ON CROSSED PRODUCTS OF HOPF ALGEBRAS

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Abstract. Let $B = A \#_a H$ denote a crossed product of the associative algebra $A$ with the Hopf algebra $H$. We investigate the weak dimension and the global dimension of $B$ and show that $\text{wdim } B \leq \text{wdim } H + \text{wdim } A$ and $\text{l.gldim } B \leq \text{r.gldim } H + \text{l.gldim } A$.

1. Introduction

Let $B = A \#_a H$ denote a crossed product of the associative algebra $A$ with the Hopf algebra $H$. We establish the following estimates for the weak dimension and the global dimension of $B$ in terms of the corresponding data for $H$ and $A$:

$$\text{wdim } B \leq \text{wdim } H + \text{wdim } A \quad \text{and} \quad \text{l.gldim } B \leq \text{r.gldim } H + \text{l.gldim } A.$$ 

The first of these estimates is a consequence of a suitable spectral sequence

$$E^2_{p,q} = \text{Tor}^H_p(k, \text{Tor}^A_q(V, W)) \Rightarrow \text{Tor}^B_p(V, W),$$

where $k$ is the trivial $H$-module (i.e., $H$ acts via the counit) and $V_B$ and $bW$ are arbitrary $B$-modules. This spectral sequence will be constructed in Section 2.3 along with an analogous spectral sequence for Ext which yields the estimate for global dimension. Since a ring is von Neumann regular precisely if its weak dimension is 0, we conclude in particular that if $H$ and $A$ are both von Neumann regular, then $B$ is likewise. Specializing to the case of global dimension 0, we also deduce the known fact that if $H$ and $A$ are both semisimple, then so is $B$ (cf. [Mont], Theorem 7.4.2). Finally, we briefly discuss relative projectivity of $B$ with respect to $A$.

Notation and basic facts. Our reference for general material about Hopf algebras are the standard texts [Abe] and [Sw]. For crossed products in particular we follow the notes [Mont]. Throughout this article, we will keep the following notation:

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$k$ denotes a commutative field; 
$H$ will be a Hopf algebra over $k$, with counit $\epsilon$; the $H$-module $k$ will always be the trivial $H$-module; 
$A$ denotes an associative $k$-algebra with identity $1$ so that there is a weak $H$-action on $A$, denoted $(h, a) \mapsto h \cdot a$ ($h \in H, a \in A$); 
$B = A \#_{\sigma} H$ will denote a crossed product, with cocycle $\sigma : H \otimes_k H \to A$.

Thus $B$ is an associative algebra such that there is an isomorphism of left $A$-modules 

$$A \otimes_k H \cong B, \quad a \otimes h \mapsto a \# h.$$ 

The map $a \mapsto a \# 1$ identifies $A$ with a subalgebra of $B$. Defining a $k$-linear map $\gamma : H \to B$ by 

$$\gamma(h) = 1 \# h \quad (h \in H),$$ 

we have $a \gamma(h) = a \# h$ for $a \in A, h \in H$. It is known (cf. [Mont], Chapter 7) that $\gamma$ is convolution invertible and satisfies the following identities, for $h, k \in H$ and $a \in A$, 

1. $\sigma(h, k) = \sum \gamma(h_1) \gamma(k_1) \gamma^{-1}(h_2 k_2)$,
2. $\gamma(h) \gamma(k) = \sum \sigma(h_1, k_1) \gamma(h_2 k_2)$,
3. $\gamma^{-1}(h k) = \sum \gamma^{-1}(k_1) \gamma^{-1}(h_1) \sigma(h_2, k_2)$,
4. $\gamma(h) a = \sum (h_1 \cdot a) \gamma(h_2)$.

2. Proofs

2.1. Action of $H$ on homomorphisms. Let $B V$ and $B W$ be left $B$-modules. For each $\phi \in \text{Hom}_A(V, W)$ and $h \in H$ define $\phi h : V \to W$ by 

$$(\phi h)(v) = \sum \gamma^{-1}(h_1) \phi(\gamma(h_2)v) \quad (v \in V).$$

Then we have the following

Lemma. The above definition makes $\text{Hom}_A(V, W)$ a right $H$-module. There is a canonical $k$-linear isomorphism 

$$\text{Hom}_H(k, \text{Hom}_A(V, W)) \cong \text{Hom}_B(V, W).$$

Furthermore, 

$$\text{Hom}_A(B, W) \cong \text{Hom}_k(H, W)$$

as right $H$-modules (where $H$ acts on the right-hand side by $(\psi h)(k) = \psi(hk)$ for $\psi \in \text{Hom}_k(H, W)$ and $h, k \in H$). Finally, if $f : V \to V'$ and $g : W \to W'$ are $B$-module maps, then $g \circ f^* : \text{Hom}_A(V', W) \to \text{Hom}_A(V, W')$ is an $H$-module map.

Proof. The fact that $\phi h : V \to W$ is $A$-linear is proved exactly as in [Mont], proof of Theorem 7.4.2. Furthermore, the map $\text{Hom}_A(V, W) \times H \to \text{Hom}_A(V, W)$, $(\phi, h) \mapsto \phi h$ is clearly $k$-bilinear. Using the identities (1a)
and (1b) we compute, for $h, k \in H$ and $v \in V$,

$$[\phi(hk)](v) = \sum \gamma^{-1}(h_1k_1)\phi(\gamma(h_2k_2)v)$$

$$= \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\sigma(h_2, k_2)\phi(\gamma(h_3k_3)v)$$

$$= \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\phi[\sigma(h_2, k_2)\gamma(h_3k_3)v]$$

$$= \sum \gamma^{-1}(k_1)\gamma^{-1}(h_1)\phi[\gamma(h_2)\gamma(k_2)v]$$

$$= [(\phi h)k](v).$$

Thus Hom$_A(V, W)$ is a right $H$-module.

In order to establish the first isomorphism, we first note that there is a canonical isomorphism of Hom$_H(k, \text{Hom}_A(V, W))$ with the $k$-space of $H$-invariants in Hom$_A(V, W)$, that is, with

$$\text{Hom}_A(V, W)^H = \{ \phi \in \text{Hom}_A(V, W) | \phi h = \epsilon(h)\phi \text{ for all } h \in H \}.$$

Thus it suffices to show that Hom$_A(V, W)^H = \text{Hom}_B(V, W)$. Let $\phi \in \text{Hom}_A(V, W)$, $h \in H$ and $v \in V$. Then

$$(\phi h)(v) = \epsilon(h)\phi(v) \leftrightarrow \sum \gamma^{-1}(h_1)\phi(\gamma(h_2)v) = \sum \gamma^{-1}(h_1)\gamma(h_2)\phi(v)$$

$$\leftrightarrow \phi(\gamma(h)v) = \gamma(h)\phi(v).$$

Since $B = A\gamma(H)$, the last condition is equivalent with $\phi \in \text{Hom}_B(V, W)$. This proves the first isomorphism.

Now consider the map $f : \text{Hom}_A(B, W) \rightarrow \text{Hom}_k(H, W)$ that is defined by

$$f(\phi)(h) = (\phi h)(1) = \sum \gamma^{-1}(h_1)\phi(\gamma(h_2))$$

for $\phi \in \text{Hom}_A(B, W)$ and $h \in H$. Then $f$ is right $H$-linear. Define a map $g : \text{Hom}_k(H, W) \rightarrow \text{Hom}_A(B, W)$ by

$$g(\psi)(\gamma(h)) = \sum \gamma(h_1)\psi(h_2)$$

for $\psi \in \text{Hom}_k(H, W)$ and $h \in H$. Note that $g(\psi)$ is well defined because $B \cong A \otimes_k \gamma(H)$ as left $A$-modules. One readily checks that $f$ and $g$ are inverse to each other, whence the second isomorphism follows.

Finally, the last assertion is trivial and so the lemma is proved. \(\Box\)

2.2. Action of $H$ on tensors. Let $V_B$ and $BW$ be $B$-modules. For $v \otimes w \in V \otimes_A W$ and $h \in H$ define $h(v \otimes w) \in V \otimes_A W$ by

$$h(v \otimes w) = \sum v\gamma^{-1}(h_1) \otimes \gamma(h_2)w.$$ 

Using identity (2), one easily checks that this is well defined, i.e., that $h(va \otimes w) = h(v \otimes aw)$ holds for all $v \in V$, $w \in W$, $h \in H$, and $a \in A$.

**Lemma.** The above definition makes $V \otimes_A W$ a left $H$-module. There is a canonical $k$-linear isomorphism

$$k \otimes_H (V \otimes_A W) \cong V \otimes_B W.$$ 

Furthermore,

$$V \otimes_A B \cong H \otimes_k V.$$
as left $H$-modules (where the $H$-action on the right-hand side is via the action on the factor $H$. So $H \otimes_k V \cong H^{(\dim_k V)}$. Finally, if $f : V \to V'$ and $g : W \to W'$ are $B$-module maps, then $g \otimes f : V \otimes_A W \to V' \otimes_A W'$ is an $H$-module map.

Proof. The module properties again follow readily from the identities (1a) and (1b). For the first isomorphism, note that

$$k \otimes_H (V \otimes_A W) \cong V \otimes_A W / (\text{Ker} \epsilon)(V \otimes_A W).$$

Now $(\text{Ker} \epsilon)(V \otimes_A W)$ is the $k$-subspace of $V \otimes_A W$ that is generated by the elements of the form $h(v \otimes w) - \epsilon(h)v \otimes w$ for $h \in H$, $v \in V$, $w \in W$. But

$$h(v \otimes w) - \epsilon(h)v \otimes w = \sum [v\gamma^{-1}(h_1) \otimes \gamma(h_2)w - v\gamma^{-1}(h_1)\gamma(h_2) \otimes w],$$

and hence $(\text{Ker} \epsilon)(V \otimes_A W)$ equals the $k$-subspace of $V \otimes_A W$ that is generated by the elements of the form $v\gamma(h) \otimes w - v \otimes \gamma(h)w$. Since $B = A\gamma(H)$, this proves the first isomorphism.

For the second isomorphism, define $f : H \otimes_k V \to V \otimes_A B$ by

$$f(h \otimes v) = h(v \otimes 1) = \sum v\gamma^{-1}(h_1) \otimes \gamma(h_2).$$

Then $f$ is clearly $H$-linear. Furthermore, since $B \cong A \otimes k \gamma(H)$ as left $A$-modules, we can define $g : V \otimes_A B \to H \otimes_k V$ by

$$g(v \otimes \gamma(h)) = \sum h_2 \otimes v\gamma(h_1).$$

One easily checks that $f$ and $g$ are inverse to each other, and hence $f$ is an isomorphism.

The last assertion is again clear and so the lemma is proved. □

2.3. Ext and Tor. The $H$-actions in Sections 2.1 and 2.2 extend to $H$-actions on Ext and Tor. We explain this for Ext, the case of Tor being entirely analogous. So let $B V$ and $B W$ be left $B$-modules and let

$$P : \cdots \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} 0$$

be a projective resolution of $V$, so $H_n(P) = 0$ for $n \neq 0$ and $H_0(P) \cong V$. Since $B$ is projective (in fact, free) as a left $A$-module, the restriction of $P$ to $A$ is a projective resolution of $A V$ and so we have $\text{Ext}^*_A(V, W) \cong H^*(\text{Hom}_A(P, W))$. By Section 2.1, the components of the complex $\text{Hom}_A(P, W)$ are right $H$-modules and the differential $(f_n^*)$ is $H$-linear. Thus the cohomology $H^*(\text{Hom}_A(P, W))$ is a right $H$-module and hence so is $\text{Ext}^*_A(V, W)$.

Proposition. (a) Let $B V$ and $B W$ be left $B$-modules. Then there is a third quadrant spectral sequence

$$E_2^{p,q} = \text{Ext}_H^p(k, \text{Ext}_A^q(V, W)) \Rightarrow \text{Ext}_B^p(V, W).$$

(b) Let $B V$ and $B W$ be $B$-modules. Then there is a first quadrant spectral sequence

$$E_2^{p,q} = \text{Tor}_H^p(k, \text{Tor}_A^q(V, W)) \Rightarrow \text{Tor}_B^p(V, W).$$

Proof. Both spectral sequences can be obtained as applications of the Grothendieck spectral sequence (cf. [Rot], Chapter 11). We let $B \mathcal{M}$ denote the category
of left $B$-modules and similarly for the other algebras under consideration and for right modules.

(a) Let $bW$ be a given left $B$-module. Define functors
$$G : \mathcal{M}_B \rightarrow \mathcal{M}_H, \quad G(V) = \text{Hom}_A(V, W)$$
and
$$F : \mathcal{M}_H \rightarrow \mathcal{M}_k, \quad F(X) = \text{Hom}_k(k, X).$$
By Lemma 2.1, $FG$ is equivalent with the functor $\text{Hom}_B(\cdot, W)$ and so the right derived functors $R^n(FG)$ are equivalent with $\text{Ext}^n_B(\cdot, W)$. Moreover, if $P \in \mathcal{M}_B$ is projective, then $(R^nF)(G(P)) = \text{Ext}^n_k(k, G(P)) = 0$ for all $n > 0$ and so $G(P)$ is right $F$-acyclic. Indeed, it suffices to check this equality for $P = B$. In this case, Lemma 2.1 and [Rot], Theorem 11.56, together imply that
$$\text{Ext}^n_k(k, G(B)) = \text{Ext}^n_k(k, \text{Hom}_k(H, W))$$
$$\cong \text{Ext}^n_k(k \otimes_H H, W)$$
$$= \text{Ext}^n_k(k, W)$$
$$= 0 \quad (n > 0).$$

The required spectral sequence now follows from [Rot], Theorem 11.38.

(b) Let $V_B$ be a given right $B$-module. Define functors
$$G : \mathcal{M}_B \rightarrow \mathcal{M}_H, \quad G(W) = V \otimes_A W$$
and
$$F : \mathcal{M}_H \rightarrow \mathcal{M}_k, \quad F(X) = k \otimes_H X.$$
By Lemma 2.2, $FG$ is equivalent with the functor $V \otimes_B \cdot$ and so the left derived functors $L^n(FG)$ are equivalent with $\text{Tor}^B_n(\cdot, V)$. Furthermore, Lemma 2.2 implies that $G$ maps projective $B$-modules to projective $H$-modules. Since projective $H$-modules are left $F$-acyclic, the required spectral sequence follows from [Rot], Theorem 11.39. □

2.4. Homological dimension. The above proposition directly implies the following estimates for the flat dimension and the projective dimension of modules, denoted $\text{fdim}$ and $\text{pdim}$, respectively.

Corollary. (a) Let $bV$ be a $B$-module. Then $\text{pdim}_B bV \leq \text{pdim}_k k_H + \text{pdim}_A V$. Consequently, $\text{l.gldim} B \leq \text{r.gldim} H + \text{l.gldim} A$. In particular, if $A$ and $H$ are both semisimple ($\text{gldim} 0$), then so is $B$ (cf. [Mont], Theorem 7.4.2).

(b) Let $V_B$ be a $B$-module. Then $\text{fdim}_B V_B \leq \text{fdim}_k k_H + \text{fdim}_A V$. Therefore, $\text{wdim} B \leq \text{wdim} H + \text{wdim} A$. In particular, if $A$ and $H$ are both von Neumann regular ($\text{wdim} 0$), then so is $B$.

We note that
$$\text{r.gldim} H = \text{pdim}_k k_H \quad \text{and} \quad \text{wdim} H = \text{fdim}_k k_H.$$
For, if $P$ is a projective resolution of $k_H$, then, for any right $H$-module $X$, $X \otimes_k P$ is a resolution of $X \otimes_k k \cong X$ which consists of projective $H$-modules. To see the latter, note that the Fundamental Theorem of Hopf modules ([Mont], Theorem 1.9.4) implies that $X \otimes_k H$ is a free $H$-module, and hence $X \otimes_k P$ is projective for any projective $H$-module $P$. Thus $\text{pdim}_H X \leq \text{pdim} k_H$ which proves the first equality. For the second equality, consider a flat resolution $P$ of $k_H$ and use the fact that flat modules are direct limits of free modules to obtain that $X \otimes_k P$ is a flat resolution of $X$. 

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2.5. **Relative projectivity.** Recall that if \( R \subseteq S \) is a pair of rings, then \( S \) is called *projective relative to \( R \) (or \( R \)-projective) if the following holds: Given \( S \)-modules \( \underline{S}W \subseteq \underline{S}V \) so that \( W \) is a direct summand of \( V \) as \( R \)-modules, then \( W \) is a direct summand of \( V \) as \( S \)-modules. The following result is identical with [Mont], Theorem 7.4.2(1), but the proof below is a nice application of the techniques of Section 2.1.

**Corollary.** If \( H \) is semisimple, then \( B \) is projective relative to \( A \).

**Proof.** First note that if \( f : X \to Y \) is an epimorphism in \( \mathcal{M}_H \) then \( f(X^H) = Y^H \), because \( f \) splits by assumption on \( H \). Now let \( B W \subseteq B V \) so that \( W \) is a direct summand of \( V \) as \( A \)-modules. Then the canonical epimorphism \( \pi : V \to V/W \) splits in \( \mathcal{M}_H \) and so the map \( \pi_* : \text{Hom}_A(V/W, V) \to \text{End}_A(V/W) \) is surjective. Since \( \pi_* \) is a map in \( \mathcal{M}_H \), we deduce from the foregoing that \( \pi_*(\text{Hom}_A(V/W, V)^H) = \text{End}_A(V/W)^H \). By Section 2.1, \( \text{Hom}_A(V/W, V)^H = \text{Hom}_B(V/W, V) \) and \( \text{End}_A(V/W)^H = \text{End}_B(V/W) \). Thus there exists \( \mu \in \text{Hom}_B(V/W, V) \) with \( \pi_*(\mu) = \pi \circ \mu = \text{Id}_{V/W} \) and so \( W \) is a direct summand of \( V \) as \( B \)-modules, as required. \( \Box \)

**References**


