Contemporary Mathematics
Volume 184, 1995

# Grothendieck Groups and Higher Class Groups of Commutative Invariants

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ABSTRACT. Using techniques from [C-F], [C-F2], we establish some qualitative results on the structure of the Grothendieck group $G_0(R)$ of finitely generated $R$-modules in the case where $R$ is the ring of invariants of the action of a finite group $G$ on a commutative Noetherian ring $S$. We also study the connections between the higher class groups, in the sense of [C-F], of $S$ and $R$.

## Introduction

This article studies the transfer of structure from the Grothendieck group $G_0(S)$ of finitely generated modules over a commutative Noetherian ring $S$ to the Grothendieck group $G_0(R)$, where $R$ is the ring of invariants under the action of a finite group $G$ on $S$. For example, sharpening some results of [Br-L], we show in 2.2 that if $S$ has finite (Krull) dimension $d$ and the trace map $\mathrm{tr} : S \to R$ is surjective, then

- the cokernel of the restriction map $\mathrm{Res}_R^S : G_0(S) \to G_0(R)$ is annihilated by $|G|^{d+1}$, and
- if $S$ is a domain with $G_0(S) = \langle [S] \rangle$, then $G_0(R) = \langle [R] \rangle \oplus F$ where $F$ is annihilated by $|G|^d$.

Surjectivity of the trace map is not essential here as long as $S$ is Noetherian as $R$-module. In this more general setting, one has to replace the powers of $|G|$ in the above by suitable invariants that are defined in terms of the ramification of primes of $R$ in $S$. We also consider the higher class groups $W_i$ of [C-F] for $R$ and $S$ and study their behavior under induction and restriction between $R$ and $S$. We show, for example, that if the extension $R \subseteq S$ is $G$-Galois in codimension $i$ (cf. 2.3 for the definition) then the kernel of the induction map $W_i(R) \to W_i(S)$ can be expressed as the first cohomology group $H^1(G, U_i(S))$ with suitable coefficients $U_i(S)$ (cf. 2.5 Proposition 2).

The first section applies to arbitrary commutative Noetherian rings $R$. Following [C-F] and [C-F2], we study a filtration $\{F_i\}$ of $G_0(R)$ which is defined in terms of a filtration $\{\mathfrak{M}_i\}$ of the category $\mathfrak{mod}$-$R$ of finitely generated $R$-modules. Here $\mathfrak{M}_i$

1991 *Mathematical Subject Classification.* Primary 16W20, 16E20, 16D90; Secondary 19A49, 13A50.

The second author was supported in part by NSF Grant DMS–9005597.

is the full subcategory of $\mathfrak{mod}$-$R$ consisting of those finitely generated $R$-modules $M$ such that $\mathrm{Supp}(M)$ consists of primes of height at least $i$. The $i$th class group $W_i$ of $R$ is defined to be the canonical image of $K_0(\mathfrak{M}_i/\mathfrak{M}_{i+1})$ in $K_0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1})$. If $R$ is an integrally closed domain, then the first class group $W_1$ is the usual class group of $R$. The interest in the higher class groups for studying $G_0(R)$ mainly stems from the following two facts:

(i) $W_i$ canonically maps onto the $i$th slice $F_i/F_{i+1}$ of the above filtration of $G_0(R)$;

(ii) $W_i$ is accessible via an explicit presentation.

The second section then applies these techniques to rings of invariants and derives the aforementioned results.

In a sequel to this article, we will extend some of the results of Section 2 to rings of invariants of finite group actions on FBN rings under the additional hypothesis that the trace map is surjective. Our approach there will be in some sense dual to the present one and will be based on the notion of an exact dimension function for finitely generated modules rather than on codimension or height. This change in strategy is necessitated by the well-known difficulties with noncommutative localization.

## 1. Higher Class Groups and Grothendieck Groups

**1.1 The Categories $\mathfrak{M}_i$.** Let $R$ be a commutative Noetherian ring. (The Noetherian hypothesis, here and throughout this note, can be relaxed, but we will content ourselves with the Noetherian case and refer to [C-F2] for a somewhat more general setting.) Following [C-F], we let $\mathfrak{M}_i = \mathfrak{M}_i(R)$ denote the category of all finitely generated $R$-modules $M$ such that $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of height less than $i$. (Bourbaki's notation for $\mathfrak{M}_i$ is $\mathcal{C}^{\geq i}$; cf. [Bou], Chap. 8 §1 no. 5) In view of [Bou], Chap. 4 §1 no. 4, one has the following four equivalent reformulations of the defining condition for $\mathfrak{M}_i$:

(i) $\mathrm{Supp}(M)$ consists of primes of height at least $i$;

(ii) $\mathrm{Ass}(M)$ consists of primes of height at least $i$;

(iii) $M$ has a series of submodules $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_t = 0$ with $M_j/M_{j+1} \cong R/P_j$ for suitable primes $P_j$ of height at least $i$;

(iv) The annihilator of $M$, $\mathrm{ann}_R M$, has height at least $i$.

Clearly, $\mathfrak{M}_i \supseteq \mathfrak{M}_j$ if $i \leq j$. In fact, $\mathfrak{M}_j$ is a Serre subcategory of $\mathfrak{M}_i$, that is, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules in $\mathfrak{M}_i$, then $M$ belongs to $\mathfrak{M}_j$ if and only if both $M'$ and $M''$ do. Thus one can form the (abelian) quotient categories $\mathfrak{M}_i/\mathfrak{M}_j$ (e.g., [F], Chapter 15). It will be convenient to put $\mathfrak{M}_\infty = \bigcap_i \mathfrak{M}_i = \{0\}$. Thus $\mathfrak{M}_i/\mathfrak{M}_\infty$ is just $\mathfrak{M}_i$.

Examples. $\mathfrak{M}_0$ is the category $\mathfrak{mod}$-$R$ of all finitely generated $R$-modules. In case $R$ is a domain, $\mathfrak{M}_1$ consists of all finitely generated torsion modules and $\mathfrak{M}_2$, for $R$ integrally closed, consists of the pseudo-zero modules of [Bou], Chap. 7 §4 no. 4. If $R$ has finite (Krull) dimension $n$, then $\mathfrak{M}_{n+1} = 0$, and all modules in $\mathfrak{M}_n$ have finite length over $R$. If all maximal ideals of $R$ have the same height $n$, then $\mathfrak{M}_n$ consists of precisely the modules of finite length.

**1.2 Change of Rings: Induction.** Let $R \subseteq S$ be an extension of commutative rings. Consider the following conditions:

$(\mathrm{NBU})_k$ For each prime ideal $\mathfrak{P}$ of $S$ of height at most $k$, one has $\mathrm{ht}(\mathfrak{P} \cap R) \leq \mathrm{ht}(\mathfrak{P})$.

and

$(\mathrm{F})_k$ For each prime ideal $\mathfrak{P}$ of $S$ of height at most $k$, the localization $S_{\mathfrak{P}}$ is flat over $R$.

Here NBU stands for *no blowing up*, as in [S], where $(\mathrm{NBU})_1$ is denoted (NBU). (PDE), for *pas d' éclatement*, is often used in the literature (e.g., [Bou], Chap. 7 §1 no. 10, or [Fo]). When $(\mathrm{F})_k$ is satisfied, the extension $R \subseteq S$ is called *k-flat* in [C-F2]. Lemma 1 below offers equivalent reformulations of $(\mathrm{NBU})_k$ and $(\mathrm{F})_k$ in terms of the module categories $\mathfrak{M}_i$.

REMARKS. (1) $(\mathrm{NBU})_k$ is satisfied, for all $k$ and with $\mathrm{ht}(\mathfrak{P} \cap R) = \mathrm{ht}(\mathfrak{P})$, in case $R \subseteq S$ is an integral extension such that $S$ has no $R$-torsion and $R$ is integrally closed. This follows from the Incomparability and Going Down theorems of Cohen and Seidenberg.

(2) $(\mathrm{NBU})_k$ is also satisfied, for all $k$ and with $\mathrm{ht}(\mathfrak{P} \cap R) = \mathrm{ht}(\mathfrak{P})$, if $R = S^G$ is the ring of invariants in $S$ under the action of a finite group $G$. Indeed, $S$ is integral over $R$ in this case and all primes of $S$ lying over a given prime of $R$ are conjugate under $G$, and hence are of the same height (cf. [Bou], Chap. 5). The above equality follows from Incomparability and Going Up.

(3) By [Bou], Chap. 2 §3 no. 4, Prop. 15, $(\mathrm{F})_k$ holds for all $k$ (or, equivalently, for $k = \dim S$ if the latter is finite) if and only if $S$ is flat over $R$.

(4) $(\mathrm{F})_k$ implies $(\mathrm{NBU})_k$ ([C-F2], Theorem 7.9). To see this, let $\mathfrak{P}$ be a prime ideal of $S$ of height at most $k$ and put $\mathfrak{p} = \mathfrak{P} \cap R$. By $(\mathrm{F})_k$, we know that $S_{\mathfrak{P}}$ is flat over $R_{\mathfrak{p}}$ (cf. [Bou], Chap. 2 §3 no. 4, Remarque). Therefore, $S_{\mathfrak{P}}$ is faithfully flat over $R_{\mathfrak{p}}$, and the inequality $\mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{P})$ follows from [Bou], Chap. 2 §2 no. 5, Cor. 4.

(5) By the first two remarks, the converse of (4) need not hold. However, in case $R$ and $S$ are Krull domains, $(\mathrm{NBU})_1$ and $(\mathrm{F})_1$ are equivalent (cf. [Fo], Theorem 6.2).

Much of the following lemma is contained in [C-F2], Proposition 7.10.

LEMMA 1. *Let $R \subseteq S$ be an extension of commutative Noetherian rings.*

(a) $(\mathrm{NBU})_k$ *holds if and only if* $(\,.\,) \otimes_R S$ *yields a functor* $\mathfrak{M}_i(R) \to \mathfrak{M}_i(S)$ *for all* $i \leq k+1$.

(b) $(\mathrm{F})_k$ *is satisfied if and only if* $\mathrm{Tor}_1^R(\,.\,, S)$ *maps* $\mathfrak{M}_0(R) = \mathfrak{mod}\text{-}R$ *to* $\mathfrak{M}_{k+1}(R)$. *In this case,* $(\,.\,) \otimes_R S$ *yields an exact functor* $\mathfrak{M}_i(R)/\mathfrak{M}_j(R) \to \mathfrak{M}_i(S)/\mathfrak{M}_j(S)$ *for all* $i \leq j \leq k+1$.

PROOF. (a) By assuming $(\mathrm{NBU})_k$, the assertion about $(\,.\,) \otimes_R S$ follows from the equality $\mathrm{Supp}(M \otimes_R S) = \{\mathfrak{P} \in \mathrm{Spec}(S) \mid \mathfrak{P} \cap R \in \mathrm{Supp}(M)\}$ for every finitely generated $R$-module $M$ (cf. [Bou], Chap. 2 §4 no. 4, Prop. 19). Conversely, let $\mathfrak{P}$ be a prime ideal of $S$ of height $i \leq k$ and put $\mathfrak{p} = \mathfrak{P} \cap R$. If $\mathfrak{p}$ has height $> i$, then $R/\mathfrak{p} \in \mathfrak{M}_{i+1}(R)$ and so, by assumption, $R/\mathfrak{p} \otimes_R S = S/\mathfrak{p}S \in \mathfrak{M}_{i+1}(S)$. Therefore, $(S/\mathfrak{p}S)_{\mathfrak{P}} = 0$, which contradicts $\mathfrak{p}S \subseteq \mathfrak{P}$.

(b) First, $(\mathrm{F})_k$ is equivalent with $\mathrm{Tor}_1^R(M, S_{\mathfrak{P}}) = \mathrm{Tor}_1^R(M,S)_{\mathfrak{P}} = 0$ for all $R$-modules $M$ and all prime ideals $\mathfrak{P}$ of $S$ having height at most $k$. Since Tor commutes with directed direct limits, one can restrict to finitely generated modules $M$ here. This proves the equivalent reformulation of $(\mathrm{F})_k$.

Next, note that $(\mathrm{F})_k$ entails $(\mathrm{NBU})_k$ by Remark 4 above. Thus, composing the functor $(\,.\,) \otimes_R S$ from (a) with the canonical functor $\mathfrak{M}_i(S) \to \mathfrak{M}_i(S)/\mathfrak{M}_j(S)$ we obtain a functor $T_{i,j} : \mathfrak{M}_i(R) \to \mathfrak{M}_i(S)/\mathfrak{M}_j(S)$ which has $\mathfrak{M}_j(R)$ in its kernel, by (a). Thus, $(\,.\,) \otimes_R S$ passes down to a functor $\mathfrak{M}_i(R)/\mathfrak{M}_j(R) \to \mathfrak{M}_i(S)/\mathfrak{M}_j(S)$, and the latter functor is exact if and only if $T_{i,j}$ is exact (cf. [F], Cor. 15.9 and Cor. 15.10). Since $T_{i,j}$ is a composite of a right exact functor and an exact functor; it suffices to show that, for every monomorphism $f : M' \hookrightarrow M$ in $\mathfrak{M}_i(R)$, the morphism $T_{i,j}f$ is mono in $\mathfrak{M}_i(S)/\mathfrak{M}_j(S)$. By [F], Lemma 15.5, the latter condition is equivalent with $\mathrm{Ker}(f \otimes_R \mathrm{Id}_S) \in \mathfrak{M}_j(S)$. But the latter module is a factor of $\mathrm{Tor}_1^R(M/M', S) \in \mathfrak{M}_{k+1}(S) \subseteq \mathfrak{M}_j(S)$, and hence it also belongs to $\mathfrak{M}_j(S)$. This completes the proof.

**1.3 Change of Rings: Restriction.** Similar remarks can be made on restriction. In fact, the procedure is easier than for induction. Assume that $R \subseteq S$ is an extension of commutative Noetherian rings such that

$$S \text{ is finitely generated as } R\text{-module.}$$

In this case, restriction yields an exact functor $\mathfrak{mod}\text{-}S \to \mathfrak{mod}\text{-}R$. Moreover, since the extension $R \subseteq S$ satisfies Incomparability, the inequality $\mathrm{ht}(\mathfrak{P} \cap R) \geq \mathrm{ht}(\mathfrak{P})$ holds for each prime ideal $\mathfrak{P}$ of $S$, and so restriction maps each $\mathfrak{M}_i(S)$ to the corresponding $\mathfrak{M}_i(R)$. Therefore, restriction yields exact functors $\mathfrak{M}_i(S)/\mathfrak{M}_j(S) \to \mathfrak{M}_i(R)/\mathfrak{M}_j(R)$ for all $i \leq j$.

**1.4 Higher Class Groups and Grothendieck groups.** Following [C-F], we now consider the Grothendieck groups $K_0(\mathfrak{M}_i/\mathfrak{M}_j)$ for $i \leq j$ $(\leq \infty)$ for a commutative Noetherian ring $R$. If $h \leq i \leq j$, then $\mathfrak{M}_i/\mathfrak{M}_j$ is a Serre subcategory of $\mathfrak{M}_h/\mathfrak{M}_j$ with quotient category equivalent to $\mathfrak{M}_h/\mathfrak{M}_i$ (cf. [C-F2], Lemma 7.7). Thus, we have a localization sequence of (Quillen) K-groups

$$(1) \quad \ldots \to K_1(\mathfrak{M}_h/\mathfrak{M}_i) \to K_0(\mathfrak{M}_i/\mathfrak{M}_j) \xrightarrow{\phi_{h,i,j}} K_0(\mathfrak{M}_h/\mathfrak{M}_j) \to K_0(\mathfrak{M}_h/\mathfrak{M}_i) \to 0.$$

Putting

$$F_i = F_i(R) = \phi_{0,i,\infty}(K_0(\mathfrak{M}_i)) \subseteq K_0(\mathfrak{M}_0) = G_0(R) \quad (i \geq 0),$$

we obtain a filtration

$$(2) \qquad G_0(R) = F_0 \supseteq F_1 \supseteq \ldots \supseteq F_i \supseteq F_{i+1} \supseteq \ldots$$

of $G_0(R)$. Note that $F_i/F_{i+1} \subseteq G_0(R)/F_{i+1} \cong K_0(\mathfrak{M}_0/\mathfrak{M}_{i+1})$ can be identified with $\phi_{0,i,i+1}(K_0(\mathfrak{M}_i/\mathfrak{M}_{i+1})) \subseteq K_0(\mathfrak{M}_0/\mathfrak{M}_{i+1})$, and $\phi_{0,i,i+1}$ factors through all $\phi_{h,i,i+1}$ with $h < i$. Following [C-F], we put

$$W_i = W_i(R) = \phi_i(K_0(\mathfrak{M}_i/\mathfrak{M}_{i+1})) \subseteq K_0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1}) \quad (i \geq 1),$$

where $\phi_i = \phi_{i-1,i,i+1}$. Thus, we have canonical epimorphisms

$$\pi_i : W_i \twoheadrightarrow \phi_{0,i,i+1}(K_0(\mathfrak{M}_i/\mathfrak{M}_{i+1})) \cong F_i/F_{i+1}, \tag{3}$$

and $\pi_1$ is in fact an isomorphism. The groups $W_i$ were introduced in [C-F] as possible candidates for higher class groups of $R$. Indeed, if $R$ is an integrally closed Noetherian domain, then $W_1(R)$ is the usual class group $C(R)$ of $R$ (cf. [C-F]). The following presentation of $W_i$, which is an immediate consequence of (1), can also be found in [C-F]:

$$K_1(\mathfrak{M}_{i-1}/\mathfrak{M}_i) \xrightarrow{\delta_i} K_0(\mathfrak{M}_i/\mathfrak{M}_{i+1}) \xrightarrow{\phi_i} W_i \to 0. \tag{4}$$

**1.5 Automorphisms.** The automorphisms of $R$ operate on the categories $\mathfrak{M}_i$ and $\mathfrak{M}_i/\mathfrak{M}_j$ and on their K-groups by conjugating modules: If $g \in \mathrm{Aut}(R)$ and an $R$-module $M$ are given, then the $g$-conjugate of $M$, denoted ${}^gM$, is isomorphic to $M$ as abelian group, say via $m \mapsto {}^gm$, but the $R$-operation is given by ${}^gm \cdot r = {}^g(mg^{-1}(r))$. This leads to exact functors on the above categories, and hence to endomorphisms of their K-groups. The maps in localization sequence (1) all commute with the action of Aut$(R)$. This follows from the obvious commutative diagram of exact functors

$$\begin{array}{ccc} \mathfrak{M}_i/\mathfrak{M}_j & \xrightarrow{\text{incl.}} & \mathfrak{M}_h/\mathfrak{M}_j \\ {}^g(\,.\,)\big\downarrow & & \big\downarrow{}^g(\,.\,) \\ \mathfrak{M}_i/\mathfrak{M}_j & \xrightarrow[\text{incl.}]{} & \mathfrak{M}_h/\mathfrak{M}_j \end{array}$$

where $g \in \mathrm{Aut}(R)$, because the localization sequence of Quillen is functorial in the pair $(\mathfrak{M}_i/\mathfrak{M}_j, \mathfrak{M}_h/\mathfrak{M}_j)$ (cf. [Q], §5, Theorem 5). In particular, all $F_i(R)$ are stable under the action of Aut$(R)$ on $G_0(R)$.

If $R \subseteq S$ is an extension of commutative rings, then we let

$$\mathrm{Aut}_R(S)$$

denote the group of all automorphisms of $S$ which act trivially on $R$.

**1.6 Induction and Restriction for K-groups.**

LEMMA 2. *Let $R \subseteq S$ be an extension of commutative Noetherian rings so that* $(\mathrm{F})_k$ *is satisfied. Then* $(\,.\,)\otimes_R S$ *yields homomorphisms* $\psi_{i,j} : K_*(\mathfrak{M}_i(R)/\mathfrak{M}_j(R)) \to$

$K_*(\mathfrak{M}_i(S)/\mathfrak{M}_j(S))$ $(i \le j \le k+1)$ *and a commutative diagram of localization sequences (1) for* $h \le i \le j \le k+1$:

$$\begin{array}{ccccccc} \dots \longrightarrow & K_*(\mathfrak{M}_i(R)/\mathfrak{M}_j(R)) & \xrightarrow{\phi_{h,i,j}(R)} & K_*(\mathfrak{M}_h(R)/\mathfrak{M}_j(R)) & \longrightarrow \dots \\ & \downarrow{\scriptstyle \psi_{i,j}} & & \downarrow{\scriptstyle \psi_{h,j}} & \\ \dots \longrightarrow & K_*(\mathfrak{M}_i(S)/\mathfrak{M}_j(S)) & \xrightarrow[\phi_{h,i,j}(S)]{} & K_*(\mathfrak{M}_h(S)/\mathfrak{M}_j(S)) & \longrightarrow \dots \end{array}$$

*Furthermore,* $\mathrm{Aut}_R(S)$ *acts trivially on the images of the maps* $\psi_{i,j}$.

Proof. In view of Lemma 1, we have the following commutative diagram of exact functors whenever $h \le i \le j \le k+1$:

$$\begin{array}{ccc} \mathfrak{M}_i(R)/\mathfrak{M}_j(R) & \xrightarrow{\text{incl.}} & \mathfrak{M}_h(R)/\mathfrak{M}_j(R) \\ \downarrow & & \downarrow \\ \mathfrak{M}_i(S)/\mathfrak{M}_j(S) & \xrightarrow[\text{incl.}]{} & \mathfrak{M}_h(S)/\mathfrak{M}_j(S) \end{array}$$

where the vertical functors are afforded by $(\,.\,) \otimes_R S$. The above commutative diagram of localization sequences now follows from functoriality of the localization sequence. Finally, since all $\mathrm{Aut}_R(S)$-conjugates of modules that are induced from $R$ are isomorphic, $\mathrm{Aut}_R(S)$ acts trivially on the images of the maps $\psi_{i,j}$.

The following corollary is essentially identical with [C-F2], Proposition 7.12.

Corollary 1. *If* $R \subseteq S$ *is an extension of commutative Noetherian rings so that* $(\mathrm{F})_k$ *is satisfied, then* $(\,.\,) \otimes_R S$ *yields homomorphisms* $\psi_i : W_i(R) \to W_i(S)$ *for all* $i \le k$ *which fit into a commutative diagram*

$$\begin{array}{ccccccc} K_1(\mathfrak{M}_{i-1}(R)/\mathfrak{M}_i(R)) & \longrightarrow & K_0(\mathfrak{M}_i(R)/\mathfrak{M}_{i+1}(R)) & \xrightarrow{\phi_i(R)} & W_i(R) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Psi'_{i-1}} & & \downarrow{\scriptstyle \Psi_i} & & \downarrow{\scriptstyle \psi_i} & & \\ K_1(\mathfrak{M}_{i-1}(S)/\mathfrak{M}_i(S)) & \longrightarrow & K_0(\mathfrak{M}_i(S)/\mathfrak{M}_{i+1}(S)) & \xrightarrow[\phi_i(S)]{} & W_i(S) & \longrightarrow & 0 \end{array}$$

*Furthermore,* $\mathrm{Aut}_R(S)$ *acts trivially on the images of the maps* $\psi_i$.

Proof. By Lemma 2, $\psi_{h,j}$ maps $\mathrm{Im}(\phi_{h,i,j}(R))$ to $\mathrm{Im}(\phi_{h,i,j}(S))$. To prove the corollary, take $h = i-1$ and $j = i+1$ and define $\Psi_i = \psi_{i,i+1}$, $\Psi'_{i-1} = \psi_{i-1,i}$, and $\psi_i = \psi_{i-1,i+1}$ restricted to $W_i(R)$.

Adapting the notation of [W], the kernel of $\psi_i$ will be denoted by $\mathcal{P}_i(R,S)$. Thus, we have an exact sequence (assuming $(\mathrm{F})_i$ holds, so that $\psi_i$ is defined)

$$(5) \qquad 0 \to \mathcal{P}_i(R,S) \to W_i(R) \xrightarrow{\psi_i} W_i(S).$$

We now briefly turn to restriction. Using (1.3) instead of Lemma 1 in the above, one obtains the following lemma.

LEMMA 3. *Let $R \subseteq S$ be an extension of commutative Noetherian rings so that $S$ is finitely generated as $R$-module. Then restriction yields homomorphisms $\rho_{i,j} : K_*(\mathfrak{M}_i(S)/\mathfrak{M}_j(S)) \to K_*(\mathfrak{M}_i(R)/\mathfrak{M}_j(R))$ $(i \leq j)$ which factor through the coinvariants of the action of $\mathrm{Aut}_R S$ on $K_*(\mathfrak{M}_i(S)/\mathfrak{M}_j(S))$. Furthermore, there is a commutative diagram of localization sequences (1) for $h \leq i \leq j$:*

$$\begin{array}{ccccccc}
\ldots \longrightarrow & K_*(\mathfrak{M}_i(S)/\mathfrak{M}_j(S)) & \xrightarrow{\phi_{h,i,j}(S)} & K_*(\mathfrak{M}_h(S)/\mathfrak{M}_j(S)) & \longrightarrow \ldots \\
 & \downarrow \rho_{i,j} & & \downarrow \rho_{h,j} & \\
\ldots \longrightarrow & K_*(\mathfrak{M}_i(R)/\mathfrak{M}_j(R)) & \xrightarrow[\phi_{h,i,j}(R)]{} & K_*(\mathfrak{M}_h(R)/\mathfrak{M}_j(R)) & \longrightarrow \ldots
\end{array}$$

*In particular, restriction yields homomorphisms $\rho_i : W_i(S) \to W_i(R)$ for all $i$ which fit into a commutative diagram*

$$\begin{array}{ccccccc}
K_1(\mathfrak{M}_{i-1}(S)/\mathfrak{M}_i(S)) & \longrightarrow & K_0(\mathfrak{M}_i(S)/\mathfrak{M}_{i+1}(S)) & \xrightarrow{\phi_i(S)} & W_i(S) & \longrightarrow & 0 \\
\downarrow \varrho'_{i-1} & & \downarrow \varrho_i & & \downarrow \rho_i & & \\
K_1(\mathfrak{M}_{i-1}(R)/\mathfrak{M}_i(R)) & \longrightarrow & K_0(\mathfrak{M}_i(R)/\mathfrak{M}_{i+1}(R)) & \xrightarrow[\phi_i(R)]{} & W_i(R) & \longrightarrow & 0
\end{array}$$

The map $\rho_{0,\infty}$ above is of course the usual restriction map $\rho = \mathrm{Res}^S_R : G_0(S) \to G_0(R)$.

**1.7 Explicit Presentation of $W_i$.** The material of this section is based on [C-F], Section 2. All rings considered here are assumed to be commutative Noetherian.

By [C-F], Proposition 2.4 (or [Bou], Chap. 8 §1 no. 5, Prop. 10), the group $K_0(\mathfrak{M}_i(R)/\mathfrak{M}_{i+1}(R))$ is isomorphic to the free abelian group $D_i(R)$ with the basis set $X^{(i)}(R)$ of primes of height $i$ of $R$. Letting $\langle\mathfrak{p}\rangle$ denote the basis element of $D_i(R)$ corresponding to the prime $\mathfrak{p} \in X^{(i)}(R)$ and $[M]_i$, the class of $M \in \mathfrak{M}_i$ in $K_0(\mathfrak{M}_i(R)/\mathfrak{M}_{i+1}(R))$; an explicit isomorphism is given by

$$(6) \qquad \begin{array}{ccc}
K_0(\mathfrak{M}_i(R)/\mathfrak{M}_{i+1}(R)) & \xrightarrow{\cong} & D_i(R) \\
{[M]_i} & \longmapsto & \sum_{\mathfrak{p} \in X^{(i)}(R)} \ell_{R_\mathfrak{p}}(M_\mathfrak{p})\langle\mathfrak{p}\rangle,
\end{array}$$

where $\ell(\,.\,)$ denotes composition length. In particular, by (1),

$$G_0(R)/F_1(R) \cong D_0(R) \cong G_0(\mathrm{Fract}(R/N)),$$

where $N$ is the nilpotent radical of $R$ and $F_1(R)$ is the kernel of the localization epimorphism $G_0(R) \xrightarrow{\cong} G_0(R/N) \twoheadrightarrow G_0(\mathrm{Fract}(R/N))$. Thus, since $D_0(R)$ is free, we obtain the following decomposition of $G_0(R)$:

$$(7) \qquad G_0(R) = \bigoplus_{\mathfrak{p} \in X^{(0)}(R)} \mathbb{Z}[R/\mathfrak{p}] \;\oplus\; F_1(R).$$

Since $(R/\mathfrak{p} \otimes_R S)_{\mathfrak{P}} \cong S_{\mathfrak{P}}/\mathfrak{p}S_{\mathfrak{P}}$, the map $\Psi_i$ of Corollary 1 now becomes

$$(8) \qquad \begin{aligned} \Psi_i : D_i(R) &\longrightarrow D_i(S) \\ \langle \mathfrak{p} \rangle &\longmapsto \sum_{\mathfrak{P} \in X^{(i)}(S)} \ell_{S_{\mathfrak{P}}}(S_{\mathfrak{P}}/\mathfrak{p}S_{\mathfrak{P}})\langle \mathfrak{P} \rangle. \end{aligned}$$

Note that the sum on the right is supported by those $\mathfrak{P} \in X^{(i)}(S)$ with $\mathfrak{P} \cap R = \mathfrak{p}$. Therefore, $\Psi_i$ is injective. $\Psi_1$ and $\psi_1$, for Krull domains $R$ and $S$, are the maps considered in [Bou], Chap. 7 §1 no. 10, Prop. 14, for example.

Furthermore, by [C-F], Prop. 2.5, one has an isomorphism

$$(9) \qquad K_1(\mathfrak{M}_{i-1}(R)/\mathfrak{M}_i(R)) \cong Q_i(R) := \bigoplus_{\mathfrak{p} \in X^{(i-1)}(R)} Q(R/\mathfrak{p})^*.$$

Thus, presentation (4) of $W_i(R)$ now reads as follows:

$$(10) \quad Q_i(R) = \bigoplus_{\mathfrak{p} \in X^{(i-1)}(R)} Q(R/\mathfrak{p})^* \xrightarrow{\delta_i} D_i(R) = \bigoplus_{\mathfrak{q} \in X^{(i)}(R)} \mathbb{Z}\langle \mathfrak{q} \rangle \xrightarrow{\phi_i} W_i(R) \to 0.$$

Here, $\delta_i$ has the following explicit description:

$$(11) \quad \delta_i((\overline{x_\mathfrak{p}}/\overline{y_\mathfrak{p}})_\mathfrak{p}) = \sum_{\mathfrak{p}} \sum_{\mathfrak{q} \in X^{(i)}(R)} (\ell_{R_\mathfrak{q}}((R/\mathfrak{p} + y_\mathfrak{p}R)_\mathfrak{q}) - \ell_{R_\mathfrak{q}}((R/\mathfrak{p} + x_\mathfrak{p}R)_\mathfrak{q}))\langle \mathfrak{q} \rangle \ ,$$

where $\overline{(\,.\,)}$ denotes residue classes of elements of $R$ mod $\mathfrak{p}$. The map $\Psi'_{i-1} : Q_i(R) \to Q_i(S)$ of Corollary 1 is given by the diagonal embeddings

$$(12) \qquad Q(R/\mathfrak{p})^* \hookrightarrow \bigoplus_{\mathfrak{P} \cap R = \mathfrak{p}} Q(S/\mathfrak{P})^* \qquad (\mathfrak{p} \in X^{(i-1)}(R)).$$

We put

$$U_i = U_i(R) = \operatorname{Ker} \delta_i.$$

STANDARD EXAMPLE. Suppose that $R$ is an integrally closed domain with field of fractions $K$. Then $Q_1(R) = K^*$, $D_1(R) = D(R)$ is the group of divisors of $R$, $U_1(R) = U(R)$ is the group of units of $R$, and (10) becomes the familiar sequence

$$(13) \qquad 0 \to U(R) \to K^* \xrightarrow{\text{div}} D(R) \to C(R) \to 0$$

(e.g., [Fo], p. 27).

## 2. Rings of Invariants

**2.1 Notations.** The following notations will be kept fixed throughout this section.

$S$ will be a commutative ring,

$G$ will be a finite group acting by automorphisms on $S$,

$R = S^G$ will denote the ring of $G$-invariants in $S$, and
$T = S * G$ will be the skew group ring of $G$ over $S$.
Multiplication in $T$ is based on the rule $gs = g(s)g$ for $s \in S, g \in G$. Furthermore, we will use the element

$$t = \sum_{g \in G} g \in T$$

and the following maps:
$f : S \otimes_R S \to T$ is the $S$-$S$-bimodule map that is defined by $f(s \otimes s') = sts'$,
$j : T \to \mathrm{End}_R(S)$ is the $R$-algebra map that is defined by $j(sg)(s') = sg(s')$, and
$\mathrm{tr} : S \to R$, $\mathrm{tr}(s) = \sum_{g \in G} g(s)$ is the trace map.
As usual, the decomposition group (stabilizer in $G$) of a prime ideal $\mathfrak{P}$ of $S$ is denoted by $G^Z(\mathfrak{P})$ and the inertia group, by $G^T(\mathfrak{P})$. For each prime $\mathfrak{p}$ of $R$, we put

$$f_\mathfrak{p} = [Q(S/\mathfrak{P}) : Q(R/\mathfrak{p})] \quad \text{and} \quad e_\mathfrak{p} = \ell_{S_\mathfrak{P}}(S_\mathfrak{P}/\mathfrak{p}S_\mathfrak{P}),$$

where $\mathfrak{P}$ is an arbitrary prime ideal of $S$ over $\mathfrak{p}$. (They are all conjugate under $G$.) By [Bou], Chap. 5 §2 no. 2, Théorème 2 and Cor. to Prop. 5, $f_\mathfrak{p} \geq |G^Z(\mathfrak{P})/G^T(\mathfrak{P})|$, and equality holds if the order of $G^T(\mathfrak{P})$ is nonzero in $R/\mathfrak{p}$ (e.g., if the trace map is surjective; cf. [Br-L], §2.1).

**2.2 The Cokernel of Restriction.** In this section, we assume that $S$ is Noetherian as $R$-module. This happens, for example, if $S$ is an affine algebra over some commutative Noetherian subring of $R$ ([Bou], Chap. 5 §1 no. 9, Théorème 2) or if $S$ is Noetherian and the trace map is surjective ([Br-L], §1.3). In this case, Lemma 3 yields restriction maps $\overline{\varrho_i}$: $K_0(\mathfrak{M}_i(S)/\mathfrak{M}_{i+1}(S))_G \to K_0(\mathfrak{M}_i(R)/\mathfrak{M}_{i+1}(R))$, where $K_0(\mathfrak{M}_i(S)/\mathfrak{M}_{i+1}(S))_G$ denotes the $G$-coinvariants in $K_0(\mathfrak{M}_i(S)/\mathfrak{M}_{i+1}(S))$. Identifying $K_0(\mathfrak{M}_i/\mathfrak{M}_{i+1})$ with $D_i$ by means of (6) and denoting the $G$-coinvariants in $D_i(S)$ by $D_i(S)_G$, we have the following lemma.

LEMMA 4. *The following sequence is exact:*

$$0 \to D_i(S)_G \xrightarrow{\overline{\varrho_i}} D_i(R) \to \bigoplus_{\mathfrak{p} \in X^{(i)}(R)} \mathbb{Z}/f_\mathfrak{p}\mathbb{Z} \to 0.$$

PROOF. Fix $\mathfrak{P} \in X^{(i)}(S)$ and put $\mathfrak{p} = \mathfrak{P} \cap R$. Then $\varrho_i$ maps the element $\langle \mathfrak{P} \rangle \in D_i(S)$ to $\ell_{R_\mathfrak{p}}((S/\mathfrak{P})_\mathfrak{p})\langle \mathfrak{p} \rangle = f_\mathfrak{p}\langle \mathfrak{p} \rangle \in D_i(R)$. The lemma follows from this.

Put

$$f_i = l.c.m.\{f_\mathfrak{p} : \mathfrak{p} \in X^{(i)}(R)\}.$$

Furthermore, assuming $d = \dim(S)$ $(= \dim(R)) < \infty$, put

$$f^{(i)} = \prod_{j=i}^{d} f_j \qquad (i = 0, \ldots, d).$$

Note that if the trace map is surjective, then $f_i$ divides $|G|$ and $f^{(i)}$ divides $|G|^{d-i+1}$. We will now apply Lemma 4 to study the restriction map

$$\rho = \mathrm{Res}_R^S : G_0(S) \to G_0(R),$$

and on the filtration $\{F_i\}$ of $G_0$ in (2). Similar results could be formulated for the other restriction maps as well (and indeed for more general ring extensions).

PROPOSITION 1. *$f_i$ annihilates the cokernel of $\rho_i : W_i(S) \to W_i(R)$. Furthermore,*

$$f_i \cdot F_i(R) \subseteq \rho(F_i(S)) + F_{i+1}(R),$$

*and, assuming $d = dim(S) < \infty$,*

$$f^{(i)} F_i(R) \subseteq \rho(F_i(S)).$$

PROOF. By Lemma 4, $f_i$ annihilates the cokernel of $\varrho_i$ and, hence, also the cokernel of $\rho_i$, by Lemma 3. The first inclusion above also follows, because $F_i(R)/F_{i+1}(R) \cong \phi_{0,i,i+1}(K_0(\mathfrak{M}_i(R)/\mathfrak{M}_{i+1}(R))) \subseteq K_0(\mathfrak{M}_0(R)/\mathfrak{M}_{i+1}(R))$. The second inclusion follows from the first by induction.

Parts (i) and (iii) of the following corollary are slight sharpenings of [Br-L], Proposition 2.3 and Theorem 2.4(i) in the situation where $S$ has finite (Krull) dimension.

COROLLARY 2. *Assume that $d = dim(S) < \infty$. Then:*

(i) *The cokernel of $\rho = \mathrm{Res}_R^S : G_0(S) \to G_0(R)$ is annihilated by $f^{(0)}$.*

(ii) *If $F_1(S)$ is a torsion group, then so is $F_1(R)$. Hence (by (7)) $F_1(R)$ is the torsion subgroup of $G_0(R)$ in this case.*

(iii) *Assume that $S$ is a domain with $G_0(S) = \langle [S] \rangle$ (so $F_1(S) = 0$). Then*

$$G_0(R) = \langle [R] \rangle \oplus F_1(R) \text{ and } f^{(1)} F_1(R) = 0.$$

One special case deserves mention. Namely, assume that $S/\mathfrak{M} = R/\mathfrak{M} \cap R$ holds for all $\mathfrak{M} \in X^{(d)}(S)$ or, equivalently,

$$f_d = 1.$$

This is automatically satisfied, for example, if $S$ is an affine algebra over an algebraically closed field $k \subseteq R$, since then we have $S/\mathfrak{M} \cong k$ for all $\mathfrak{M} \in X^{(d)}(S)$. In case the trace map is surjective, the assumption $f_d = 1$ is equivalent with $G^Z(\mathfrak{M}) = G^T(\mathfrak{M})$ for all $\mathfrak{M} \in X^{(d)}(S)$. We concentrate on this particular case in the following corollary.

COROLLARY 3. *Assume that the trace map is surjective and that $G^Z(\mathfrak{M}) = G^T(\mathfrak{M})$ holds for all $\mathfrak{M} \in X^{(d)}(S)$.*

(i) *If $W_d(S) = 0$, then $W_d(R) = 0$.*

(ii) *If $F_1(S) = 0$; then, for $i = 1, \dots, d$,*

$$|G|^{d-i} F_i(R) = 0.$$

PROOF. (i) $W_d(R)$ is the cokernel of $\rho_d$ and hence is annihilated by $f_d = 1$.

(ii) Since $f^{(i)}$ is a divisor of $|G|^{d-i}$ in our situation, the assertion is immediate from the proposition.

**2.3 Galois Extensions.** The extension $R \subseteq S$ is called *G-Galois* if the following equivalent conditions are satisfied (cf. [C-H-R]):

(i) $S$ is finitely generated projective over $R$ and the map $j : T \to \mathrm{End}_R(S)$ is an isomorphism.

(ii) The $S$-$S$-bimodule map $f : S \otimes_R S \to T$ is surjective (and hence is actually an isomorphism; cf. [B], Theorem 3.4 on p. 62).

(iii) $G^T(\mathfrak{P}) = \langle 1 \rangle$ holds for all maximal ideals $\mathfrak{P}$ of $S$.

In this case, by (iii) say, $G$ acts faithfully on $S$. Furthermore, by [C-H-R], Lemma 1.6, p. 7, the trace map is surjective and, consequently, $R$ is Morita equivalent with $T$. (This is no longer true in a noncommutative setting.) We will say that the extension $R \subseteq S$ is *G-Galois in codimension k* if the following condition is satisfied:

$(G)_k$ For each prime ideal $\mathfrak{p}$ of $R$ of height at most $k$, the extension $R_\mathfrak{p} \subseteq S_\mathfrak{p}$ is $G$-Galois.

Note that $R_\mathfrak{p}$ is the fixed ring of the action of $G$ on $S_\mathfrak{p}$ (cf. [Bou], Chap. 5 §1 no. 9, Prop. 23).

REMARKS. (1) By (iii) above, $(G)_k$ is equivalent with the following condition:

For each prime ideal $\mathfrak{P}$ of $S$ of height at most $k$, one has $G^T(\mathfrak{P}) = \langle 1 \rangle$.

This follows from the equality $G^T(\mathfrak{P}S_{\mathfrak{P}\cap R}) = G^T(\mathfrak{P})$. In particular, if $S$ is a domain, then $(G)_0$ holds if and only if $G$ acts faithfully on $S$. Furthermore, the extension $R \subseteq S$ is $G$-Galois if and only if it is $G$-Galois in all codimensions (or, equivalently, in codimension $\dim S$ in case the latter is finite). For more precise information, see Lemma 5 below.

(2) Extensions that are $G$-Galois in codimension 1 coincide with the *pseudo-Galois* extensions that are considered in [W] (in a more general context).

(3) $(G)_k$ implies $(F)_k$. Indeed, if $\mathfrak{P}$ is a prime ideal of $S$ of height at most $k$ and $\mathfrak{p} = \mathfrak{P} \cap R$, then $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,\mathfrak{P}$ (cf. Section 2, Remark 2), and so $(G)_k$ implies that $S_\mathfrak{p}$ is finitely generated projective over $R_\mathfrak{p}$. Therefore, $S_\mathfrak{P}$ is flat over $R_\mathfrak{p}$ and, hence, over $R$ (cf. [Bou], Chap. 2 §3 no. 4, Remarque). Thus, summarizing, the various conditions are related as follows:

$$\boxed{(G)_k \Rightarrow (F)_k \Rightarrow (NBU)_k}$$

We now put

$$I = \mathrm{Im}(f) = TtT$$

with $t$ and $T$ as in (2.1) and $f$ as in (ii) above. Furthermore, for each $g \in G$, we put

$$I(g) = \{s - g(s) : s \in S\}S.$$

Then, by [Br-L], Lemma 2.2,

$$\prod_{g \in G \setminus \{1\}} I(g) \subseteq I \cap S \subseteq \bigcap_{g \in G \setminus \{1\}} I(g). \tag{14}$$

By definition, the extension $R \subseteq S$ is $G$-Galois if and only if $I = T$ or, equivalently, $I \cap S = S$. The following lemma gives more precise information.

LEMMA 5. *$(G)_k$ is satisfied for all $k < \mathrm{ht}(I \cap S) = \min\{\mathrm{ht}(I(g)) : 1 \neq g \in G\}$, but not for $k = \mathrm{ht}(I \cap S)$ in case $I \cap S \neq S$.*

PROOF. The formula for $\mathrm{ht}(I \cap S)$ is clear from (14). Let $\mathfrak{P}$ be a prime ideal of $S$. Then $G^T(\mathfrak{P}) \neq \langle 1 \rangle$ is equivalent with $I(g) \subseteq \mathfrak{P}$ for some $1 \neq g \in G$. In this case, $\mathrm{ht}(\mathfrak{P}) \geq \mathrm{ht}(I(g)) \geq \mathrm{ht}(I \cap S)$, whence $(G)_k$ holds for all $k < \mathrm{ht}(I \cap S)$. If $I \cap S \neq S$, then $I(g) \subseteq \mathfrak{P}$ for some $1 \neq g \in G$ and some prime $\mathfrak{P}$ of $S$ with $\mathrm{ht}(\mathfrak{P}) = \mathrm{ht}(I \cap S)$, and so $G^T(\mathfrak{P}) \neq \langle 1 \rangle$.

**2.4 The Map $\Psi_i : D_i(R) \to D_i(S)$.** The group $G$ acts as a permutation group on $X^{(i)}(S)$ and, hence, on $D_i(S)$. The image of $\Psi_i$ lies in the subgroup $D_i(S)^G$ of $G$-invariants in $D_i(S)$, and $D_i(S)^G$ is the free abelian group with basis, the $G$-orbits in $X^{(i)}(S)$. If $\mathfrak{p} \in X^{(i)}(R)$ is given, then the primes $\mathfrak{P}$ of $S$ with $\mathfrak{p} = \mathfrak{P} \cap R$ form a $G$-orbit in $X^{(i)}(S)$, and all $G$-orbits in $X^{(i)}(S)$ arise in this fashion. Thus, the map $\Psi_i$ in (8) leads to a map

$$\text{(8')} \qquad \begin{aligned} \Psi_i' : \ D_i(R) &\longrightarrow D_i(S)^G \\ \langle \mathfrak{p} \rangle &\longmapsto e_{\mathfrak{p}}[\mathfrak{P}], \end{aligned}$$

where $[\mathfrak{P}]$ denotes the element of $D_i(S)^G$ corresponding to the $G$-orbit of a fixed prime $\mathfrak{P}$ of $S$ with $\mathfrak{p} = \mathfrak{P} \cap R$, and where $e_{\mathfrak{p}}$ is as in (2.1).

This proves the existence of the exact sequence in the following lemma.

LEMMA 6. *The following sequence is exact:*

$$0 \to D_i(R) \xrightarrow{\Psi_i'} D_i(S)^G \to \bigoplus_{\mathfrak{p} \in X^{(i)}(R)} \mathbb{Z}/e_{\mathfrak{p}}\mathbb{Z} \to 0.$$

*Moreover, if $(G)_k$ is satisfied, then $\Psi_i'$ is an isomorphism for all $i \leq k$.*

PROOF. It remains to show that $(G)_k$ implies that $e_{\mathfrak{p}} = 1$ holds for all $\mathfrak{p} \in X^{(i)}(R)$ or, equivalently, $\mathfrak{p}S_{\mathfrak{P}} = \mathfrak{P}S_{\mathfrak{P}}$, where $\mathfrak{P}$ is an arbitrary prime of $S$ lying over $\mathfrak{p}$. In view of [A-B], Prop. 2.2, this follows from the fact that $(G)_k$ implies that $S_{\mathfrak{p}}$ is a separable $R_{\mathfrak{p}}$-algebra ([C-H-R], Theorem 1.3). Here is a short direct argument. Surjectivity of $f_{\mathfrak{p}}$ implies that

$$\mathfrak{b} = (\mathfrak{b} \cap R_{\mathfrak{p}})S_{\mathfrak{p}}$$

holds for each $G$-stable ideal $\mathfrak{b}$ of $S_{\mathfrak{p}}$. To see this, write $1 = \sum_j x_j t y_j$ in $T_{\mathfrak{p}}$ with $x_j, y_j \in S_{\mathfrak{p}}$. Then, using the map $j_{\mathfrak{p}} : T_{\mathfrak{p}} \to \mathrm{End}_{R_{\mathfrak{p}}}(S_{\mathfrak{p}})$, we see that $s = j_{\mathfrak{p}}(1)(s) = \sum_j x_j \mathrm{tr}(y_j s)$ holds for each $s \in S_{\mathfrak{p}}$. Taking $s \in \mathfrak{b}$, we conclude that $\mathfrak{b} \subseteq S_{\mathfrak{p}}\mathrm{tr}(\mathfrak{b}) \subseteq S_{\mathfrak{p}}(\mathfrak{b} \cap R_{\mathfrak{p}})$, and so equality holds throughout. In particular, if $\mathfrak{a}$ is a semiprime ideal of $R_{\mathfrak{p}}$, then $\mathfrak{b} = \mathfrak{a}S_{\mathfrak{p}}$ is semiprime in $S_{\mathfrak{p}}$, because $\sqrt{\mathfrak{b}} = \{s \in S_{\mathfrak{p}} \mid s^n \in \mathfrak{b} \text{ for some } n\}$ satisfies $\sqrt{\mathfrak{b}} = (\sqrt{\mathfrak{b}} \cap R_{\mathfrak{p}})S_{\mathfrak{p}} = \mathfrak{a}S_{\mathfrak{p}}$. Consequently, if $\mathfrak{p}$ and $\mathfrak{P}$ are as above, then $\mathfrak{p}S_{\mathfrak{p}} = \bigcap_{g \in G} g(\mathfrak{P})S_{\mathfrak{p}}$, and so $\mathfrak{p}S_{\mathfrak{P}} = \bigcap_{g \in G} g(\mathfrak{P})S_{\mathfrak{P}} = \mathfrak{P}S_{\mathfrak{P}}$, as required.

**2.5 Galois Descent: The Induction Map $\psi_i' : W_i(R) \to W_i(S)^G$.** We keep the notations of the previous sections, but we will now assume in addition that $S$ and $R$ are Noetherian. Our goal is to extend [S], Theorem 1.1 on p. 55 (or see [Fo], Theorem 16.1) to study the kernel $\mathcal{P}_i(R,S)$ of the maps $\psi_i : W_i(R) \to W_i(S)$ in (5). The material of this section can surely be extended to Hopf algebra actions, as done in [W] for $i = 1$ (and Krull domains), but we will concentrate on group actions here. Moreover, the Noetherian hypothesis can be weakened, as in [C-F2].

Note that, since the image of $\psi_i$ above lies in the $G$-invariants $W_i(S)^G$ of $W_i(R)$; by Corollary 1, we can refine (5) as follows:

$$0 \to \mathcal{P}_i(R,S) \to W_i(R) \xrightarrow{\psi_i'} W_i(S)^G. \tag{5'}$$

Furthermore, $G$ acts in an obvious fashion on $Q_i(S)$ in (9), and the maps $\delta_i$ and $\phi_i$ in (10) respect the $G$-actions on $Q_i(S)$, $D_i(S)$, and $W_i(S)$. In particular, $G$ operates on $U_i(S) = \operatorname{Ker}\delta_i$.

The following proposition extends Samuel's theorem. The assumption on the orders of the groups $G^T(\mathfrak{P})$ in (i) is always satisfied if the trace map is surjective (cf. [Br-L], §2.1) or if $(\mathrm{G})_{i-1}$ holds (Remark 1 in (2.3)).

PROPOSITION 2. (i) *Assume that the extension $R = S^G \subseteq S$ satisfies* $(\mathrm{F})_i$ *and that, for each $\mathfrak{P} \in X^{(i-1)}(S)$, the order of $G^T(\mathfrak{P})$ is nonzero in $S/\mathfrak{P}$. Then there is an exact sequence*

$$0 \to \mathcal{P}_i(R,S) \to \mathcal{H}_i \to \bigoplus_{\mathfrak{p}\in X^{(i)}(R)} \mathbb{Z}/e_{\mathfrak{p}}\mathbb{Z} \to \operatorname{Coker}(\psi_i'),$$

*where $\mathcal{H}_i$ is the kernel of the canonical map $H^1(G, U_i(S)) \to H^1(G, Q_i(S))$.*

(ii) *If* $(\mathrm{G})_{i-1}$ *is satisfied, then $H^1(G, Q_i(S)) = 0$ and, hence,*

$$\mathcal{H}_i = H^1(G, U_i(S)).$$

(iii) *If* $(\mathrm{G})_i$ *is satisfied, then we obtain an isomorphism*

$$\mathcal{P}_i(R,S) \cong H^1(G, U_i(S)).$$

PROOF. (i) Put $V_i = \delta_i(Q_i) \subseteq D_i$. Then, in view of Lemma 6 and (5'), the exact sequence follows from the Snake Lemma applied to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_i(R) & \xrightarrow{\text{incl.}} & D_i(R) & \xrightarrow{\phi_i(R)} & W_i(R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \Psi_i'} & & \downarrow{\scriptstyle \psi_i'} & & \\ 0 & \longrightarrow & V_i(S)^G & \xrightarrow{\text{incl.}} & D_i(S)^G & \xrightarrow{\phi_i(S)} & W_i(S)^G & & \end{array}$$

provided we can show that $\mathcal{H}_i$ is isomorphic to $V_i(S)^G/\Psi_i'(V_i(R))$. For this, note that the map $\Psi_{i-1}' : Q_i(R) \to Q_i(S)$ of (12) has image $Q_i(S)^G$. Indeed, if $\mathfrak{p} \in X^{(i-1)}(R)$ is given, then all primes $\mathfrak{P}$ of $S$ with $\mathfrak{P} \cap R = \mathfrak{p}$ are conjugate under $G$, and so

$$\Big(\bigoplus_{\mathfrak{P}\cap R=\mathfrak{p}} Q(S/\mathfrak{P})^*\Big)^G = \big(Q(S/\mathfrak{P})^*\big)^{G^Z(\mathfrak{P})},$$

where, on the right, $\mathfrak{P}$ is a fixed prime of $S$ over $\mathfrak{p}$. Our assumption on the order of $G^T(\mathfrak{P})$ implies that the right hand side is equal to the canonical image of $Q(R/\mathfrak{p})^*$ (cf. [Bou], Chap. 5 §2 no. 2, Cor. to Prop. 5). Therefore, $\Psi'_{i-1}(Q_i(R)) = Q_i(S)^G$ and, hence,

$$\delta_i(S)(Q_i(S)^G) = (\Psi_i \circ \delta_i(R))(Q_i(R)) = \Psi_i(V_i(R)) = \Psi'_i(V_i(R)).$$

Thus, $V_i(S)^G/\Psi'_i(V_i(R)) = V_i(S)^G/\delta_i(S)(Q_i(S)^G)$, and this is isomorphic with $\mathcal{H}_i$ in view of the long cohomology sequence

$$0 \to U_i(S)^G \to Q_i(S)^G \to V_i(S)^G \to H^1(G, U_i(S)) \to H^1(G, Q_i(S)) \to \dots .$$

This proves (i).

(ii) We claim that $H^1(G, Q_i(S))$ is annihilated by the least common multiple of the orders $|G^T(\mathfrak{P})|$ for $\mathfrak{P} \in X^{(i-1)}(S)\}$. Since these orders are all equal to 1 if $(G)_{i-1}$ is satisfied; by Remark 1 in (2.3), part (ii) will be a consequence. First, Schanuel's Lemma gives

$$H^1(G, Q_i(S)) \cong \bigoplus_{\mathfrak{P}} H^1(G^Z(\mathfrak{P}), Q(S/\mathfrak{P})^*),$$

where $\mathfrak{P}$ ranges over a full set of non-$G$-conjugate primes in $X^{(i-1)}(S)$. By Hilbert's Theorem 90, $H^1(G^Z(\mathfrak{P})/G^T(\mathfrak{P}), Q(S/\mathfrak{P})^*) = 0$ and, hence,

$$H^1(G^Z(\mathfrak{P}), Q(S/\mathfrak{P})^*) \cong H^1(G^T(\mathfrak{P}), Q(S/\mathfrak{P})^*)^{G^Z(\mathfrak{P})/G^T(\mathfrak{P})}.$$

Since this is a $|G^T(\mathfrak{P})|$-torsion group, our claim is proved.

(iii) Since $(G)_i$ implies $(F)_i$; by Remark 3 in (2.3), part (iii) follows from (i), (ii), and the last assertion of Lemma 6.

COROLLARY 4 (SAMUEL). *Assume that $S$ is a UFD and that $G$ acts faithfully on $S$. Then there is an exact sequence*

$$0 \to C(R) \to H^1(G, U(S)) \to \bigoplus_{\mathfrak{p} \in X^{(1)}(R)} \mathbb{Z}/e_{\mathfrak{p}}\mathbb{Z} \to 0.$$

PROOF. First, $S$ and $R$ are both integrally closed, and so $(F)_1$ is satisfied by Remarks (2) and (5) in (1.2). Furthermore, since $G$ acts faithfully, $(G)_0$ holds too. Therefore, $\mathcal{H}_1 = H^1(G, U(S))$. Finally, $W_1(S) = C(S) = 0$ since $S$ is a UFD, and hence $\psi'_1 = 0$ and $\mathcal{P}_1(R, S) = C(R)$.

**2.6 Multiplicative Actions.** Let $S = kA$ be the group algebra of the free abelian group $A$ of rank $d$ over the field $k$, and let $G$ act on $S$ via a homomorphism $\alpha : G \to \mathrm{GL}(A) = \mathrm{GL}_d(\mathbb{Z})$ ("multiplicative action"). In this situation, all $I(g)$ in (2.3) are contained in the augmentation ideal of $S$, and so $I \cap S \neq S$. By Lemma 5, we conclude that

(15) $\qquad (G)_k$ is satisfied $\iff k < \delta := \mathrm{ht}(I \cap S)$.

By [Br-L], (3.2), $I(g)$ is the kernel $\omega([A,g])$ of the canonical map $kA \twoheadrightarrow kA/[A,g]$, where

$$[A,g] = \{[a,g] = a^{-1}g(a) : a \in A\}$$

is a subgroup of $A$. Thus, by Lemma 5,

$$\begin{aligned} \delta &= \min\{\operatorname{rank}([A,g]) : 1 \neq g \in G\} \\ &= \min\{\operatorname{rank}(\alpha(g) - 1) : 1 \neq g \in G\}. \end{aligned} \tag{16}$$

If $\alpha$ is mono, which we may assume without essential loss ("faithful action"), then $\delta \geq 1$.

Example 1: Fixed-Point-Free Actions. The action of $G$ is called *fixed-point-free* or *free* if $a \neq g(a)$ holds for all $g \in G \setminus \{1\}$ and $a \in A \setminus \{1\}$. By (16) above, this occurs precisely if $\delta = d$. Thus:

$$G \text{ acts fixed-point-freely} \iff (\mathrm{G})_k \text{ holds for all } k < d = \operatorname{rank}(A). \tag{17}$$

Example 2: Reflections. An element $g \in G$ is called a *reflection* if and only if the matrix $1 - \alpha(g) \in \mathrm{GL}(A) \subseteq \mathrm{GL}_d(\mathbb{Q})$ has rank 1 or, equivalently, if $[A,g]$ is a cyclic subgroup of $A$. Thus:

$$\delta > 1 \iff G \text{ contains no reflections.} \tag{18}$$

In the setting of multiplicative actions, $W_i(S) = 0$ holds for all $i \geq 1$ ([C-F], Theorems 7.7 and 6.5). Also, $G_0(S) = \langle [S] \rangle \cong \mathbb{Z}$ and $F_1(S) = 0$. Therefore, Proposition 1 implies that

$$f_i \text{ annihilates } W_i(R) \text{ and } F_i(R)/F_{i+1}(R) \text{ for all } i = 1, \dots, d.$$

Moreover, we conclude from Proposition 2 and (15) that

$$W_i(R) \cong H^1(G, U_i(S)) \qquad (i = 1, \dots, \delta - 1).$$

Since $U(S) = k^* \times A$, Corollary 4 gives

$$W_1(R) = C(R) \subseteq H^1(G, U(S)) \cong \operatorname{Hom}(G^{ab}, k^*) \oplus H^1(G, A).$$

Finally, if $|G|^{-1} \in k$ and $G^Z(\mathfrak{M}) = G^T(\mathfrak{M})$ holds for all maximal ideals $\mathfrak{M}$ of $S$ (e.g., if $k$ is algebraically closed), then Corollary 3 yields

$$W_d(R) = 0 \text{ and } |G|^{d-i} F_i(R) = 0.$$

In particular, in [Br-L], Example 5.3 with $d = 3$, we get $F_3(R) = 0$ and $F_2(R) \cong (\mathbb{Z}/2\mathbb{Z})^7$ (at least if $k$ is algebraically closed). This answers the question that was left open there.

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