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ON THE GLOBAL DIMENSION OF FIXED RINGS

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ABSTRACT. Let G be a finite group acting on a k-algebra R, and let $S = R^G$ denote the fixed subring of R. Our main interest is in the case where $|G|$ is *not* invertible in R. Instead, we assume that R is flat over S and that the trivial kG-module k has a periodic projective resolution. (For a field k of characteristic p, the latter condition holds precisely if the Sylow p-subgroups of G are cyclic or generalized quaternion.) We use a periodicity result for Ext-groups, established here in a more general setting that is independent of group actions, to estimate the global dimension of S in this case.

INTRODUCTION

Let G be a finite group acting by automorphisms on a k-algebra R (k some commutative ring), and let $S = R^G$ denote the fixed subring of R. In this note, we are concerned with bounding the (right) global dimension of S in terms of the global dimension of R and other related data.

In case the global dimension of R is at most 1 and $|G|$ is invertible in R, one knows that $\mathrm{r.gldim}(S) \le \mathrm{r.gldim}(R)$, by results of Levitzki [L] and Bergman [Be]. The situation becomes worse for larger global dimensions and for $|G|^{-1} \notin R$. For example, if $R = k[x, y]$ is the polynomial ring over a field k of characteristic $\neq 2$, and $G = C_2$ acts on R by $x \mapsto -x$, $y \mapsto -y$, then R has global dimension 2 whereas $S = k[x^2, xy, y^2]$ has infinite global dimension [R]. Also, taking R to be the ring of 2×2-matrices over a field k with $\operatorname{char} k = 2$, and $G = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle \cong C_2$ acting by conjugation on R, one obtains $S \cong k \oplus k\varepsilon$ with $\varepsilon^2 = 0$. So $\mathrm{r.gldim}(R) = 0$, but $\mathrm{r.gldim}(S) = \infty$.

Nevertheless, if $|G|^{-1} \in R$, then one has the following estimate (see 2.2 below):

$$\mathrm{r.gldim}(S) \le \mathrm{r.gldim}(R) + \mathrm{pdim}(R_S).$$

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The aim of this article is to establish a similar estimate without the assumption on $|G|$. Our main result is as follows.

Theorem. *Put* $\overline{S} = S/\operatorname{Tr}(R)$, *where* $\operatorname{Tr}: R \to S$ *is the usual trace map, and*

$$\rho = \text{r.gldim}(R) + \text{pdim}(R_S) \quad \textit{and} \quad \sigma = \text{r.gldim}(\overline{S}) + \text{pdim}(\overline{S}_S).$$

Assume that

(1) ${}_S R$ *is flat, and*
(2) *the trivial* kG*-module* k *has a resolution* $0 \to k \to X_c \to \cdots \to X_1 \to k \to 0$ *with all* X_i *projective over* kG.

Then either $\text{r.gldim}(S) \leq \max\{\rho, \sigma\}$ *or* $\text{r.gldim}(S) = \infty$.

Assumption (2) here is satisfied if $|G|^{-1} \in k$ (see 2.3). More interestingly, if k is a field of characteristic p, then (2) holds precisely if the Sylow p-subgroups of G are cyclic or generalized quaternion. Thus the important special case of one automorphism of finite order acting on R is covered by (2). For further examples, see 2.4 below. The precise meaning of (1) is less clear, but R is known to be projective over S in some cases that arise naturally (see 2.6).

The above result is a consequence of a more precise periodicity result for Ext-groups that holds in the situation of the above theorem (see Theorem 2.7). In §1, we establish such a periodicity result for Ext-groups in a more general abstract setting, independent of group actions. This result is then applied, in §2, to the case of fixed subrings.

1. Periodicity for ext

1.1 **Lemma.** *Let* $S \to \overline{S}$ *be a surjective ring homomorphism with kernel* I. *Let* M_S *be an* S*-module with* $M \cdot I^n = 0$ *for some* $n \geq 1$. *Then* $\text{pdim}(M_S) \leq \text{r.gldim}(S) + \text{pdim}(\overline{S}_S)$.

Proof. Arguing by induction on n, one reduces to the case $n = 1$, where the assertion is well known (e.g., [Ro, Theorem 9.32]). □

1.2 **Lemma.** *Let* S *be a ring and let*

$$\mathbf{T}: 0 \to T_{c+1} \to T_c \to \cdots \to T_1 \to T_0 \to 0$$

be a complex of right S*-modules, with* $c \geq 1$. *Assume that*

$$\delta = \max_{0 \leq n \leq c+1} \{\text{pdim}\, H_n(\mathbf{T})\} \quad \textit{and} \quad \tau = \max_{1 \leq m \leq c} \{\text{pdim}(T_m)\}$$

are both finite. Let V_S *be an* S*-module. Then there are homomorphisms*

$$f^q: \operatorname{Ext}_S^{q+c}(T_0, V) \to \operatorname{Ext}_S^q(T_{c+1}, V) \qquad (q > \tau)$$

such that f^q *is epi for* $q > \max\{\tau, \delta\}$, *and* f^q *is an isomorphism for* $q > \max\{\tau, \delta + 1\}$.

Proof. We first introduce a number of (cochain-) complexes. Let $\mathbf{Q} = (Q^i)_{i\geq 0}$ be an injective resolution of V_S. So $H^i(\mathbf{Q}) = 0$ for $i > 0$ and $H^0(\mathbf{Q}) \cong V$. Form the double complex

$$\mathbf{B} = \mathbf{Hom}_S(\mathbf{T}, \mathbf{Q}) = (B^{p,q})_{p,q\geq 0}, \qquad B^{p,q} = \mathrm{Hom}_S(T_p, Q^q)$$

and its associated total complex

$$\mathbf{C} = \mathrm{Tot}(\mathbf{B}) = (C^n)_{n\geq 0}, \qquad C^n = \bigoplus_{p+q=n} B^{p,q}.$$

Note that $B^{p,q} = 0$ for $p > c+1$ as $T_p = 0$ in this case.

Step 1. $H^n(\mathbf{C}) = 0$ for $n > \delta + c + 1$.

Proof. Since $\mathbf{Q}_S$ is injective, there is a Künneth spectral sequence $\{E_r\}$ with

$$E_2^{p,q} \cong \bigoplus_{i+j=q} \mathrm{Ext}_S^p(H_i(\mathbf{T}),\ H^j(\mathbf{Q})) \underset{p}{\Rightarrow} H^{p+q}(\mathbf{C})$$

(see [Ro, Theorem 11.34] or [G, Theorem 5.4.1]). Thus

$$E_2^{p,q} = \mathrm{Ext}_S^p(H_q(\mathbf{T}), V) = 0$$

if either $q > c+1$ or $p > \delta$. Therefore, $H^{p+q}(\mathbf{C}) = 0$ if $p + 9 > \delta + c + 1$.

Step 2. The maps f^q.

Consider the first filtration of $\mathbf{C}$, as in [C-E, p. 330] (omitting the subscript I):

$$(F^p\mathbf{C})^n = \bigoplus_{\substack{r\geq p \\ r+s=n}} B^{r,s}.$$

The corresponding spectral sequence $\{E'_r\}$ converges to $H^*(\mathbf{C})$ and has E'_2-term (notation as in [C-E, pp. 330–331])

$$E'_2 \cong H_{\mathrm{I}} H_{\mathrm{II}}(\mathbf{B}),$$

where

$$H_{\mathrm{II}}^{p,q}(\mathbf{B}) = H^q(B^{p,*}) = H^q(\mathrm{Hom}_S(T_p, \mathbf{Q})) = \mathrm{Ext}_S^q(T_p, V).$$

In particular, since $T_p = 0$ for $p > c+1$, we have $H_{\mathrm{II}}^{p,*}(\mathbf{B}) = 0$ for $p > c+1$ and so $(E'_2)^{p,q} = 0$ for $p > c+1$. Since the differential d_r of E'_r has bidegree $(r, 1-r)$, it follows that $d_{c+2} = 0$ and so

$$E'_\infty = E'_{c+2}.$$

Moreover, by definition of τ, $H_{\mathrm{II}}^{p,q}(\mathbf{B}) = 0$ for $q > \tau$ and $p \neq 0,\ c+1$. Consequently, for $q > \tau$, we have

$$(E'_2)^{p,q} = \begin{cases} 0 & \text{if } p \neq 0, c+1 \\ \mathrm{Ext}_S^q(T_0, V) & \text{if } p = 0 \\ \mathrm{Ext}_S^q(T_{c+1}, V) & \text{if } p = c+1 \end{cases} \qquad (q > \tau).$$

By considering the bidegree of the differential d_r of E'_r, one sees that

$$q \geq \tau + c \Rightarrow d_r^{0,q} = 0 \quad \text{for } 2 \leq r \leq c$$
$$\Rightarrow (E'_2)^{0,q} = (E'_3)^{0,q} = \cdots = (E'_{c+1})^{0,q}$$

and

$$q \leq \tau \Rightarrow d_r^{c+1-r,q-(1-r)} = 0 \quad \text{for } 2 \leq r \leq c$$
$$\Rightarrow (E'_2)^{c+1,q} = (E'_3)^{c+1,q} = \cdots = (E'_{c+1})^{c+1,q}.$$

Therefore, for $q \geq \tau$, the differential d_{c+1} yields a homomorphism

$$(E'_2)^{0,q+c} \xrightarrow{\sim} (E'_{c+1})^{0,q+c} \xrightarrow{d_{c+1}} (E'_{c+1})^{c+1,q} \xrightarrow{\sim} (E'_2)^{c+1,q}.$$

For $q > \tau$, this is the required homomorphism

$$f^q \colon \operatorname{Ext}_S^{q+c}(T_0, V) \to \operatorname{Ext}_S^q(T_{c+1}, V).$$

Step 3. Surjectivity and injectivity.

Since $H^n(\mathbf{C}) = 0$ for $n > \delta + c + 1$, by Step 1, we know that $(E'_\infty)^{p,q} = 0$ for $p + q > \delta + c + 1$. In particular,

$$0 = (E'_\infty)^{c+1,q} = (E'_{c+2})^{c+1,q} = (E'_{c+1})^{c+1,q} / d_{c+1}(E'_{c+1})^{0,q+c}$$

for $c + 1 + q > \delta + c + 1$, and so f^q is surjective for $q > \delta$, $q > \tau$. Also,

$$0 = (E'_\infty)^{0,q+c} = (E'_{c+2})^{0,q+c} = \operatorname{Ker}(d_{c+1}|_{(E'_{c+1})^{0,q+c}})$$

for $q + c > \delta + c + 1$. Thus f^q is injective for $q > \delta + 1$, $q > \tau$. This proves the lemma. □

1.3 **Proposition.** *Let* $S \subseteq R$ *be an inclusion of rings and let* $S \to \overline{S}$ *be a surjective ring homomorphism. Assume that*

$$\rho = \operatorname{r.gldim}(R) + \operatorname{pdim}(R_S) \quad \textit{and} \quad \sigma = \operatorname{r.gldim}(\overline{S}) + \operatorname{pdim}(\overline{S}_S)$$

are both finite. Suppose that there is a complex of (S, S)*-bimodules*

$$\mathbf{P} \colon 0 \to P_{c+1} \to P_c \to \cdots \to P_1 \to P_0 \to 0$$

with $c \geq 1$, *such that*

(1) *each* ${}_SP_i$ *is flat;*
(2) *for* $1 \leq i \leq c$, P_i *is an* (S, S)*-bimodule direct summand of some* (S, R)*-bimodule;*
(3) *the right* S*-action on* $H_*(\mathbf{P})$ *factors through* $\overline{S}$.

Then, for any two S*-modules* V_S *and* W_S, *there exist homomorphisms*

$$f^q \colon \operatorname{Ext}_S^{q+c}(W \otimes_S P_0, V) \to \operatorname{Ext}_S^q(W \otimes_S P_{c+1}, V) \qquad (q > \rho)$$

such that f^q *is epi for* $q > \max\{\rho, \sigma\}$ *and an isomorphism for* $q > \max\{\rho, \sigma + 1\}$.

Proof. Put $\mathbf{T} = W \otimes_S \mathbf{P}$, a complex of right S-modules. We estimate the numbers δ and τ in Lemma 1.2. First, by assumption (2), T_i is an S-module direct summand of some right R-module, say Q_i $(i = 1, \dots, c)$. Therefore, by [Ro, Theorem 9.32],

$$\operatorname{pdim}(T_i)_S \le \operatorname{pdim}(Q_i)_S \le \operatorname{pdim}(Q_i)_R + \operatorname{pdim}(R_S) \le \rho .$$

Hence

$$\tau \le \rho .$$

Next, we claim that $H_n(\mathbf{T}) \cdot I^{n+1} = 0$ for $0 \le n \le c+1$, where I denotes the kernel of the map $S \to \overline{S}$. Indeed, since ${}_S\mathbf{P}$ is flat, by (1), there is a Künneth spectral sequence $\{E^r\}$ with

$$E^2_{p,q} \cong \operatorname{Tor}^S_p(W, H_q(\mathbf{P})) \underset{p}{\Rightarrow} H_{p+q}(\mathbf{T})$$

([Ro, Theorem 11.34] or [M, Chapter XII, Theorem 12.1]). By (3), we have $E^2_{p,q} \cdot I = 0$. The spectral sequence yields a chain

$$0 = F^{-1}H_n \subseteq F^0 H_n \subseteq \cdots \subseteq F^n H_n = H_n$$

of S-submodules of $H_n = H_n(\mathbf{T})$ such that each $F^p H_n / F^{p-1} H_n$ is isomorphic to a subquotient of $E^2_{p,n-p}$. Therefore, $H_n(\mathbf{T}) \cdot I^{n+1} = 0$, as we have claimed.

Lemma 1.1 now yields $\delta \le \sigma$, and so the proposition follows from Lemma 1.2. □

1.4 *Remarks.* The proof of Proposition 1.3 works without essential changes if (3) is weakened to

(3′) Some power of $I = \operatorname{Ker}(S \to \overline{S})$ annihilates each $H_n(\mathbf{P})$.

Moreover, arguing somewhat differently (without invoking the spectral sequence) in the last part of the proof, the conclusion of Proposition 1.3 still holds if (1) is replaced by

(1′) Some power of I annihilates each $\operatorname{Tor}^S_i(W, P_j)$, for all $i > 0$.

However, these generalizations don't seem to be useful, at least not for our applications to fixed rings in §2.

1.5 **Corollary.** *In the situation of Proposition* 1.3, *assume in addition that* S *is an* (S, S)*-bimodule direct summand of* P_{c+1}. *Then, for any* S*-module* V_S, *either* $\operatorname{idim}(V_S) \le \max\{\rho, \sigma\}$ *or* $\operatorname{idim}(V_S) = \infty$. *Consequently, either* $\operatorname{r.gldim}(S) \le \max\{\rho, \sigma\}$ *or* $\operatorname{r.gldim}(S) = \infty$.

Proof. Assume that $\operatorname{idim}(V_S) > \max\{\rho, \sigma\}$, so $\operatorname{Ext}^q_S(W, V) \ne 0$ for some S-module W_S and some $q > \max\{\rho, \sigma\}$. Proposition 1.3 yields an epimorphism $\operatorname{Ext}^{q+c}_S(W \otimes_S P_0, V) \to \operatorname{Ext}^q_S(W \otimes_S P_{c+1}, V)$ and the latter group is nonzero, since it maps onto $\operatorname{Ext}^q_S(W, V)$, by assumption on P_{c+1}. Therefore,

$$\operatorname{Ext}^{q+c}_S(W \otimes_S P_0, V) \ne 0,$$

and we can continue with $W \otimes_S P_0$ in place of W. □

2. Periodic resolutions for groups

2.1 *Notations.* The following notations will be kept throughout this section. R will be an algebra over some commutative ring k, and G will be a finite group acting on R by k-algebra automorphisms that will be written $r \mapsto r^g$ ($r \in R, g \in G$). We let $S = R^G$ denote the fixed subring of R. As usual, $\mathrm{Tr}: R \to S$ is the trace map given by $\mathrm{Tr}(r) = \sum_{g \in G} r^g$ ($r \in R$). Since Tr is an (S, S)-bimodule map, its image $\mathrm{Tr}(R)$ is an ideal of S. We put $\overline{S} = S/\mathrm{Tr}(R)$. The augmentation ideal of the group algebra kG will be denoted by ωG, and we put $\widehat{G} = \sum_{g \in G} g \in kG$. Viewing R as a right kG-module via the given G-action, the trace map can be written as $\mathrm{Tr}(r) = r^{\widehat{G}}$ ($r \in R$).

2.2 *The case* $|G|^{-1} \in R$. If $|G|^{-1} \in R$, then the map $|G|^{-1} \cdot \mathrm{Tr}: R \to S$ is the identity on S. thus S is an (S, S)-bimodule direct summand of R, and [K, Theorem 5 on p. 173] implies

$$\mathrm{r.gldim}(S) \leq \mathrm{r.gldim}(R) + \mathrm{pdim}(R_S).$$

Thus, in the following, our emphasis will be on the case where $|G|^{-1} \notin R$, although the case $|G|^{-1} \in R$ is covered by the following, too.

2.3 *Periodic resolutions.* Following [A], we say that the trivial kG-module k is *periodic* if there exists an exact sequence of left (say) kG-modules

$$\mathbf{X}: 0 \to k \xrightarrow{\mu} X_c \xrightarrow{\phi_c} \cdots \to X_i \xrightarrow{\phi_i} \cdots \to X_1 \xrightarrow{\varepsilon} k \to 0,$$

where all X_i are projective over kG. The smallest such c is called the *period* of k. (In [A], these concepts are introduced, in the same way, for arbitrary kG-modules.) The case where k is projective over kG or, equivalently, $|G|^{-1} \in k$, is included here via the obvious sequence $0 \to k \to X_1 = k \oplus k \to k \to 0$.

2.4 *Remarks and Examples.* (a) If k is periodic with period c, then the (Tate) cohomology ring $\widehat{H}^*(G, k)$ has an invertible element $u \in \widehat{H}^c(G, k)$. In particular, using the fact that $\widehat{H}^*(G, k)$ is anticommutative, one sees that, if $2 \neq 0$ in $\widehat{H}^0(G, k) = k/|G|k$, then c must be even. But, of course, the case of odd c also occurs, e.g. $c = 1$ when $|G| = 2$ and $2 = 0$ in k.

(b) If $\mathbf{Z}$ is periodic for $\mathbf{Z}G$, then any k is periodic for kG. In this case, all abelian subgroups of G must be cyclic ([Br, Theorem 9.5 on p. 157]). All finite groups of this type have been classified by Zassenhaus [Z] in the solvable case (see also [W, Theorem 6.1.11]) and by Suzuki [S] in general.

A construction of periodic free $\mathbf{Z}G$-resolutions $\mathbf{X}$ for $\mathbf{Z}$ starting with a fixed-point-free complex representation of G (that is, a finite-dimensional $\mathbf{C}G$-module V such that $\{v \in V | gv = v\} = 0$ holds for all $1 \neq g \in G$) is described in [Br, p. 154]. The complete classification of all groups admitting a fixed-point-free representation can be found in [W, Theorems 6.1.11 and 6.3.1]. These groups include in particular the finite subgroups of the multiplicative group of the quaternions $\mathbf{H}$:

—the finite cyclic groups,

—the generalized quaternion groups $Q_{4m} = \langle x, y | y^2 = x^m, \ xyx = y \rangle$ $(m \geq 2)$,

—the binary tetrahedral group (order 24), the binary octahedral group (order 48), and the binary icosahedral group $SL(2,5)$ (order 120).

For further examples and more details see [Br, pp. 154–156].

(c) If k is a field of characteristic $p > 0$, then the trivial kG-module k is periodic if and only if all elementary abelian p-subgroups of G are cyclic (see [A-E]). The latter condition is satisfied if and only if the Sylow p-subgroups of G are either cyclic or generalized quaterion (for $p = 2$ only) (see [Br, p. 157]).

2.5 **Lemma.** *Assume that the trivial kG-module k is periodic with period c. Then there exists a complex of (S,S)-bimodules*

$$\mathbf{P}: 0 \to P_{c+1} \to P_c \to \cdots \to P_1 \to P_0 \to 0$$

such that

(1) $P_0 \cong S \cong P_{c+1}$;

(2) *for* $1 \leq i \leq c$, P_i *is an* (S,S)*-bimodule direct summand of some* $R^{(\alpha_i)}$;

(3) *the left and right* S*-actions on* $H_*(\mathbf{P})$ *factor through* $\overline{S}$.

Proof. Let

$$\mathbf{X}: 0 \to k \xrightarrow{\mu} X_c \xrightarrow{\phi_c} \ldots \xrightarrow{\phi_2} X_1 \xrightarrow{\varepsilon} k \to 0$$

be a periodic kG-resolution as in 2.3. Expand $\mathbf{X}$ into a complete periodic resolution

$$\widehat{\mathbf{X}}: \cdots \to X_{c+1} = X_1 \xrightarrow{\phi_{c+1}=\mu\circ\varepsilon} X_c \xrightarrow{\phi_c} \cdots \xrightarrow{\phi_2} X_1 \xrightarrow{\phi_1=\mu\circ\varepsilon} X_0 = X_c \to \cdots.$$

Consider the complex $\mathbf{Q} = R \otimes_{kG} \widehat{\mathbf{X}}$, with maps $\psi_i = \mathrm{id}_R \otimes \phi_i$. The (S,S)-bimodule structure on R makes $\mathbf{Q}$ a complex of (S,S)-bimodules. Moreover, since each X_i is a direct summand of some $kG^{(\alpha_i)}$, $Q_i = R \otimes_{kG} X_i$ is an (S,S)-bimodule direct summand of $R \otimes_{kG} kG^{(\alpha_i)} = R^{(\alpha_i)}$. The homology of $\mathbf{Q}$ is given by

$$H_i(\mathbf{Q}) = \widehat{H}_{i-1}(G,R),$$

the Tate homology of G with coefficients in R, up to an index shift. The (S,S)-bimodule structure on $H_*(\mathbf{Q}) = \widehat{H}_{*-1}(G,R)$ can be viewed as the (left and right) cap product action of $S = H^0(G,R)$ on $\widehat{H}_*(G,R)$, and this action factors through $\overline{S} = \widehat{H}^0(G,R)$.

We now modify $\mathbf{X}$ and $\mathbf{Q}$ to obtain $\mathbf{P}$. Note that $\mathrm{Im}(\mu) \subseteq \mathrm{ann}_{X_c}(\omega G) = \widehat{G} \cdot X_c$ and so, identifying k with $k \cdot \widehat{G} \subseteq kG$, we have $\mu(\widehat{G}) = \widehat{G} \cdot \xi$ for some $\xi \in X_c$. Define a kG-linear map $\mu' : kG \to X_c$ by $\mu'(1) = \xi$, so $\mu'|_{k\widehat{G}} = \mu$. Therefore, $\phi_c \circ \mu'|_{k\widehat{G}} = 0$, and hence $\mathrm{Im}(\phi_c \circ \mu') \subseteq \mathrm{ann}_{X_c}(\widehat{G}) = (\omega G) \circ X_{c-1}$.

Furthermore, define $\varepsilon': X_1 \to kG$ to be the composite $X_1 \xrightarrow{\varepsilon} k \cong k \cdot \widehat{G} \hookrightarrow kG$. Note that $\phi_1: X_1 \to X_0 = X_c$ factors as $f_1 = \mu' \circ \varepsilon'$.

Now define (S, S)-bimodule maps $\tilde{\mu}: R = R \otimes_{kG} kG \to Q_c = R \otimes_{kG} X_c$ by $\tilde{\mu} = \mathrm{id}_R \otimes \mu'$ and $\tilde{\varepsilon}: Q_1 = R \otimes_{kG} X_1 \to R = R \otimes_{kG} kG$ by $\tilde{\varepsilon} = \mathrm{id}_R \otimes \varepsilon'$. Then the image of $\tilde{\varepsilon}$ is equal to the image of $R \otimes_{kG} k\widehat{G} \to R \otimes_{kG} kG = R$, that is

$$\tilde{\varepsilon}(Q_1) = \mathrm{Tr}(R).$$

Next, we show that $\tilde{\mu}$ is mono on $\mathrm{Tr}(R) \subseteq R$. To see this, embed X_c as a direct summand in a free module $F = kG^{(I)}$, with corresponding map $\lambda: X_c \to F$. Then, putting $\phi = \lambda \circ \mu': kG \to F$ and $\widetilde{\phi} = \mathrm{id}_R \otimes \phi: R \to R \otimes_{kG} F \cong R^{(I)}$, we have $\mathrm{Ker}(\widetilde{\mu}) = \mathrm{Ker}(\widetilde{\phi})$. The map

$$\phi|_{k\widehat{G}}: k\widehat{G} \xrightarrow{\mu} \widehat{G} \cdot X_c \xrightarrow{\lambda} \widehat{G} \cdot F = (\widehat{G}k)^{(I)}$$

is k-split, since $\mathbf{X}$ splits over k, and hence μ does. Therefore, $\phi(\widehat{G}) = (\widehat{G}\xi_i)_{i\in I}$ for certain $\xi_i \in k$ (almost all 0) with $\sum_i \xi_i \tau_i = 1$ for suitable $\tau_i \in k$. Now, for any $r^{\widehat{G}} \in \mathrm{Tr}(R)$, we have in $R \otimes_{kG} F = R^{(I)}$,

$$\begin{aligned}\widetilde{\phi}(r^{\widehat{G}}) &= r^{\widehat{G}} \otimes \phi(1) = r \otimes \phi(\widehat{G}) \\ &= r \otimes (\widehat{G}\xi_i)_{i\in I} = (r^{\widehat{G}}\xi_i)_{i \in I},\end{aligned}$$

and so, if $\widetilde{\phi}(r^{\widehat{G}}) = 0$, then $r^{\widehat{G}} = \sum_i r^{\widehat{G}} \xi_i \tau_i = 0$, as we have claimed. It follows that $\mathrm{Tr}(R) \cdot \mathrm{Ker}(\widetilde{\mu}|_S) = 0 = \mathrm{Ker}(\widetilde{\mu}|_S) \cdot \mathrm{Tr}(R)$.

Finally, we claim that $\widetilde{\mu}(S) \subseteq \mathrm{Ker}(\psi_c)$. Indeed, since

$$\mathrm{Im}(\phi_c \circ \mu') \subseteq (\omega G) \cdot X_{c-1},$$

it follows that $\psi_c \circ \widetilde{\mu}(S)$ is contained in the canonical image of $S \otimes_{kG} (\omega G) X_{c-1}$ in $Q_{c-1} = R \otimes_{kG} X_{c-1}$, and this image equals 0, since $S^{\omega G} = 0$. Thus we have the following complex of (S, S)-bimodules

$$\mathbf{P}: 0 \to P_{c+1} = S \xrightarrow{\widetilde{\mu}|_S} P_c = Q_c \xrightarrow{\psi_c} \cdots \xrightarrow{\psi_2} P_1 = Q_1 \xrightarrow{\widetilde{\varepsilon}} P_0 = S \to 0.$$

Its homology is: $H_0(\mathbf{P}) = \overline{S}$, $H_1(\mathbf{P}) = H_1(\mathbf{Q})$ (because $\psi_1 = \widetilde{\mu} \circ \widetilde{\varepsilon}$ and $\widetilde{\mu}$ is injective on $\mathrm{Im}(\widetilde{\varepsilon}) = \mathrm{Tr}(R)$), and $H_i(\mathbf{P}) = H_i(\mathbf{Q})$ $(i = 1, \ldots, c-1)$. Moreover, $H_{c+1}(\mathbf{P}) = \mathrm{Ker}(\widetilde{\mu}|_S)$ is annihilated, on both sides, by $\mathrm{Tr}(R)$ and $H_c(\mathbf{P}) = \mathrm{Ker}(\psi_c)/\widetilde{\mu}(S)$ is an image of $H_c(\mathbf{Q}) = \mathrm{Ker}(\psi_c)/\psi_1(Q_1) = \mathrm{Ker}(\psi_c)/\widetilde{\mu}(\widetilde{\varepsilon}(Q_1)) = \mathrm{Ker}(\psi_c)/\widetilde{\mu}(\mathrm{Tr}(R))$. Thus (3) is satisfied, and so $\mathbf{P}$ has all the required properties. □

2.6 *Flatness and projectivity.* If the trivial kG-module k is periodic, then Lemma 2.5 guarantees that the hypotheses of Proposition 1.3 are satisfied, except possibly for the flatness of all ${}_S P_i$. For this, it suffices to assume that ${}_S R$ is flat. I am not aware of any results specifically in this direction, but there do exist quite a few results ensuring projectivity of ${}_SR$. For example, ${}_SR$ is known to be projective in each of the following cases:

(a) If the skew group ring $T = R * G$ is a simple ring (it actually suffices to assume that $T = T\widehat{G}T$, where $\widehat{G} = \sum_{g\in G} g \in T$), then R is finitely generated projective over S, on both sides (e.g., [Mo, proof of Theorem 2.4]). This happens, for example, if R has no nontrivial G-invariant ideals and G consists of outer automorphisms of R (use [Mo, Lemma 3.16]).

(b) If R is a finite direct product of simple rings and $|G|^{-1} \in R$, then R is finitely generated projective over S [H-R].

(c) Assume that R is hereditary and $|G|^{-1} \in R$. Then each of the following implies that R is projective over S, although not necessarily finitely generated in all cases: R is semiprime Noetherian, R is commutative von Neumann regular, R is reduced von Neumann regular and G is solvable [J].

2.7 **Theorem.** *Let R be an algebra over a commutative ring k and let G be a finite group of k-algebra automorphisms of R. Put $S = R^G$, $\overline{S} = S/\operatorname{Tr}(R)$ and assume that*

$$\rho = \operatorname{r.gldim}(R) + \operatorname{pdim}(R_S) \quad \textit{and} \quad \sigma = \operatorname{r.gldim}(\overline{S}) + \operatorname{pdim}(\overline{S}_S).$$

are both finite. Assume further that

(1) *${}_SR$ is flat, and*

(2) *the trivial kG-module k is periodic of period c.*

Then, for any S-modules V_S and W_S, there exist homomorphisms

$$f^q\colon \operatorname{Ext}_S^{q+c}(W, V) \to \operatorname{Ext}_S^q(W, V) \qquad (q > \rho)$$

such that f^q is epi for $q > \max\{\rho, \sigma\}$ and is an isomorphism for $q > \max\{\rho, \sigma+1\}$. Consequently, either $\operatorname{r.gldim}(S) \le \max\{\rho, \sigma\}$ or $\operatorname{r.gldim}(S) = \infty$.

Proof. In view of the remarks in 2.6, this follows from Lemma 2.5 and Proposition 1.3. □

2.8 *The case $|G| = 2$.* We end this article by illustrating Theorem 2.7 in the special case where $|G| = 2$, $2k = 0$, and the skew group ring $T = T * G$ is simple, or at least $T = T\widehat{G}T$. In this case, we have $\rho = \operatorname{r.gldim}(R)$ (see 2.6). Moreover, the exact sequence of S-modules

$$0 \to S \xrightarrow{\text{incl.}} R \xrightarrow{\operatorname{Tr}} S \to \overline{S} \to 0$$

shows that $\operatorname{pdim}(\overline{S}_S) \le 2$. Therefore, $\sigma \le \operatorname{r.gldim}(\overline{S}) + 2$. Theorem 2.7 now implies that

—either $\operatorname{r.gldim}(S) = \infty$

—or $\operatorname{r.gldim}(S) \le \max\{\operatorname{r.gldim}(R), \operatorname{r.gldim}(\overline{S}) + 2\}$.

For a lower bound, we note that if $\operatorname{Tr}(R) \ne S$, then $\operatorname{pdim}(\overline{S}) \ge 2$, because otherwise $\operatorname{Tr}(R)$ would be projective over S, and hence S would be an S-module direct summand of R. But the equality $T = T\widehat{G}T$ implies that $R = R \cdot \operatorname{Tr}(R)$, which leads to a contradiction, since $\operatorname{Tr}(R) \ne S$.

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