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# ON CROSSED PRODUCTS OF HOPF ALGEBRAS

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ABSTRACT. Let $B = A\#_\sigma H$ denote a crossed product of the associative algebra $A$ with the Hopf algebra $H$. We investigate the weak dimension and the global dimension of $B$ and show that $\operatorname{wdim} B \le \operatorname{wdim} H + \operatorname{wdim} A$ and $\operatorname{l.gldim} B \le \operatorname{r.gldim} H + \operatorname{l.gldim} A$.

## 1. INTRODUCTION

Let $B = A\#_\sigma H$ denote a crossed product of the associative algebra $A$ with the Hopf algebra $H$. We establish the following estimates for the weak dimension and the global dimension of $B$ in terms of the corresponding data for $H$ and $A$:

$$\operatorname{wdim} B \le \operatorname{wdim} H + \operatorname{wdim} A \qquad \text{and} \qquad \operatorname{l.gldim} B \le \operatorname{r.gldim} H + \operatorname{l.gldim} A .$$

The first of these estimates is a consequence of a suitable spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^H_p(k, \operatorname{Tor}^A_q(V, W)) \underset{p}{\Longrightarrow} \operatorname{Tor}^B_n(V, W) ,$$

where $k$ is the trivial $H$-module (i.e., $H$ acts via the counit) and $V_B$ and ${}_BW$ are arbitrary $B$-modules. This spectral sequence will be constructed in Section 2.3 along with an analogous spectral sequence for Ext which yields the estimate for global dimension. Since a ring is von Neumann regular precisely if its weak dimension is 0, we conclude in particular that if $H$ and $A$ are both von Neumann regular, then $B$ is likewise. Specializing to the case of global dimension 0, we also deduce the known fact that if $H$ and $A$ are both semisimple, then so is $B$ (cf. [Mont], Theorem 7.4.2). Finally, we briefly discuss relative projectivity of $B$ with respect to $A$.

**Notation and basic facts.** Our reference for general material about Hopf algebras are the standard texts [Abe] and [Sw]. For crossed products in particular we follow the notes [Mont]. Throughout this article, we will keep the following notation:

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| $k$ | denotes a commutative field; |
| $H$ | will be a Hopf algebra over $k$, with counit $\epsilon$; the $H$-module $k$ will always be the trivial $H$-module; |
| $A$ | denotes an associative $k$-algebra with identity 1 so that there is a weak $H$-action on $A$, denoted $(h, a) \mapsto h \cdot a \quad (h \in H, a \in A)$; |
| $B = A\#_\sigma H$ | will denote a crossed product, with cocycle $\sigma : H \otimes_k H \to A$. |

Thus $B$ is an associative algebra such that there is an isomorphism of left $A$-modules

$$A \otimes_k H \xrightarrow{\cong} B, \qquad a \otimes h \mapsto a\#h .$$

The map $a \mapsto a\#1$ identifies $A$ with a subalgebra of $B$. Defining a $k$-linear map $\gamma : H \to B$ by

$$\gamma(h) = 1\#h \qquad (h \in H) ,$$

we have $a\gamma(h) = a\#h$ for $a \in A$, $h \in H$. It is known (cf. [Mont], Chapter 7) that $\gamma$ is convolution invertible and satisfies the following identities, for $h, k \in H$ and $a \in A$,

(1) $$\sigma(h, k) = \sum \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2) ,$$

(1a) $$\gamma(h)\gamma(k) = \sum \sigma(h_1, k_1)\gamma(h_2k_2) ,$$

(1b) $$\gamma^{-1}(hk) = \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\sigma(h_2, k_2) ,$$

(2) $$\gamma(h)a = \sum (h_1 \cdot a)\gamma(h_2) .$$

## 2. Proofs

2.1. **Action of $H$ on homomorphisms.** Let ${}_BV$ and ${}_BW$ be left $B$-modules. For each $\phi \in \mathrm{Hom}_A(V, W)$ and $h \in H$ define $\phi h : V \to W$ by

$$(\phi h)(v) = \sum \gamma^{-1}(h_1)\phi(\gamma(h_2)v) \qquad (v \in V) .$$

Then we have the following

**Lemma.** *The above definition makes* $\mathrm{Hom}_A(V, W)$ *a right* $H$*-module. There is a canonical* $k$*-linear isomorphism*

$$\mathrm{Hom}_H(k, \mathrm{Hom}_A(V, W)) \cong \mathrm{Hom}_B(V, W) .$$

*Furthermore,*

$$\mathrm{Hom}_A(B, W) \cong \mathrm{Hom}_k(H, W)$$

*as right* $H$*-modules (where* $H$ *acts on the right-hand side by* $(\psi h)(k) = \psi(hk)$ *for* $\psi \in \mathrm{Hom}_k(H, W)$ *and* $h, k \in H$*). Finally, if* $f : V \to V'$ *and* $g : W \to W'$ *are* $B$*-module maps, then* $g_* \circ f^* : \mathrm{Hom}_A(V', W) \to \mathrm{Hom}_A(V, W')$ *is an* $H$*-module map.*

*Proof*. The fact that $\phi h : V \to W$ is $A$-linear is proved exactly as in [Mont], proof of Theorem 7.4.2. Furthermore, the map $\mathrm{Hom}_A(V, W) \times H \to \mathrm{Hom}_A(V, W)$ , $(\phi, h) \mapsto \phi h$ is clearly $k$-bilinear. Using the identities (1a) and (1b) we compute, for $h, k \in H$ and $v \in V$,

$$\begin{aligned}[\phi(hk)](v) &= \sum \gamma^{-1}(h_1k_1)\phi(\gamma(h_2k_2)v) \\ &\underset{(1b)}{=} \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\sigma(h_2, k_2)\phi(\gamma(h_3k_3)v) \\ &= \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\phi[\sigma(h_2, k_2)\gamma(h_3k_3)v] \\ &\underset{(1a)}{=} \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\phi[\gamma(h_2)\gamma(k_2)v] \\ &= [(\phi h)k](v) .\end{aligned}$$

Thus $\operatorname{Hom}_A(V, W)$ is a right $H$-module.

In order to establish the first isomorphism, we first note that there is a canonical isomorphism of $\operatorname{Hom}_H(k, \operatorname{Hom}_A(V, W))$ with the $k$-space of $H$-invariants in $\operatorname{Hom}_A(V, W)$, that is, with

$$\operatorname{Hom}_A(V, W)^H = \{\phi \in \operatorname{Hom}_A(V, W) \mid \phi h = \epsilon(h)\phi \text{ for all } h \in H\} .$$

Thus it suffices to show that $\operatorname{Hom}_A(V, W)^H = \operatorname{Hom}_B(V, W)$. Let $\phi \in \operatorname{Hom}_A(V, W)$, $h \in H$ and $v \in V$. Then

$$\begin{aligned}(\phi h)(v) = \epsilon(h)\phi(v) &\Leftrightarrow \sum \gamma^{-1}(h_1)\phi(\gamma(h_2)v) = \sum \gamma^{-1}(h_1)\gamma(h_2)\phi(v) \\ &\Leftrightarrow \phi(\gamma(h)v) = \gamma(h)\phi(v) .\end{aligned}$$

Since $B = A\gamma(H)$, the last condition is equivalent with $\phi \in \operatorname{Hom}_B(V, W)$. This proves the first isomorphism.

Now consider the map $f : \operatorname{Hom}_A(B, W) \to \operatorname{Hom}_k(H, W)$ that is defined by

$$f(\phi)(h) = (\phi h)(1) = \sum \gamma^{-1}(h_1)\phi(\gamma(h_2))$$

for $\phi \in \operatorname{Hom}_A(B, W)$ and $h \in H$. Then $f$ is right $H$-linear. Define a map $g : \operatorname{Hom}_k(H, W) \to \operatorname{Hom}_A(B, W)$ by

$$g(\psi)(\gamma(h)) = \sum \gamma(h_1)\psi(h_2)$$

for $\psi \in \operatorname{Hom}_k(H, W)$ and $h \in H$. Note that $g(\psi)$ is well defined because $B \cong A \otimes_k \gamma(H)$ as left $A$-modules. One readily checks that $f$ and $g$ are inverse to each other, whence the second isomorphism follows.

Finally, the last assertion is trivial and so the lemma is proved. □

2.2. **Action of $H$ on tensors.** Let $V_B$ and ${}_BW$ be $B$-modules. For $v \otimes w \in V \otimes_A W$ and $h \in H$ define $h(v \otimes w) \in V \otimes_A W$ by

$$h(v \otimes w) = \sum v\gamma^{-1}(h_1) \otimes \gamma(h_2)w .$$

Using identity (2), one easily checks that this is well defined, i.e., that $h(va \otimes w) = h(v \otimes aw)$ holds for all $v \in V$, $w \in W$, $h \in H$, and $a \in A$.

**Lemma.** *The above definition makes $V \otimes_A W$ a left $H$-module. There is a canonical $k$-linear isomorphism*

$$k \otimes_H (V \otimes_A W)) \cong V \otimes_B W .$$

*Furthermore,*

$$V \otimes_A B \cong H \otimes_k V$$

*as left $H$-modules (where the $H$-action on the right-hand side is via the action on the factor $H$. So $H \otimes_k V \cong H^{(\dim_k V)}$). Finally, if $f : V \to V'$ and $g : W \to W'$ are $B$-module maps, then $g \otimes f : V \otimes_A W \to V' \otimes_A W'$ is an $H$-module map.*

*Proof.* The module properties again follow readily from the identities (1a) and (1b). For the first isomorphism, note that

$$k \otimes_H (V \otimes_A W)) \cong V \otimes_A W / (\operatorname{Ker} \epsilon)(V \otimes_A W) .$$

Now $(\operatorname{Ker} \epsilon)(V \otimes_A W)$ is the $k$-subspace of $V \otimes_A W$ that is generated by the elements of the form $h(v \otimes w) - \epsilon(h) v \otimes w$ for $h \in H$, $v \in V$, $w \in W$. But

$$h(v \otimes w) - \epsilon(h) v \otimes w = \sum [v\gamma^{-1}(h_1) \otimes \gamma(h_2) w - v\gamma^{-1}(h_1)\gamma(h_2) \otimes w] ,$$

and hence $(\operatorname{Ker} \epsilon)(V \otimes_A W)$ equals the $k$-subspace of $V \otimes_A W$ that is generated by the elements of the form $v\gamma(h) \otimes w - v \otimes \gamma(h) w$. Since $B = A\gamma(H)$, this proves the first isomorphism.

For the second isomorphism, define $f : H \otimes_k V \to V \otimes_A B$ by

$$f(h \otimes v) = h(v \otimes 1) = \sum v\gamma^{-1}(h_1) \otimes \gamma(h_2) .$$

Then $f$ is clearly $H$-linear. Furthermore, since $B \cong A \otimes_k \gamma(H)$ as left $A$-modules, we can define $g : V \otimes_A B \to H \otimes_k V$ by

$$g(v \otimes \gamma(h)) = \sum h_2 \otimes v\gamma(h_1) .$$

One easily checks that $f$ and $g$ are inverse to each other, and hence $f$ is an isomorphism.

The last assertion is again clear and so the lemma is proved. □

2.3. **Ext and Tor.** The $H$-actions in Sections 2.1 and 2.2 extend to $H$-actions on Ext and Tor. We explain this for Ext, the case of Tor being entirely analogous. So let ${}_B V$ and ${}_B W$ be left $B$-modules and let

$$\mathbf{P} : \cdots \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} 0$$

be a projective resolution of $V$, so $H_n(\mathbf{P}) = 0$ for $n \neq 0$ and $H_0(\mathbf{P}) \cong V$. Since $B$ is projective (in fact, free) as a left $A$-module, the restriction of $\mathbf{P}$ to $A$ is a projective resolution of ${}_A V$ and so we have $\operatorname{Ext}^*_A(V, W) \cong H^*(\operatorname{Hom}_A(\mathbf{P}, W))$. By Section 2.1, the components of the complex $\operatorname{Hom}_A(\mathbf{P}, W)$ are right $H$-modules and the differential $(f_n^*)_n$ is $H$-linear. Thus the cohomology $H^*(\operatorname{Hom}_A(\mathbf{P}, W))$ is a right $H$-module and hence so is $\operatorname{Ext}^*_A(V, W)$.

**Proposition.** (a) *Let ${}_B V$ and ${}_B W$ be left $B$-modules. Then there is a third quadrant spectral sequence*

$$E_2^{p,q} = \operatorname{Ext}^p_H(k, \operatorname{Ext}^q_A(V, W)) \underset{p}{\Longrightarrow} \operatorname{Ext}^n_B(V, W) .$$

(b) *Let $V_B$ and ${}_B W$ be $B$-modules. Then there is a first quadrant spectral sequence*

$$E^2_{p,q} = \operatorname{Tor}^H_p(k, \operatorname{Tor}^A_q(V, W)) \underset{p}{\Longrightarrow} \operatorname{Tor}^B_n(V, W) .$$

*Proof.* Both spectral sequences can be obtained as applications of the Grothendieck spectral sequence (cf. [Rot], Chapter 11). We let ${}_B\mathfrak{M}$ denote the category of left $B$-modules and similarly for the other algebras under consideration and for right modules.

(a) Let ${}_BW$ be a given left $B$-module. Define functors

$$G: {}_B\mathfrak{M} \to \mathfrak{M}_H, \quad G(V) = \mathrm{Hom}_A(V, W)$$

and

$$F: \mathfrak{M}_H \to \mathfrak{M}_k, \quad F(X) = \mathrm{Hom}_H(k, X).$$

By Lemma 2.1, $FG$ is equivalent with the functor $\mathrm{Hom}_B(\,.\,, W)$ and so the right derived functors $R^n(FG)$ are equivalent with $\mathrm{Ext}^n_B(\,.\,, W)$. Moreover, if $P \in {}_B\mathfrak{M}$ is projective, then $(R^nF)(G(P)) = \mathrm{Ext}^n_H(k, G(P)) = 0$ for all $n > 0$ and so $G(P)$ is right $F$-acyclic. Indeed, it suffices to check this equality for $P = B$. In this case, Lemma 2.1 and [Rot], Theorem 11.56, together imply that

$$\begin{aligned}\mathrm{Ext}^n_H(k, G(B)) &= \mathrm{Ext}^n_H(k, \mathrm{Hom}_k(H, W)) \\ &\cong \mathrm{Ext}^n_k(k \otimes_H H, W) \\ &= \mathrm{Ext}^n_k(k, W) \\ &= 0 \qquad (n > 0).\end{aligned}$$

The required spectral sequence now follows from [Rot], Theorem 11.38.

(b) Let $V_B$ be a given right $B$-module. Define functors

$$G: {}_B\mathfrak{M} \to {}_H\mathfrak{M}, \quad G(W) = V \otimes_A W$$

and

$$F: {}_H\mathfrak{M} \to {}_k\mathfrak{M}, \quad F(X) = k \otimes_H X.$$

By Lemma 2.2, $FG$ is equivalent with the functor $V \otimes_B (\,.\,)$ and so the left derived functors $L_n(FG)$ are equivalent with $\mathrm{Tor}^B_n(V, \,.\,)$. Furthermore, Lemma 2.2 implies that $G$ maps projective $B$-modules to projective $H$-modules. Since projective $H$-modules are left $F$-acyclic, the required spectral sequence follows from [Rot], Theorem 11.39. □

2.4. **Homological dimension.** The above proposition directly implies the following estimates for the flat dimension and the projective dimension of modules, denoted fdim and pdim, respectively.

**Corollary.** (a) *Let* ${}_BV$ *be a* $B$*-module. Then* $\mathrm{pdim}\,{}_BV \le \mathrm{pdim}\,k_H + \mathrm{pdim}\,{}_AV$. *Consequently,* $\mathrm{l.gldim}\,B \le \mathrm{r.gldim}\,H + \mathrm{l.gldim}\,A$. *In particular, if* $A$ *and* $H$ *are both semisimple* (gldim 0), *then so is* $B$ (*cf.* [Mont], *Theorem* 7.4.2).

(b) *Let* $V_B$ *be a* $B$*-module. Then* $\mathrm{fdim}\,V_B \le \mathrm{fdim}\,k_H + \mathrm{fdim}\,V_A$. *Therefore,* $\mathrm{wdim}\,B \le \mathrm{wdim}\,H + \mathrm{wdim}\,A$. *In particular, if* $A$ *and* $H$ *are both von Neumann regular* (wdim 0), *then so is* $B$.

We note that

$$\mathrm{r.gldim}\,H = \mathrm{pdim}\,k_H \qquad \text{and} \qquad \mathrm{wdim}\,H = \mathrm{fdim}\,k_H.$$

For, if $\mathbf{P}$ is a projective resolution of $k_H$, then, for any right $H$-module $X$, $X \otimes_k \mathbf{P}$ is a resolution of $X \otimes_k k \cong X$ which consists of projective $H$-modules. To see the latter, note that the Fundamental Theorem of Hopf modules ([Mont], Theorem 1.9.4) implies that $X \otimes_k H$ is a free $H$-module, and hence $X \otimes_k P$ is projective for any projective $H$-module $P$. Thus $\mathrm{pdim}\,X_H \le \mathrm{pdim}\,k_H$ which proves the first equality. For the second equality, consider a flat resolution $\mathbf{P}$ of $k_H$ and use the fact that flat modules are direct limits of free modules to obtain that $X \otimes_k \mathbf{P}$ is a flat resolution of $X$.

2.5. **Relative projectivity.** Recall that if $R \subseteq S$ is a pair of rings, then $S$ is called *projective relative to* $R$ (or $R$*-projective*) if the following holds: Given $S$-modules ${}_S W \subseteq {}_S V$ so that $W$ is a direct summand of $V$ as $R$-modules, then $W$ is a direct summand of $V$ as $S$-modules. The following result is identical with [Mont], Theorem 7.4.2(1), but the proof below is a nice application of the techniques of Section 2.1.

**Corollary.** *If $H$ is semisimple, then $B$ is projective relative to $A$.*

*Proof.* First note that if $f : X \to Y$ is an epimorphism in $\mathfrak{M}_H$ then $f(X^H) = Y^H$, because $f$ splits by assumption on $H$. Now let ${}_B W \subseteq {}_B V$ so that $W$ is a direct summand of $V$ as $A$-modules. Then the canonical epimorphism $\pi : V \to V/W$ splits in ${}_A\mathfrak{M}$ and so the map $\pi_* : \mathrm{Hom}_A(V/W, V) \to \mathrm{End}_A(V/W)$ is surjective. Since $\pi_*$ is a map in $\mathfrak{M}_H$, we deduce from the foregoing that $\pi_*(\mathrm{Hom}_A(V/W, V)^H) = \mathrm{End}_A(V/W)^H$. By Section 2.1, $\mathrm{Hom}_A(V/W, V)^H = \mathrm{Hom}_B(V/W, V)$ and $\mathrm{End}_A(V/W)^H = \mathrm{End}_B(V/W)$. Thus there exists $\mu \in \mathrm{Hom}_B(V/W, V)$ with $\pi_*(\mu) = \pi \circ \mu = \mathrm{Id}_{V/W}$ and so $W$ is a direct summand of $V$ as $B$-modules, as required. □

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