GOLDIE RANKS OF PRIME POLYCYCLIC CROSSED PRODUCTS

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Abstract. We use Moody's "Brauer induction theorem" for crossed products $S^*\Gamma$ of polycyclic-by-finite groups Γ over right Noetherian rings S to determine the Goldie rank of $S^*\Gamma$ in certain cases.

Introduction. In [6], S. Rosset stated a conjecture on the structure of the Grothendieck group $G_0(k[\Gamma])$ of all finitely generated modules over the group ring $k[\Gamma]$ of a polycyclic-by-finite group Γ over a commutative right Noetherian domain k. He also showed that this conjecture implies the so- called Goldie rank conjecture for prime polycyclic group algebras. Recently, J. A. Moody [3] has confirmed Rosset's conjecture on G_0, even for *crossed products* $S^*\Gamma$, where Γ is a polycyclic-by-finite group and S is a right Noetherian ring. In this note, we apply Moody's result to obtain estimates for the Goldie rank of $S^*\Gamma$ in the case where $S^*\Gamma$ is prime. These estimates do in particular yield a slightly generalized version of the Goldie rank conjecture for group rings, as well as a zero divisor theorem for polycyclic crossed products. Our approach to Goldie rank differs from Rosset's in that it is non-homological and is based directly on Goldie's reduced rank function.

1.

Let $S^*\Gamma$ be a crossed product of the group Γ over the ring S, and assume that $S^*\Gamma$ is right Noetherian. Then, for any finitely generated $S^*\Gamma$-module V, we can define the *normalized reduced rank* of V by

$$\chi_{S^*\Gamma}(V) = \rho(V)/\rho(S^*\Gamma),$$

where ρ denotes the usual (Goldie-) reduced rank function.

Lemma. *Suppose that* $S^*\Gamma$ *is prime, and let* H *be a subgroup of* Γ *having finite index in* Γ. *Then, for any finitely generated* $S^*\Gamma$*-module* V, *we have*

$$\chi_{S^*H}(V) = [\Gamma : H]\cdot \chi_{S^*\Gamma}(V).$$

Proof. Fix a normal subgroup N of Γ with $N \subseteq H$ and with $[\Gamma : N]$ finite. Let C denote the set of regular elements of S^*N. Since $S^*\Gamma$ is prime, S^*N must be semiprime (in fact, Γ/N-prime). Thus C is a right Ore set in S^*N, and also in S^*H and $S^*\Gamma$ (cf. [4], proof of Lemma 13.3.5(ii)). Moreover, letting $Q(.)$ denote classical rings of fractions, we have

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