*–modules,

$$T^*=\bigoplus_{g\in\mathcal{T}_H} e_gT^*=\bigoplus_{g\in\mathcal{T}_H} g^{-1}e_1T^*\cong(e_1T^*)^h\,,$$

and so, as rings,

$$T^*\cong \mathrm{End}_{T^*}(T^*)\cong M_h(\mathrm{End}_{T^*}(e_1T^*))\cong M_h(e_1T^*e_1)\,.$$

An explicit isomorphism is given as follows:

$$T^* \to M_h(e_1T^*e_1): \quad a \mapsto (e_1xay^{-1}e_1)_{x,y\in\mathcal{T}_H}\,.$$

Under this isomorphism t maps to the matrix whose (x,y) – position equals $\sum_{g\in G} e_1xgy^{-1}e_1$. Since $e_1xgy^{-1}e_1 = e_1xgy^{-1}$ if $xgy^{-1}\in N$ and $e_1xgy^{-1}e_1 = 0$ otherwise, the above sum equals e_1t_N or, identifying $e_1T^*e_1$ with $(S/I(H))*N$, just t_N. This proves the claim.

The claim implies in particular that

$$T^*/(t) \cong M_h(S*N/(I(H),\, t_N))\,.$$

Consider the following commutative diagram of rings:

$$\begin{array}{ccccccc}
T & \twoheadrightarrow & T(H) & \xrightarrow{\psi} & T^*/(t) & \cong & M_h(S*N/(I(H),t_N)) \\
\Big\| {\scriptstyle \mathrm{id}_T} & & & & & & \uparrow {\scriptstyle \mathrm{can.}} \\
T & \twoheadrightarrow & T^0 & \hookrightarrow & T^* & \cong & M_h((S/I(H))*N)
\end{array}$$

This yields a commutative diagram

$$\begin{array}{ccccccc}
G_0(T) & \xleftarrow{\mathrm{Infl}^T_{T(H)}} & G_0(T(H)) & \xleftarrow{\rho} & G_0(T^*/(t)) & \cong & G_0(S*N/(I(H),t_N)) \\
\Big\| {\scriptstyle \mathrm{id}} & & & & & & \downarrow {\scriptstyle \mathrm{Infl}} \\
G_0(T) & \xleftarrow{\mathrm{Infl}^T_{T^0}} & G_0(T^0) & \xleftarrow{\mathrm{Res}} & G_0(T^*) & \cong & G_0((S/I(H))*N)
\end{array}$$

Thus it is enough to show that the image of

$$G_0(S*N/(I(H),t_N)) \xrightarrow{\mathrm{Infl}} G_0((S/I(H))*N) \cong G_0(T^*) \xrightarrow{\mathrm{Res}} G_0(T^0) \xrightarrow{\mathrm{Infl}} G_0(T)$$

coincides with the image of

$$G_0(S*N/(I(H),t_N)) \xrightarrow{\mathrm{Infl}} G_0(S*N) \xrightarrow{\mathrm{Ind}^T_{S*N}} G_0(T)\,.$$

This will follow if we can show that the diagram

$$\begin{array}{ccccc}
G_0((S/I(H))*N) & \xrightarrow{\cong} & G_0(T^*) & \xrightarrow{\mathrm{Res}} & G_0(T^0) \\
\downarrow {\scriptstyle \mathrm{Infl}} & & & & \downarrow {\scriptstyle \mathrm{Infl}} \\
G_0(S*N) & & \xrightarrow{\mathrm{Ind}^T_{S*N}} & & G_0(T)
\end{array}$$

commutes. So let W be a finitely generated $(S/I(H))*N$–module. Proceeding clockwise, $[W]$ successively gets mapped as follows:

$$[W] \overset{\cong}{\longmapsto} [W \underset{S/I(H)*N}{\otimes} e_1T^*] \overset{\mathrm{Infl}\circ\mathrm{Res}}{\longmapsto} [W \underset{S/I(H)*N}{\otimes} e_1T^*_T] \in G_0(T)\,.$$

Now, as right T–modules,

$$e_1 T^* = (S/I(H))T^* \cong T/I(H)T \cong \mathrm{Ind}_{S*N}^{T}((S/I(H)) * N)\,,$$

and so

$$W \underset{S/I(H)*N}{\otimes} e_1 T_T^* \cong \mathrm{Ind}_{S*N}^{T} \circ \mathrm{Infl}_{S/I(H))*N}^{S*N}(W)$$

as required. This completes the proof of the lemma. □

1.5 Lemma. *Let $\psi : A \to B$ be a homomorphism of right Noetherian rings so that B is finitely generated as right A–module. Suppose further that, for some ideal X of A and some $n \geqq 1$,*

$$X^n \cdot \ker\psi \cdot X^n = (0) \quad \textit{and} \quad \operatorname{coker}\psi \cdot X^n = (0)\,.$$

Then

$$G_0(A) = \rho(G_0(B)) + \mathrm{Infl}\, G_0(A/X)\,,$$

where ρ denotes restriction along ψ.

Proof. Since inflation yields isomorphisms $G_0(A/X) \cong G_0(A/X^n)$ for every $n \geqq 1$, we may assume that $n = 1$. Let M be a finitely generated right A–module. Then $[M] - [MX] = [M/MX] \in \mathrm{Infl}\, G_0(A/X)$, and so it suffices to show that

$$[MX] \in \rho(G_0(B)) + \mathrm{Infl}\, G_0(A/X)\,.$$

Applying $MX \otimes_A (\,.\,)$ to the A–A–exact sequence

$$0 \to \psi A \to B \to \operatorname{coker}\psi \to 0$$

gives the following exact sequence of finitely generated right A–modules:

$$\mathrm{Tor}_1^A(MX, \operatorname{coker}\psi) \to MX \underset{A}{\otimes} \psi A \to MX \underset{A}{\otimes} B \to MX \underset{A}{\otimes} \operatorname{coker}\psi \to 0\,.$$

Here, the first and the last term are A/X–modules, and $MX \otimes_A B$ is the restriction of a B–module. Thus

$$[MX \underset{A}{\otimes} \psi A] \in \rho(G_0(B)) + \mathrm{Infl}\, G_0(A/X)\,.$$

Finally, $MX \otimes_A \psi A \cong MX/MX \cdot \ker\psi$ and $MX \cdot \ker\psi$ is a finitely generated right A/X–module. Thus

$$[MX] = [MX \underset{A}{\otimes} \psi A] + [MX \cdot \ker\psi] \in \rho(G_0(B)) + \mathrm{Infl}\, G_0(A/X)\,,$$

and the proof is complete. □

1.6 Proposition. (i) *Let $\mathscr{X}_H$ denote the image of the map*

$$G_0(S*N/(I(H),t_N)) \xrightarrow{\text{Infl}} G_0(S*N) \xrightarrow{\text{Ind}} G_0(T)\,,$$

where $N = N_G(H)$. Then

$$\mathscr{I}_H = \mathscr{X}_H + \sum_{1 \neq x \in \mathscr{T}_H} \mathscr{I}_{\langle H,H^x\rangle}\,.$$

(ii) *Let $\mathscr{Y}_H$ denote the image of the map*

$$G_0(S*N/(I(H),t_N)) \xrightarrow{\text{Infl}} G_0(S*N) \xrightarrow[\cong]{\pi_N} G_0(kN) \xrightarrow{\text{Ind}} G_0(kG)\,,$$

where $N = N_G(H)$. Then

$$\mathscr{K}_H = \mathscr{Y}_H + \sum_{1 \neq x \in \mathscr{T}_H} \mathscr{K}_{\langle H,H^x\rangle}\,.$$

Proof. (i) By Lemmas 1.4 and 1.5,

$$G_0(T(H)) = \rho(G_0(T^*(H)/(t))) + \text{Infl}\, G_0(T(H)/Z)\,,$$

where we have put $Z = (YT + J(H))/J(H)$. Since $Z \subseteq J(\langle H,H^x\rangle)/J(H)$ and $\prod_{1 \neq x \in \mathscr{T}_H} J(\langle H,H^x\rangle)/J(H) \subseteq Z$, by definition of Y, we have

$$\begin{aligned} G_0(T(H)/Z) &= \sum_{1 \neq x \in \mathscr{T}_H} \text{Infl}\, G_0(T/J(\langle H,H^x\rangle)) \\ &= \sum_{1 \neq x \in \mathscr{T}_H} \text{Infl}\, G_0(T(\langle H,H^x\rangle))\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Infl}^T_{T(H)/Z}\, G_0(T(H)/Z) &= \sum_{1 \neq x \in \mathscr{T}_H} \text{Infl}\, G_0(T(\langle H,H^x\rangle)) \\ &= \sum_{1 \neq x \in \mathscr{T}_H} \mathscr{I}_{\langle H,H^x\rangle}\,. \end{aligned}$$

Since $(\text{Infl}^T_{T(H)} \circ \rho)(G_0(T^*(H)/(t))) = \mathscr{X}_H$, by Lemma 1.4, the proof of (i) is complete.

(ii) is immediate from (i), using the commutative diagram

$$\begin{array}{ccc} G_0(S*N) & \xrightarrow{\pi_N} & G_0(kN) \\ \downarrow{\scriptstyle \text{Ind}^T_{S*N}} & & \downarrow{\scriptstyle \text{Ind}^{kG}_{kN}} \\ G_0(T) & \xrightarrow{\pi} & G_0(kG) \end{array}\,.$$

To check commutativity, it suffices to test a *projective* $S*N$–module X, since $S*N$ has finite global dimension ([McC-R], Theorem 7.5.6(iii)). But then, proceeding clockwise, $[X]$ gets mapped as follows:

$$[X] \mapsto [X \underset{S*N}{\otimes} kN] \longmapsto [X \underset{S*N}{\otimes} kN \underset{kN}{\otimes} kG] = [X \underset{S*N}{\otimes} kG]\,,$$

and the counterclockwise route gives

$$[X] \mapsto [X \underset{S*N}{\otimes} T] \longmapsto [X \underset{S*N}{\otimes} T \underset{T}{\otimes} kG] = [X \underset{S*N}{\otimes} kG]\,,$$

as needed. □

1.7 Note that the algebra $S * N/(I(H), t_N)$ which features in the terms $\mathscr{X}_H$ and $\mathscr{Y}_H$ in Proposition 1.6 can be thought of as $T(H)$, but with G replaced by $N = N_G(H)$. Thus the image of the map

$$G_0(S * N/(I(H), t_N)) \xrightarrow{\text{Infl}} G_0(S * N)$$

corresponds to $\mathscr{I}_H$, again with $N = N_G(H)$ instead of G. In order to describe $\mathscr{X}_H$ and $\mathscr{Y}_H$ in more detail, we can therefore assume that H is normal in G. This case will be dealt with in the next section.

2 Normal subgroups

2.1 Throughout this section, we assume that H is a *normal* subgroup of G. Thus, using the notation of **1.1**, $I(H) = I^0(H)$ is a G–stable ideal of S, and

$$T^0(H) = T/I(H)T \cong (S/I(H)) * G\ .$$

We define

$$I_1(H) := I(H)T + t_H T\ ,$$

where $t_H = \sum_{h \in H} h \in T$ and

$$I_2(H) := I(H)T + \sum_{h \in H} (h-1)T + Tt'_H T\ ,$$

where $t'_H \in T$ is the sum over a fixed transversal for H in G. The choice of this transversal is immaterial because of the term $\sum_{h \in H}(h-1)T$. Note that t_H is central modulo $I(H)T$ and that $\sum_{h \in H}(h-1)T \equiv \sum_{h \in H} T(h-1)$ modulo $I(H)T$. Thus $I_1(H)$ and $I_2(H)$ are twosided ideals of T. Moreover, $t_H(h-1) = 0$ holds for all $h \in H$, and $t = t_H t'_H$. Therefore, with $J(H) = I(H)T + I$ as in **1.1**, we have

$$I_1(H) \cap I_2(H) \supseteq J(H)$$

and

$$I_1(H) I_2(H) \subseteq J(H)\ .$$

Recall that $T(H) = T/J(H) = T^0(H)/(t)$. Putting

$$T_1(H) = T/I_1(H) \cong T^0(H)/(t_H)$$

and

$$T_2(H) = T/I_2(H) \cong (S/I(H)) * (G/H)/(t_{G/H})$$

we obtain that

$$G_0(T(H)) = \text{Infl}\, G_0(T_1(H)) + \text{Infl}\, G_0(T_2(H))\ .$$

Hence we have the following

Lemma. *Let $\mathscr{K}_H$ be as in* 1.2. *Then* $\mathscr{K}_H = \operatorname{Im} \varphi_1 + \operatorname{Im} \varphi_2$, *where*

$$\varphi_\ell : G_0(T_\ell(H)) \xrightarrow{\text{Infl}} G_0(T) \xrightarrow{\pi} G_0(kG) \quad (\ell = 1, 2).\ \square$$

2.2 Put

$$V(H) := k\text{–subspace of } V \text{ generated by the elements}$$
$$v - v^h \quad (v \in V, h \in H)\,.$$

This is a G–invariant subspace of V with $I(H) = V(H)T$. Moreover, $V = V(H) \oplus V^H$, since kG is semisimple, and

$$S/I(H) \cong S(V/V(H)) \cong S(V^H)\,,$$

the symmetric algebra of $V/V(H) \cong V^H$. Define

$$\alpha_H := \sum_{i \geqq 0} (-1)^i [\wedge^i V(H)] \in G_0(kG)\,,$$

and

$$\begin{aligned} G^*(H) &:= \text{image of Infl} : G_0(kG/(t_H)) \to G_0(kG) \\ &= \text{subgroup of } G_0(kG) \text{ generated by the irreducible } kG\text{–} \\ &\quad\ \text{modules } X \text{ with } X^H = (0)\,. \end{aligned}$$

Indeed, the above modules X provide a $\mathbb{Z}$–basis for $G^*(H)$.

Lemma. $\operatorname{Im} \varphi_1 = \alpha_H \cdot G^*(H)$.

Proof. Put $e_H = |H|^{-1} t_H \in kG \subseteq T^0(H)$, a central idempotent of $T^0(H)$ with $T_1(H) \cong T^0(H)/(e_H) \cong (1 - e_H)T^0(H)$. The Quillen induction isomorphism

$$\operatorname{Ind}_{kG}^{T^0(H)} : G_0(kG) \xrightarrow{\cong} G_0(T^0(H))$$

(see Lemma 1.2(i)) maps the direct factor $G_0(kG/(t_H)) \cong G_0((1 - e_H)kG) = G^*(H)$ of $G_0(kG)$ to the direct factor $G_0((1 - e_H)T^0(H)) \cong G_0(T_1(H))$ of $G_0(T^0(H))$. Therefore, $\operatorname{Im} \varphi_1$ is identical with the image of the map

$$\lambda : G_0(kG/(t_H)) \underset{\cong}{\xrightarrow{\text{Ind}}} G_0(T_1(H)) \xrightarrow{\varphi_1} G_0(kG)\,.$$

To describe the latter, put

$$\widetilde{T} = S(V(H)) * G \subseteq T$$

and consider the following commutative diagram of rings:

$$\begin{array}{ccc} \widetilde{T} & \xrightarrow{\text{can.}} & kG/(t_H) = \widetilde{T}/(V(H), t_H) \\ \downarrow & & \downarrow \\ T & \xrightarrow{\text{can.}} & T_1(H) = T/(V(H), t_H) \end{array}$$

This yields a commutative diagram

$$\begin{array}{ccc} G_0(\widetilde{T}) & \xleftarrow{\text{Infl}} & G_0(kG/(t_H)) \\ \big\downarrow \text{ind} & & \big\downarrow \text{ind} \\ G_0(T) & \xleftarrow{\text{Infl}} & G_0(T_1(H)) \end{array}$$

To check commutativity, note that

$$T_1(H) \cong (\widetilde{T}/(V(H), t_H)) \underset{\widetilde{T}}{\otimes} T \cong kG/(t_H) \underset{\widetilde{T}}{\otimes} T$$

as $kG/(t_H)$–T–bimodules. Thus, for any $[W] \in G_0(kG/(t_H))$,

$$\begin{aligned}\mathrm{Ind}_{\widetilde{T}}^{T} \circ \mathrm{Infl}_{kG/(t_H)}^{\widetilde{T}}([W]) &= [W \underset{\widetilde{T}}{\otimes} T] = [W \underset{kG/(t_H)}{\otimes} kG/(t_H) \underset{\widetilde{T}}{\otimes} T] \\ &= \mathrm{Infl}_{T_1(H)}^{T}([W \underset{kG/(t_H)}{\otimes} T_1(H)]) \\ &= \mathrm{Infl}_{T_1(H)}^{T} \circ \mathrm{Ind}_{kG/(t_H)}^{T_1(H)}([W]),\end{aligned}$$

as required.

Therefore, we obtain

$$\begin{aligned}\lambda &= \pi \circ \mathrm{Ind}_{\widetilde{T}}^{T} \circ \mathrm{Infl}_{kG/(t_H)}^{\widetilde{T}} \\ &= \sum_{i \geqq 0} (-1)^i \, \mathrm{Tor}_i^T (\mathrm{Ind}_{\widetilde{T}}^{T} \circ \mathrm{Infl}_{kG/(t_H)}^{\widetilde{T}}(\,\cdot\,), kG) \\ &= \sum_{i \geqq 0} (-1)^i \, \mathrm{Tor}_i^{\widetilde{T}} (\mathrm{Infl}_{kG/(t_H)}^{\widetilde{T}}(\,\cdot\,), kG) \\ &= \widetilde{\pi} \circ \mathrm{Infl}_{kG/(t_H)}^{\widetilde{T}},\end{aligned}$$

where the third equality follows from Shapiro's Lemma and where $\widetilde{\pi} : G_0(\widetilde{T}) \to G_0(kG)$ denotes the usual isomorphism. By Lemma 1.2(iii), with V replaced by $V(H)$ and α_V by $\alpha_{V(H)} = \alpha_H$, we know that

$$\widetilde{\pi} \circ \mathrm{Infl}_{kG}^{\widetilde{T}} = \text{ multiplication by } \alpha_H .$$

Therefore,

$$\begin{aligned}\lambda &= \widetilde{\pi} \circ \mathrm{Infl}_{kG}^{\widetilde{T}} \circ \mathrm{Infl}_{kG/(t_H)}^{kG} \\ &= (\text{mult. by } \alpha_H) \circ \mathrm{Infl}_{kG/(t_H)}^{kG},\end{aligned}$$

which proves the lemma. □

2.3 For φ_2, note that

$$T_2(H) \cong S(V/V(H)) * (G/H) \big/ (t_{G/H})$$

is analogous with T/I, but with V replaced by $V/V(H)$ and G by G/H. Thus, putting

$$T_{G/H} := S(V/V(H)) * (G/H) \cong T \Big/ \Big(I(H)T + \sum_{h \in H} (h-1)T \Big)$$

and letting $\pi_{G/H} : G_0(T_{G/H}) \to G_0(k(G/H))$ denote the usual isomorphism, we will be able to assume in Sect. 3 that, for $1 \neq H$,

$$\mathscr{K}(G/H) := \mathrm{Im} \left\{ \psi_{G/H} : G_0(T_2(H)) \xrightarrow{\mathrm{Infl}} G_0(T_{G/H}) \xrightarrow{\pi_{G/H}} G_0(k(G/H)) \right\}$$

is known by induction (on $|G|$).

Lemma. $\operatorname{Im}\varphi_2 = \alpha_H \cdot \operatorname{Infl}_{k(G/H)}^{kG}(\mathscr{K}(G/H))$.

Proof. We will show that

$$\varphi_2 = (\text{multiplication by } \alpha_H) \circ \operatorname{Infl}_{k(G/H)}^{kG} \circ \psi_{G/H} .$$

Since $\varphi_2 = \pi \circ \operatorname{Infl}_{T_{G/H}}^{T} \circ \operatorname{Infl}_{T_2(H)}^{T_{G/H}}$, it suffices to show that

$$\pi \circ \operatorname{Infl}_{T_{G/H}}^{T} = (\text{mult. by } \alpha_H) \circ \operatorname{Infl}_{k(G/H)}^{kG} \circ \pi_{G/H} .$$

Using the fact that $\pi_{G/H}$ is inverse to $\operatorname{Ind}_{k(G/H)}^{T_{G/H}}$ (Lemma 1.2), this amounts to proving that

$$\pi \circ \operatorname{Infl}_{T_{G/H}}^{T} \circ \operatorname{Ind}_{k(G/H)}^{T_{G/H}} = (\text{mult. by } \alpha_H) \circ \operatorname{Infl}_{k(G/H)}^{kG} .$$

So let X be a finite-dimensional $k(G/H)$-module and put

$$W = \operatorname{Ind}_{k(G/H)}^{T_{G/H}}(X) = X \underset{k(G/H)}{\otimes} T_{G/H} \cong X \underset{k}{\otimes} S(V/V(H)) .$$

Here, the last term has the $T_{G/H}$-module structure given by the diagonal action of G/H and the regular action of $S(V/V(H))$ on itself, so $S(V/V(H)) \cong k \otimes_{k(G/H)} T_{G/H}$ as $T_{G/H}$-modules. Thus, $\operatorname{Infl}_{T_{G/H}}^{T} S(V/V(H)) \cong k \otimes_{\widetilde{T}} T$, where $\widetilde{T} = S(V(H)) * G$ is as in the proof of Lemma 2.2. Using the $G_0(kG)$-module structure on $G_0(T)$ as explained in **1.2**, we obtain

$$\operatorname{Infl}_{T_{G/H}}^{T}[W] = \operatorname{Infl}_{k(G/H)}^{kG}([X]) \cdot [k \underset{\widetilde{T}}{\otimes} T] .$$

Using $G_0(kG)$-linearity of π (Lemma 1.2), one computes

$$\begin{aligned}
\pi(\operatorname{Infl}_{T_{G/H}}^{T}[W]) &= \operatorname{Infl}_{kG/H}^{kG}([X]) \cdot \pi([k \underset{\widetilde{T}}{\otimes} T]) \\
&= \operatorname{Infl}_{k(G/H)}^{kG}([X]) \cdot \sum_{i \geqq 0}(-1)^i [\operatorname{Tor}_i^T(k \underset{\widetilde{T}}{\otimes} T, kG)] \\
&= \operatorname{Infl}_{k(G/H)}^{kG}([X]) \cdot \sum_{i \geqq 0}(-1)^i [\operatorname{Tor}_i^{\widetilde{T}}(k, kG)] \\
&= \operatorname{Infl}_{k(G/H)}^{kG}([X]) \cdot \widetilde{\pi}([k]) \\
&= \operatorname{Infl}_{k(G/H)}^{kG}([X]) \cdot \alpha_H ,
\end{aligned}$$

where the third equality uses Shapiro's Lemma and the last follows from Lemma 1.2(iii). This proves the desired equality. □

2.4 Combining Lemmas 2.1, 2.2 and 2.3 we obtain the following

Proposition. *If* $H \lhd G$ *then* $\mathscr{K}_H = \alpha_H \cdot G^*(H) + \alpha_H \cdot \operatorname{Infl}_{k(G/H)}^{kG}(\mathscr{K}(G/H))$.

3 Proof of the main result

3.1 Let H be a subgroup of G and put

$$N = N_G(H).$$

Let $V(H)$ be defined as in **2.2.** Then $V(H)$ is a module over kN. Also, the remaining definitions of **2.2** now become

$$\alpha_H = \sum_{i \geq 0} (-1)^i [\wedge^i V(H)] \in G_0(kN)$$

and

$$\begin{aligned} G^*(H) &= \text{image of Infl} : G_0(kN/(t_H)) \to G_0(kN) \\ &= \text{subgroup of } G_0(kN) \text{ generated by the irreducible} \\ &\qquad kN\text{-modules } X \text{ with } X^H = (0). \end{aligned}$$

In addition, we put

$$\mathcal{R}_H := \operatorname{Ind}_{kN}^{kG}(\alpha_H G^*(H)) \subseteq G_0(kG).$$

Lemma. *Let $\mathcal{Y}_H$ be as in Proposition 1.6*(ii). *Then*

$$\mathcal{Y}_H = \mathcal{R}_H + \operatorname{Ind}_{kN}^{kG}\left(\alpha_H \operatorname{Infl}_{k(N/H)}^{kN}(\mathcal{K}(N/H))\right),$$

where $\mathcal{K}(N/H)$ is as in 2.3. *In particular,*

$$\mathcal{R}_H \subseteq \mathcal{K}_H \subseteq \mathcal{K}.$$

Proof. By definition of $\mathcal{Y}_H$ and $\mathcal{R}_H$, the expression for $\mathcal{Y}_H$ will follow if we can show that the map

$$G_0(S * N/(V(H), t_N)) \xrightarrow{\text{Infl}} G_0(S * N) \xrightarrow{\pi_N} G_0(kN)$$

has image

$$\alpha_H G^*(H) + \alpha_H \operatorname{Infl}_{k(N/H)}^{kN}(\mathcal{K}(N/H)).$$

But this is the content of Proposition 2.4, with N in the role of G. (See also **1.7.**) □

3.2 Lemma. *Suppose that $H \subseteq E \subseteq \widehat{E} = N \cap N_G(E)$.*

(i) *Let $\bar{\alpha}_{E/H} \in G_0(k(\widehat{E}/H))$ be defined in the same way as α_E, but for E/H acting on $V/V(H)$. Then*

$$(\operatorname{Res}_{k\widehat{E}}^{kN_G(H)} \alpha_H) \cdot (\operatorname{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}} \bar{\alpha}_{E/H}) = \operatorname{Res}_{k\widehat{E}}^{kN_G(E)} \alpha_E.$$

(ii) *Let $G^*(E/H)$ denote the subgroup of $G_0(k(\widehat{E}/H))$ generated by all irreducible $k(\widehat{E}/H)$-modules X with $X^{E/H} = (0)$. Then*

$$\operatorname{Ind}_{k\widehat{E}}^{kN_G(E)} \circ \operatorname{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}}(G^*(E/H)) \subseteq G^*(E).$$

Proof. (i) Note that $(V/V(H))(E/H) = V(E)/V(H)$ and $V(E) \cong V(H) \oplus (V(E)/V(H))$ as $k\widehat{E}$–modules, since $k\widehat{E}$ is semisimple. Thus (i) is a consequence of the following general fact which holds for any finite group D. For any finite-dimensional kD–module U, define

$$\alpha_U = \sum_{i \geqq 0} (-1)^i [\wedge^i U] \in G_0(kD) .$$

Claim. *Let U and W be finite-dimensional kD–modules. Then $\alpha_{U \oplus W} = \alpha_U \cdot \alpha_W$.*

Proof of claim. For all $i \geqq 0$, $\wedge^i(U \oplus W) \cong \oplus_{a+b=i} \wedge^a U \otimes_k \wedge^b W$ as kD–modules, and so

$$[\wedge^i(U \oplus W)] = \sum_{a+b=i} [\wedge^a U][\wedge^b W]$$

holds in $G_0(kD)$. The claim follows from this, and hence (i) is proved.

(ii) Clearly, $\mathrm{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}}(G^*(E/H))$ is contained in the subgroup of $G_0(k\widehat{E})$ that is generated by all finite-dimensional $k\widehat{E}$–modules X with $X^E = (0)$. Letting $\mathscr{E}$ denote a set of right coset representatives of $\widehat{E}$ in $N_G(E)$ we have, for any such X,

$$(\mathrm{Ind}_{k\widehat{E}}^{kN_G(E)} X)\Big|_E \cong \bigoplus_{g \in \mathscr{E}} X^g \Big|_E ,$$

where X^g denotes the g–conjugate of X. Taking E–fixed points we obtain

$$(\mathrm{Ind}_{k\widehat{E}}^{kN_G(E)} X)^E \cong \bigoplus_{g \in \mathscr{E}} (X^g)^E \cong (X^E)^{|\mathscr{E}|} = (0) .$$

This proves (ii). □

3.3 Since $G_0(R) \cong G_0(kG)/\mathscr{K}$ by **1.1**, the following proves the main result of this article.

Theorem. $\mathscr{K} = \sum_{1 \neq H \subseteq G} \mathscr{R}_H$.

Proof. In view of Lemma 3.1, we need only show that $\mathscr{K} \subseteq \sum_H \mathscr{R}_H$. Furthermore, since $\mathscr{K} = \sum_{1 \neq H \subseteq G} \mathscr{K}_H$, by Proposition 1.2, it suffices to establish the following assertion.

$$(*) \qquad \textit{For each } 1 \neq H \leqq G,\ \mathscr{K}_H \subseteq \sum_{H \subseteq E \subseteq G} \mathscr{R}_E .$$

We argue by induction, first on $|G|$ and then on $[G : H]$. The assertion being vacuous for $|G| = 1$, assume that $G \neq \langle 1 \rangle$ and that $(*)$ is true for all nonidentity subgroups of groups of smaller order than G. In particular, the theorem holds for all these groups. Furthermore, if $H = G$, then Proposition 1.6(ii) and

Lemma 3.1 yield

$$\mathscr{K}_G = \mathscr{Y}_G = \mathscr{R}_G ,$$

since $\mathscr{K}(\langle 1\rangle) = (0)$. Thus $(*)$ holds for $H = G$, and hence we may assume that $G \neq H$ and that $(*)$ holds for all subgroups of G having smaller index than H, in particular for the subgroups of the form $\langle H, H^x\rangle$ with $x \in G\backslash N$ (where $N = N_G(H)$, as usual). Thus, from Proposition 1.6(ii) and Lemma 3.1, we obtain that

$$\mathscr{K}_H \subseteq \mathscr{R}_H + \mathrm{Ind}_N^G\left(\alpha_H \,\mathrm{Infl}_{k(N/H)}^{kN}(\mathscr{K}(N/H))\right) + \sum_{x\in G\backslash N}\ \sum_{<H,H^x>\subseteq E\subseteq G} \mathscr{R}_E .$$

Therefore, it suffices to show that

$$\mathscr{Z}_H := \mathrm{Ind}_{kN}^{kG}\left(\alpha_H \,\mathrm{Infl}_{k(N/H)}^{kN}(\mathscr{K}(N/H))\right) \subseteq \sum_{H\subseteq E\subseteq G} \mathscr{R}_E .$$

Since N/H has smaller order than G, we know by induction that

$$\mathscr{K}(N/H) = \sum_{H\subseteq E\subseteq N} \mathscr{R}_{E/H} .$$

For each such E, put $\widehat{E} = N \cap N_G(E)$. Then

$$\mathscr{R}_{E/H} = \mathrm{Ind}_{k(\widehat{E}/H)}^{k(N/H)}\left(\bar{\alpha}_{E/H} G^*(E/H)\right) ,$$

where $\bar{\alpha}_{E/H}$ and $G^*(E/H)$ are as in Lemma 3.2. Therefore, since $\mathrm{Infl}_{k(N/H)}^{kN} \circ \mathrm{Ind}_{k(\widehat{E}/H)}^{k(N/H)} = \mathrm{Ind}_{k\widehat{E}}^{kN} \circ \mathrm{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}}$, we have

$$\begin{aligned}
\mathscr{Z}_H &= \sum_{H\subseteq E\subseteq N} \mathrm{Ind}_{kN}^{kG}\left(\alpha_H \cdot (\mathrm{Infl}_{k(N/H)}^{kN} \circ \mathrm{Ind}_{k(\widehat{E}/H)}^{k(N/H)})(\bar{\alpha}_{E/H} G^*(E/H))\right) \\
&= \sum_{H\subseteq E\subseteq N} \mathrm{Ind}_{kN}^{kG}\left(\alpha_H \cdot \mathrm{Ind}_{k\widehat{E}}^{kN}(\mathrm{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}} \bar{\alpha}_{E/H} \cdot \mathrm{Infl}_{k\widehat{E}/H}^{k\widehat{E}} G^*(E/H))\right) \\
&= \sum_{H\subseteq E\subseteq N} \mathrm{Ind}_{k\widehat{E}}^{kG}\left(\alpha_H|_{\widehat{E}} \cdot \mathrm{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}} \bar{\alpha}_{E/H} \cdot \mathrm{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}} G^*(E/H)\right) ,
\end{aligned}$$

where the last equality follows from Mackey's tensor product theorem ([C-R], Corollary 10.20). By Lemma 3.2(i),

$$\alpha_H|_{\widehat{E}} \cdot \mathrm{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}} \bar{\alpha}_{E/H} = \alpha_E|_{\widehat{E}}$$

and so, using Mackey's tensor product theorem again, the last sum above can be rewritten as follows:

$$\begin{aligned}
&\sum_{H\subseteq E\subseteq N} \mathrm{Ind}_{k\widehat{E}}^{kG}\left(\alpha_E|_{\widehat{E}} \cdot \mathrm{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}} G^*(E/H)\right) \\
&\qquad = \sum_{H\subseteq E\subseteq N} \mathrm{Ind}_{kN_G(E)}^{kG}\left(\alpha_E \cdot \mathrm{Ind}_{k\widehat{E}}^{kN_G(E)} \mathrm{Infl}_{k(\widehat{E}/H)}^{k\widehat{E}} G^*(E/H)\right) ,
\end{aligned}$$

and by Lemma 3.2(ii) this last expression is contained in

$$\sum_{H\subseteq E\subseteq N} \mathrm{Ind}_{kN_G(E)}^{kG}(\alpha_E G^*(E)) = \sum_{H\subseteq E\subseteq N} \mathscr{R}_E .$$

This shows that $\mathscr{X}_H \subseteq \sum_{H\subseteq E\subseteq N} \mathscr{R}_E$, and the proof of the theorem is complete. □

3.4 Not all subgroups $H \subseteq G$ are actually required in **3.3**. First, $\mathscr{R}_{\langle 1\rangle} = (0)$ because $G^*(\langle 1\rangle) = (0)$. Next, $\mathscr{R}_{H^g} = \mathscr{R}_H$ holds for all $H \subseteq G$ and $g \in G$. Finally, for a given $H \subseteq G$, put

$$\widetilde{H} = \{g \in G : V^{1-g} \subseteq V(H)\} .$$

Clearly, $\widetilde{H}$ is the unique largest subgroup of G with $V(\widetilde{H}) = V(H)$. It follows easily that $N_G(\widetilde{H}) = \mathrm{Stab}_G\, V(H)$ and so, in particular, $N_G(H) \subseteq N_G(\widetilde{H})$. Using Lemma 3.2(i) with $E = \widetilde{H}$ (and so $\widehat{E} = N_G(H)$) we obtain

$$\alpha_H = \mathrm{Res}_{kN_G(H)}^{kN_G(\widetilde{H})} \alpha_{\widetilde{H}} ,$$

and so Mackey's tensor product theorem ([C-R], Corollary 10.20) yields

$$\mathscr{R}_H = \mathrm{Ind}_{N_G(H)}^{kG}(\alpha_H G^*(H)) = \mathrm{Ind}_{kN_G(\widetilde{H})}^{kG}\left(\alpha_{\widetilde{H}}\, \mathrm{Ind}_{kN_G(H)}^{kN_G(\widetilde{H})} G^*(H)\right) .$$

As shown in the proof of Lemma 3.2(ii) (here with $E = \widetilde{H}$, $\widehat{E} = N_G(H)$), we have

$$\mathrm{Ind}_{kN_G(H)}^{kN_G(\widetilde{H})} G^*(H) \subseteq G^*(\widetilde{H}) .$$

Therefore

$$\mathscr{R}_H \subseteq \mathscr{R}_{\widetilde{H}} ,$$

and consequently

$$\mathscr{K} = \sum_H \mathscr{R}_{\widetilde{H}} ,$$

where H runs over a full set of non-conjugate non-identity subgroups of G.

4 Remarks, examples, and questions

4.1 The elements α_H

Recall that $\alpha_H = \sum(-1)^i[\Lambda^i V(H)] \in G_0(kN_G(H))$. More generally, let G be any finite group, k any field and W an aribitrary finite dimensional kG–module. Define, as in the proof of Lemma 3.2,

$$\alpha_W := \sum_{i\geq 0}(-1)^i[\Lambda^i W] \in G_0(kG) .$$

We determine the Brauer character χ_{α_W} of α_W. Fix a p–modular system (K, A, k) and a finitely generated A–free AG–module $\widetilde{W}$ with $W \cong \widetilde{W} \otimes_A k$ ([C-R], Theorem 22.1). Put $W_0 = \widetilde{W} \otimes_A K$ and, for each $g \in G$, let g_{W_0} denote the K–endomorphism of W_0 given the action of g. Then

$$\chi_{\alpha_W}(g) = \sum(-1)^i \operatorname{Trace}(\wedge^i g_{W_0}) = \det(1 - g_{W_0}) . \tag{1}$$

The second of the above equalities follows from the fact that the characteristic polynomial of any $\phi \in \operatorname{End}_K(W_0)$ is given by $\sum_{i=0}^{m}(-1)^i \operatorname{Trace}(\wedge^i \phi) X^{m-i}$, where $m = \dim_K W_0 = \dim_k W$. Note in particular that it follows from (1) that if W is kG–isomorphic to $W_1 \oplus \cdots \oplus W_t$ then

$$\alpha_W = \prod_{i=1}^{t} \alpha_{W_i} ; \tag{2}$$

this was already established by other means in the proof of Lemma 3.2.

4.2 The subgroup $\mathcal{R}_G$

By definition, $\mathcal{R}_G = \alpha_G G^*(G)$, where $G^*(G)$ is the subgroup of $G_0(kG)$ generated by all the non-trivial irreducible kG–modules. Thus $[kG] = 1 - \beta$ where $\beta \in G^*(G)$ and 1 denotes the class of the trivial module. Since $\alpha_G \cdot [kG] = \operatorname{Ind}_{\langle 1 \rangle}^{kG}(\alpha_G|_{\langle 1 \rangle}) = 0$, it follows that $\alpha_G = \alpha_G \beta \in \mathcal{R}_G$. Hence

$$\mathcal{R}_G = \alpha_G G_0(kG) ;$$

in particular $\mathcal{R}_G$ is an *ideal* of $G_0(kG)$ contained in $\mathcal{K}$. As the following result demonstrates, this inclusion is usually strict. (This is in fact a restatement of [Br-L2], Theorem 2.4(ii). See also [M].)

Proposition. *The equality $\mathcal{K} = \alpha_G G_0(kG)$ holds if and only if G acts fixed-point-freely on $V(G)$.*

Proof. Suppose G acts fixed-point-freely on $V(G)$. Then, for each non-identity subgroup H of G, $V(H) = V(G)$. Thus, in the notation of **3.4**, $\widetilde{H} = G$ for each such H; and hence $\mathcal{K} = \mathcal{R}_G$.

Suppose now that G does not act fixed-point-freely on $V(G)$, so that $V(G)^H \neq \{0\}$ for some non-identity subgroup H of G. Since, as kH–modules, $V(G) = V(G)^H \oplus V(H)$, **4.1**(2) implies that

$$\operatorname{Res}_H^G \alpha_G = \alpha_{V(G)^H} \cdot \alpha_{V(H)} = 0 \cdot \alpha_{V(H)} = 0 .$$

In other words, $\operatorname{Res}_H^G : G_0(kG) \to G_0(kH)$ factors through $G_0(kG)/(\alpha_G)$. However, $\operatorname{Res}_H^G[k] = [k_H]$ and $\operatorname{Res}_H^G[kG] = [G : H][kH]$ are $\mathbb{Z}$–independent, so $\operatorname{Im}(\operatorname{Res}_H^G)$ has torsion-free rank at least 2. Hence, $G_0(kG)/(\alpha_G)$ has rank at least 2, whereas $G_0(R) \cong G_0(kG)/\mathcal{K}$ has rank 1 by [Br-L2], Theorem 2.5(i). Therefore (α_G) must be properly contained in $\mathcal{K}$. □

4.3 Deleting trivial constituents

Since $V = V^G \oplus V(G)$, $V(H) = V(G)(H)$ for every subgroup H of G. Thus $\mathscr{K}$ (and so also $G_0(R)$) are unchanged on replacing V by $V(G)$. An alternative way to observe this is as follows: Write $R_1 = S(V(G))^G$, so that $R = R_1 \otimes_k S(V^G)$, a polynomial ring in $\dim_k V^G$ variables over the coefficient ring R_1. By Grothendieck's theorem ([B], Theorem XII.4.1), $G_0(R) \cong G_0(R_1)$.

4.4 The torsion subgroup of $G_0(R)$

Let the dimension homomorphism $\dim : G_0(kG) \to \mathbb{Z} : [W] \mapsto \dim_k W$ have kernel $\Omega = \Omega_G$. Thus Ω is an ideal of $G_0(kG)$, with $\mathbb{Z}$–module generators $\{\dim_k(W)1 - [W] : W \text{ irreducible}\}$, and

$$G_0(kG) = \mathbb{Z} \cdot \langle 1 \rangle \oplus \Omega\,.$$

Since $\dim \alpha_H = \sum_i (-1)^i \dim_k \wedge^i V(H) = \sum (-1)^i \begin{pmatrix} \dim_k V(H) \\ i \end{pmatrix} = 0$, we have $\alpha_H G^*(H) \subseteq \Omega_{N_G(H)}$. Furthermore, $\mathrm{Ind}^G_{N_G(H)}$ maps $\Omega_{N_G(H)}$ to Ω_G, so by Theorem 3.3

$$\mathscr{K} \subseteq \Omega\,.$$

Thus,

$$G_0(R) \cong G_0(kG)/\mathscr{K} \cong \mathbb{Z} \cdot 1 \oplus \Omega/\mathscr{K}.$$

As explained in the Introduction, (or see also [Br–L2], 2.4), the first of these isomorphisms, from right to left, is given by $\mu := (\,\cdot\,)^G \circ \mathrm{Ind}^T_{kG}$. Thus [1] is mapped by it to $[(k \otimes_{kG} T)^G] = [S^G] = [R]$. Since $G_0(R) = \mathbb{Z} \cdot [R] \oplus F$ where F is finite by [Br-L2], Theorem 2.5(i), μ identifies F with $\Omega/\mathscr{K}$; so by the same result again, (and with $n = \dim_k V$),

$$\Omega/\mathscr{K} \cong G_0(R)_{\text{torsion}} \text{ is a finite group of exponent dividing } |G|^n. \qquad (1)$$

Recall that an element g of G is a *pseudoreflection* (on V) if the endomorphism $1-g$ of V has rank 1 (or, equivalently, if $V(\langle g \rangle)$ has dimension 1). Let H be the (normal) subgroup of G generated by the pseudoreflections on V. By [S], we have

$$\mathrm{Cl}(R) \cong H^1(G/H, U(S^H)) \cong \mathrm{Hom}(G/H, k^*)\,, \qquad (2)$$

where $\mathrm{Cl}(R)$ is the class group of R and $U(X)$ is the group of units of the ring X. (In fact, Singh's result is stated and proved in the context of a *local* ring S, but it is easy to check that his proof can be adapted to the present setting.) Now by [Br-L], $\mathrm{Cl}(R)$ is a factor of $\Omega/\mathscr{K}$, and indeed is the top factor given by a filtration of $\Omega/\mathscr{K}$ whose successive factors are quotients of the higher class groups of R – for details, see [Br–L]. At most $(\dim_k V) - 1$ of these class groups can be non-trivial when k is algebraically closed; in particular, when $\dim_k V = 2$ and k is algebraically closed,

$$\Omega/\mathscr{K} \cong \mathrm{Cl}(R)\,. \qquad (3)$$

This fact was already observed in [H-S], where it was used to list the Grothendieck groups for the canonical action of the finite subgroups of $SL(2,\mathbb{C})$ on $\mathbb{C}^{(2)}$.

Singh's result quoted above determines in particular when R is a UFD. In this context, it is natural to ask the

Question: When is $\Omega/\mathscr{K}$ trivial?

By the Sheppard–Todd–Chevalley theorem [Sp], this certainly happens when G is generated by pseudoreflections. Indeed when G is abelian and k is a splitting field for G it is easy to see that $\Omega/\mathscr{K}$ is trivial if and only if G is generated by pseudoreflections. However the following example (a simplified treatment of [Br-L2], Theorem 6.3) shows that this fails in the absence of the hypothesis on the field.

Example A. Let p be a prime, k a field of characteristic not p which does *not* contain a root of unity of order p, G a group of order p and $V = kG$, the regular representation. Thus

$$V \cong k[X]/(X^p - 1) \cong k[X]/(X-1) \oplus W = k \oplus W,$$

where $W = k[X]/(1 + X + \cdots + X^{p-1})$ is irreducible. Thus

$$G_0(kG) = \mathbb{Z}[k] \oplus \mathbb{Z}[W] \cong \mathbb{Z}^{(2)}.$$

Now $V(G) = W$ and G acts fixed-point-freely on this module. Hence, by Proposition 4.2,

$$G_0(R) \cong G_0(kG)/(\alpha) \quad \text{with} \quad \alpha = \sum(-1)^i[\wedge^i W]\,.$$

We claim that

$$\alpha = (p-1)[k] - [W]\,.$$

Denote the right hand side here by β, so that $\beta = p[k]-[kG]$. Fix a p–modular system (K, A, k); the Brauer character χ of β is thus given by

$$\chi_\beta(1) = 0, \quad \chi_\beta(g) = p, \quad 1 \neq g \in G\,.$$

To compute the Brauer character χ_α of α, put $W_0 = K[X]/(1+X+\cdots+X^{p-1})$ and let g_{W_0} denote the K–endomorphism of W_0 given by the action of $g \in G$. Clearly $\chi_\alpha(1) = 0$, and for $1 \neq g \in G$ **4.1**(1) shows that

$$\chi_\alpha(g) = \sum(-1)^i \mathrm{Trace}(\wedge^i g_{W_0}) = \det(1 - g_{W_0})$$
$$= (1-\omega)(1-\omega)^2 \cdots (1-\omega^{p-1})\,,$$

where ω is a primitive pth root of unity in some algebraic closure K. Since $(1-\omega)(1-\omega^2)\cdots(1-\omega^{p-1}) = p$, we get $\chi_\beta = \chi_\alpha$ and so $\alpha = \beta$ as claimed.

Therefore,

$$G_0(R) \cong (\mathbb{Z}[k] \oplus \mathbb{Z}[W])/((p-1)[k] - [W]) \cong \mathbb{Z}\,.$$

For non-abelian groups we do not know the answer to the above question, even over algebraically closed fields. As the following example illustrates, the triviality of $\Omega/\mathscr{K}$ does not force G to be generated by pseudoreflections.

Example B. Let G be the binary icosahedral group. Then it is well known that G has an irreducible fixed point free complex representation V of dimension 2; see, for example, [W, page 181]. Since $\dim_{\mathbb{C}} V = 2$, (2) and (3) show that $\Omega/\mathscr{K}$ is $\mathrm{Hom}(G/H, \mathbb{C}^*)$, where H is the subgroup of G generated by the pseudoreflections on V. But $H = 1$ since the action is fixed point free, and so, since G is perfect, $\Omega/\mathscr{K}$ is trivial.

4.5 Power series rings

This case has predominated in much recent work on Grothendieck groups of invariant rings; see, for example, [A-R2], [H-S], [H-M-W] and [M]. Here, the notation is as in the Introduction, but one considers the invariants $\widehat{R}$ of the action of G on the power series ring $\widehat{S} = k[[X_1,\ldots,X_n]]$. All the results proved in this paper are valid also in this setting, with essentially the same proofs. Note that in this case the isomorphism of $G_0(kG)$ with $G_0(\widehat{S} * G)$ via the induction map is afforded by [*B*], Proposition IX. 1.3.(1), since $\widehat{S} * G$ is a complete semilocal ring with radical $V\widehat{S} * G$.

4.6 Abelian groups with a splitting field

Let G be an Abelian group and assume that k contains a root of unity of order equal to the exponent of G. Then

$$V \cong \bigoplus_{i=1}^{n} k_{\chi_i}$$

for suitable $\chi_i \in G^* = \mathrm{Hom}(G, k^*)$. Here, $k_{\chi_i} = k$ with $cg = c\chi_i(g)$ for $c \in k$, $g \in G$. We may assume that G acts faithfully, or – equivalently – that $G^* = \langle \chi_i : 1 \leqq i \leqq n \rangle$. Now, $G^* \cong G$ and there is a ring isomorphism

$$G_0(kG) \xrightarrow{\cong} \mathbb{Z}G^* : [k_\chi] \mapsto \chi\,,$$

where $\mathbb{Z}G^*$ is the integral group ring. Note that the ideal Ω of *4.4* corresponds to the augmentation ideal of $\mathbb{Z}G^*$ under this isomorphism.

For each subgroup H of G, write

$$\mathscr{G}^*(H) = \{\chi \in G^* : \chi|_H \neq 1|_H\}\,,$$

so that $G^*(H) = \mathbb{Z}\mathscr{G}^*(H) \subseteq \mathbb{Z}G^*$. Then

$$\alpha_H = \prod_{\substack{i=1 \\ \chi_i \in \mathscr{G}^*(H)}}^{n} (1 - \chi_i)$$

as follows from **4.1**, since $V(H) \cong \bigoplus_i \{k_{\chi_i} : k_{\chi_i} \in G^*(H)\}$. For a given subgroup H of G, the subgroup $\widetilde{H}$ of **3.4** is given by

$$\widetilde{H} = \bigcap_{i=1}^{n} \{\ker \chi_i : \chi_i \notin \mathscr{G}^*(H)\}$$

Thus, in this notation, **3.3** becomes

$$\mathscr{K} = \sum_{1 \neq H \subseteq G} \alpha_{\widetilde{H}} \mathbb{Z}\mathscr{G}^*(\widetilde{H}) . \tag{1}$$

4.7 Further remarks on the Abelian case

(1) The result **4.6**(1) for G abelian was the main theorem of [H-M-W]. (As mentioned in the introduction, one can without difficulty pass from polynomial rings to power series rings, and vice versa.)

(2) *Groups of prime order* ([Br–L2], Sect. 3) Let G have order p, prime, and assume that k contains a primitive p-th root of unity. By *4.3*, we may assume that $V \cong \oplus_{i=1}^{n} k_{\chi_i}$, with each irreducible character χ_i non-trivial. Thus we are in the situation of both *4.2* and *4.6*, and so

$$\mathscr{K} = \alpha_G \mathbb{Z}G^* = \prod_{i=1}^{n} \mathbb{Z}G^*(1 - \chi_i) = \Omega^n.$$

Therefore, by *3.3*,

$$G_0(R) \cong \mathbb{Z} \oplus (\Omega/\Omega^n) .$$

Let $n - 1 = (p - 1)q + u$, where q and u are non-negative integers with $0 \leqq u < p - 1$. Then it was shown by Passi [P] that

$$\Omega/\Omega^n \cong (\mathbb{Z}/p^{q+1}\mathbb{Z})^u \oplus (\mathbb{Z}/p^q\mathbb{Z})^{p-1-u}.$$

4.8 Factoring out pseudoreflections

Let G, k, V and S be as in the introduction, and let H be the subgroup of G generated by the pseudoreflections on V. The theorem of Sheppard, Todd and Chevalley guarantees that S^H is also a polynomial ring (in n variables since H is finite), and of course $R = S^G = (S^H)^{G/H}$. The following easy and presumably well-known lemma, (whose proof is left to the reader), shows that the action of G/H on S^H is linear.

Lemma. *With the above notation, G/H acts linearly and without pseudoreflections on S^H.*

It follows that *in calculating $G_0(R)$ we may apply the Main Theorem 3.3 to G/H acting on S^H*. In particular, we can improve **4.4**(1):

$\Omega/\mathscr{K} \cong G_0(R)_{\text{torsion}}$ *is a finite group of exponent dividing* $|G/H|^n$.

4.9 Dihedral groups of order $2p$ (p an odd prime).

Let

$$G = \langle x, y : x^p = 1 = y^2, xy = yx^{-1} \rangle,$$

and assume that char $k \nmid 2p$ and that k contains a root of unity ω of order p. Let $L = k_\phi$ be the non-trivial one-dimensional kG–module (so $\phi(y) = -1$ and $\phi(x) = 1$). For $i = 0, \ldots, p-1$ define $\psi_i \in \langle x \rangle^*$ by $\psi_i(x) = \omega^i$ and set

$$M_i := \mathrm{Ind}_{k\langle x \rangle}^{kG} k_{\psi_i} ,$$

so M_i is irreducible for $i \neq 0$, $M_0 \cong k \oplus L$ and $M_i \cong M_{p-i}$. In $G_0(kG)$ write 1 for $[k]$ and μ_i for $[M_i]$, $0 \leqq i \leqq q := (p-1)/2$, so that

$$G_0(kG) = \mathbb{Z}1 \oplus \bigoplus_{i=0}^{q} \mathbb{Z}\mu_i .$$

Reading subscripts modulo p, multiplication in $G_0(kG)$ is given by

$$\mu_i \mu_j = \mu_{i+j} + \mu_{i-j} \quad (0 \leqq i, j \leqq q) .$$

Let

$$[V] = r1 + s[L] + \sum_{i=1}^{q} t_i \mu_i$$

for $r, s, t_i \geqq 0$, where we shall assume that some t_i (say t_1) is non-zero. (Otherwise, $G/\langle x \rangle$ acts on V). Up to conjugacy the non-identity subgroups of G are $\langle x \rangle, \langle y \rangle$ and G; we determine $\mathscr{R}_H$ for each of these.

(i) $\underline{H = \langle x \rangle}$: Here $N_G(\langle x \rangle) = G$, $V(\langle x \rangle) \cong \oplus_{i=1}^{q} M_i^{(t_i)}$, and (in the notation of **4.1**),

$$\alpha_{M_i} = 1 - \mu_i + \det M_i = \mu_0 - \mu_i .$$

Thus, writing $\alpha_i := \mu_0 - \mu_i$ ($i = 1, \ldots, q$),

$$\alpha_{\langle x \rangle} = \prod_{i=1}^{q} \alpha_i^{t_i} ;$$

since $G^*(\langle x \rangle) = \langle \mu_i : i = 1, \ldots, q \rangle$, we have

$$\mathscr{R}_{\langle x \rangle} = \langle \alpha_{\langle x \rangle} \mu_j : j = 1, \ldots, q \rangle.$$

(ii) $\underline{H = \langle y \rangle}$: Here $N_G(\langle y \rangle) = \langle y \rangle$ and $V(\langle y \rangle) \cong k_\chi^{(s+\sum t_i)}$ where $\chi = \phi|_{\langle y \rangle}$. Thus $G_0(k\langle y \rangle) = \mathbb{Z}1 \oplus \mathbb{Z}\chi$ and

$$\alpha_{\langle y \rangle} = (1-\chi)^{s+\sum t_i} = 2^{s-1+\sum t_i}(1-\chi) .$$

Since $G^*(\langle y \rangle) = \langle \chi \rangle$ and $\alpha_{\langle y \rangle}\chi = -\alpha_{\langle y \rangle}$, we obtain

$$\begin{aligned} \mathscr{R}_{\langle y \rangle} &= \langle 2^{s-1+\sum t_i} \mathrm{Ind}_{k\langle y \rangle}^{kG}(1-\chi) \rangle \\ &= \langle 2^{s-1+\sum t_i}(2-\mu_0) \rangle , \end{aligned}$$

where the second equality follow from

$$\mathrm{Ind}_{k\langle y\rangle}^{G}(1-\chi) = (1-[L])\,\mathrm{Ind}_{k\langle y\rangle}^{kG} 1_{\langle y\rangle} = (2-\mu_0)\left(1+\sum_{i=1}^{q}\mu_i\right) = 2-\mu_0\,.$$

(iii) $\underline{H=G}$: Since $V(G) \cong L^{(s)} \oplus \oplus_{i=1}^{q} M_i^{(t_i)}$, we have

$$\alpha_G = (1-[L])^s \alpha_{\langle x\rangle} = \begin{cases} \alpha_{\langle x\rangle} & \text{if } s=0 \\ 0 & \text{if } s\neq 0 \end{cases},$$

since $[L]\mu_i = (\mu_0 - 1)\mu_i = \mu_i$ for $i = 0,\ldots,q$, and hence $[L]\alpha_{\langle x\rangle} = \alpha_{\langle x\rangle}$. Furthermore, $G^*(G) = \langle [L], \mu_j : j = 1,\ldots,q\rangle$ and so $\alpha_{\langle x\rangle}G^*(G) = \langle \alpha_{\langle x\rangle}\mu_j : j = 1,\ldots,q\rangle$. Set $\beta_j = \alpha_1\mu_j = \alpha_{j+1}+\alpha_{j-1}-2\alpha_j$. One checks easily that $\{\beta_1,\ldots,\beta_q\}$ forms a $\mathbb{Z}$-basis for $\langle \alpha_1,\ldots,\alpha_q\rangle$, so that we have $\alpha_{\langle x\rangle}G^*(G) = \mathscr{R}_{\langle x\rangle}$. Thus

$$\mathscr{R}_G = \alpha_G G^*(G) = \begin{cases} 0 & \text{if } s\neq 0 \\ \mathscr{R}_{\langle x\rangle} & \text{if } s=0 \end{cases}.$$

Combining the contributions (i), (ii) and (iii) gives

$$\mathscr{K} = \langle 2^{s-1+\Sigma t_i}(2-\mu_0),\ \alpha_{\langle x\rangle}\mu_j : j=1,\ldots,q\rangle\,.$$

Using the $\mathbb{Z}$–basis $\{1, 2-\mu_0, \alpha_1,\ldots,\alpha_q\}$ for $G_0(kG)$ and noting that $\langle\alpha_1,\ldots,\alpha_q\rangle$ is an ideal of $G_0(kG)$ which therefore contains all but the first of the above relations, we find

$$G_0(R) \cong G_0(kG)/\mathscr{K} \cong \mathbb{Z} \oplus \left(\mathbb{Z}\big/2^{s-1+\Sigma t_i}\mathbb{Z}\right) \oplus A\,,$$

where

$$A \cong \Big(\bigoplus_{i=1}^{q} \mathbb{Z}\alpha_i\Big)\big/\langle \alpha_{\langle x\rangle}\mu_j : j=1,\ldots,q\rangle\,.$$

We claim that

$$|A| = p^{\Sigma t_i - 1}\,.$$

To see this, put $\alpha'_{\langle x\rangle} = \alpha_1^{t_1-1}\prod_{i=2}^{q}\alpha_i^{t_i}$ and $\beta_j = \alpha_1\mu_j$. Then $\alpha_{\langle x\rangle}\mu_j = \alpha_{\langle x\rangle}\beta_j$ for $j=1,\ldots,q$. One checks that $\{\beta_1,\ldots,\beta_q\}$ forms a $\mathbb{Z}$–basis for $\langle\alpha_1,\ldots,\alpha_q\rangle$ and that the endomorphsim of this group given by multiplication by a fixed α_i has determinant p. The claim follows.

In particular, when $G = S_3$ (so $p=3$, $q=1$), then

$$A \cong \mathbb{Z}/3^{\Sigma t_i - 1}\mathbb{Z}\,.$$

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Note added in proof. The second author has implemented a computer programme using GAP to calculate Grothendieck groups of invariant rings using the Main Theorem. Details can be obtained from him.

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