FROBENIUS RECIPROCITY AND G_0 OF SKEW GROUP RINGS

Martin Lorenz
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3, Fed. Rep. Germany

The aim of these notes is to generalize the main result of the author's article [5] while also substantially simplifying the K_0-theoretic part of its proof. In particular, the Morita context for group actions that played a central role in [5] doesn't occur here and is replaced by a suitable version of Frobenius reciprocity.

Our notations and conventions follow [5]. All modules will be right modules. Further assumptions will be introduced as we go along.

1. DIAGONAL ACTIONS ON TENSOR PRODUCTS.

Let R_i $(i=1,2)$ be algebras over some commutative ring k and let G be a group acting on each R_i by k-algebra automorphisms. The corresponding skew group rings will be denoted by $S_i = R_i * G$. They are k-algebras, hence we can form the k-algebra $S_1 \otimes_k S_2$. Indeed, $S_1 \otimes_k S_2$ is isomorphic to the skew group ring of $G \times G$ over $R_1 \otimes_k R_2$ with respect to the obvious action of $G \times G$ on $R_1 \otimes_k R_2$. The maps $R_1 \longrightarrow R_1 \otimes_k R_2$, $r \longmapsto r \otimes 1$, and $G \longrightarrow G \times G$, $g \longmapsto (g,g)$, give rise to a k-algebra map $\mu_1 : S_1 \longrightarrow S_1 \otimes_k S_2$. Explicitly, μ_1 is given by

$$\mu_1\left(\sum_{g\in G} r_g g\right) = \sum_{g\in G} r_g\, g \otimes g \quad ,$$

and $\mu_2 : S_2 \longrightarrow S_1 \otimes_k S_2$ is defined similarly.

Now let V_i be S_i-modules $(i=1,2)$. Then $V_1 \otimes_k V_2$ is a module over $S_1 \otimes_k S_2$ in the usual fashion, and hence via μ_i a module over each S_i. Specifically we have for S_1, say,

$$(v_1 \otimes v_2) r_1 g = v_1 r_1 g \otimes v_2 g \qquad (v_i \in V_i, r_1 \in R_1, g \in G) \quad .$$

2. FROBENIUS RECIPROCITY.

Let H be a subgroup of G and let $T_i = R_i * H \subseteq S_i$ be the corresponding skew group rings. If V is an S_1-module and W is a T_2-module, then

$$V \otimes_k \left(W \otimes_{T_2} S_2\right) \cong \left(V\big|_{T_1} \otimes_k W\right) \otimes_{T_i} S_i$$

as S_i-modules $(i=1,2)$, with the diagonal module operations on tensor products over k, as defined in Section 1.

PROOF. The k-linear map $W \longrightarrow W \otimes_{T_2} S_2$, $w \longmapsto w \otimes 1$, yields a k-linear map $V \otimes_k W \longrightarrow V \otimes_k (W \otimes_{T_2} S_2)$, $v \otimes w \longmapsto v \otimes (w \otimes 1)$. This map is linear over $T_1 \otimes_k T_2$:

$$(v \otimes w)(r_1h_1 \otimes r_2h_2) = vr_1h_1 \otimes wr_2h_2 \longmapsto vr_1h_1 \otimes (wr_2h_2 \otimes 1) =$$
$$= vr_1h_1 \otimes (w \otimes 1)r_2h_2 = (v \otimes (w \otimes 1))(r_1h_1 \otimes r_2h_2) \quad .$$

Therefore, this map is linear over each T_i (acting via μ_i). Since $V \otimes_k (W \otimes_{T_2} S_2)$ is a module over S_i, we obtain S_i-linear maps

$$\varphi_i : \left(V\big|_{T_1} \otimes_k W\right) \otimes_{T_i} S_i \longrightarrow V \otimes_k \left(W \otimes_{T_2} S_2\right)$$
$$(v \otimes w) \otimes s \longmapsto (v \otimes (w \otimes 1)) \cdot s \quad .$$

In particular, we have for $i=1,2$:

$$\varphi_i((v \otimes w) \otimes g) = vg \otimes (w \otimes g) \qquad (v \in V, w \in W, g \in G) .$$

Note that, as modules over the group algebra $kG \subseteq S_i$, we have

$$(V \otimes_k W) \otimes_{T_i} S_i\Big|_{kG} \cong (V \otimes_k W) \otimes_{kH} kG$$

and

$$V \otimes_k \left(W \otimes_{T_2} S_2\right)\Big|_{kG} \cong V \otimes_k \left(W \otimes_{kH} kG\right) \quad .$$

Moreover, the map φ_i is the usual Frobenius reciprocity isomorphism between the modules on the right hand sides. Hence, in particular, it is bijective (cf. e.g. [9, Thm.2.2 on p.15]).

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3. G_0 and K_0.

From now on, we assume that $R_2 = k$ is a field and that the group G is finite. We will write $R_1 = R$ and $S = R * G$. Our goal is to study the Grothendieck groups $G_0(S)$ and $K_0(S)$ of fin.gen. S-modules, resp. fin.gen. projective S-modules, and their analogs for R and kG. Henceforth, we will implicitly assume that S, or equivalently R, is right Noetherian so that $G_0(S)$ and $G_0(R)$ are defined.

$G_0(kG)$ is a commutative ring with 1, with multiplication afforded by $\otimes_k$ and $1 = [k]$, k the "trivial" kG-module. Clearly, if W is a kG-module then $(.) \otimes_k W$ transforms exact sequences of S-modules into exact sequences of S-modules, where S operates via $\mu = \mu_1 : S \longrightarrow S \otimes_k kG$. Moreover, if W is fin.gen. over kG and V is fin.gen. over S then $V \otimes_k W$ is also fin.gen. over S. Therefore, $[V] \longmapsto [V \otimes_k W]$ yields an endomorphism of $G_0(S)$. It is easy to check that setting

$$[V] \cdot [W] := [V \otimes_k W]$$

we obtain a well-defined module action of $G_0(kG)$ on $G_0(S)$: $[V \otimes_k W]$ depends only on the class $[W]$ of W in $G_0(kG)$, $V_1 \otimes_k (W_1 \otimes_k W_2) \cong (V \otimes_k W_1) \otimes_k W_2$ holds for all kG-modules W_i, and $V \otimes_k k \cong V$ as S-modules.

The same definitions also make $K_0(S)$ a module over $G_0(kG)$. For this, one has to check that if V is fin.gen. projective over S then so is $V \otimes_k W$, for any fin.gen. kG-module W. It suffices to do this for $V = S$: Using Frobenius reciprocity with $H = \langle 1 \rangle$ we get

$$S \otimes_k W = (R \otimes_R S) \otimes_k W \cong (R \otimes_k W) \otimes_R S \cong R^{\dim_k W} \otimes_R S \cong S^{\dim_k W}$$

as required. - The canonical Cartan map $c : K_0(S) \longrightarrow G_0(S)$ is a $G_0(kG)$-module homomorphism.

LEMMA 1. The map $\mathrm{Ind}_R^S \circ \mathrm{Res}_R^S : G_0(S) \longrightarrow G_0(R) \longrightarrow G_0(S)$ is multiplication by $[kG] \in G_0(kG)$ on $G_0(S)$. The same also holds for K_0 instead of G_0 .

PROOF. If V is a fin.gen. S-module then, using Frobenius reciprocity with $H = \langle 1 \rangle$, we otain

$$\mathrm{Ind}_R^S \circ \mathrm{Res}_R^S (V) = V \otimes_R S = (V \otimes_k k) \otimes_R S \cong V \otimes_k (k \otimes_k kG) \cong V \otimes_k kG \quad ,$$

which proves the lemma.

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We note one particular consequence of the lemma that will be used in the next section: The ring R becomes an S-module via the obvious isomorphism

$$\left(\sum_{x \in G} x\right) S = \left(\sum_{x \in G} x\right) R \cong R \quad .$$

Clearly, $\mathrm{Ind}_R^S \circ \mathrm{Res}_R^S (R_S) = \mathrm{Ind}_R^S (R_R) = S_S$, and so the lemma implies the following

CORROLARY. $[S] = [R_S] \cdot [kG]$ holds in $G_0(S)$.

4. p-GROUPS IN CHARACTERISTIC p .

If $\mathrm{char}\, k = p > 0$ and G_p is a Sylow p-subgroup of G , then $G_0(kG_p) = \langle [k] \rangle \cong \mathbb{Z}$ and, in particular, $[kG_p] = |G_p| \cdot [k]$. Therefore, in $G_0(kG)$ we have $[kG] = |G_p| \cdot \mathrm{Ind}_{kG_p}^{kG} [k]$, and the above corollary gives

$$[S] = |G_p| \cdot [R_S] \cdot \mathrm{Ind}_{kG_p}^{kG} [k]$$

in $G_0(S)$. The following lemma is now obvious.

LEMMA 2. Let $\operatorname{char} k = p > 0$ and let G_p be a Sylow p-subgroup of G . Then, for any homomorphism $\rho : G_0(S) \longrightarrow \mathbb{Z}$, we have $|G_p| \mid \rho(S)$.

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Here we have written $\rho(S) = \rho([S])$, for simplicity. Perhaps the most commonly used homomorphism $G_0(T) \longrightarrow \mathbb{Z}$, for any right Noetherian ring T , is Goldie's reduced rank function (cf. [2, Sect.2]). A quick definition of this function can be given as follows. Let N denote the nilpotent radical of T . Then the canonical inflation (or restriction) map $G_0(T/N) \longrightarrow G_0(T)$ is an isomorphism ([1, p.454]). Moreover, T/N has an Artinian ring of quotients $Q = Q(T/N)$. The reduced rank function is the composite function

$$G_0(T) \xrightarrow{\sim} G_0(T/N) \xrightarrow{\cdot \otimes_{T/N} Q} G_0(Q) \xrightarrow[\text{length over } Q]{\text{composition}} \mathbb{Z}$$

The following is the main result of this note. It extends [5, Theorem 2.4].

THEOREM. Assume that

(a) $\operatorname{char} k = p$ and G is a finite p-group $\neq \langle 1 \rangle$,

(b) $K_0(R) = \langle [R] \rangle$, that is, all fin.gen. projective R-modules are stable free,

and (c) $1 \notin [S,S] = \{\Sigma_i s_i t_i - t_i s_i \mid s_i, t_i \in S\}$, that is, $S = R * G$ has a trace function which does not vanish on 1 .

Then, for any homomorphism $\rho : G_0(S) \longrightarrow \mathbb{Z}$, one has $p \mid \rho(P)$ for all fin.gen. projective S-modules P .

PROOF. Let P be a fin.gen. projective S-module with $p \nmid \rho(P)$. By (b), $[P_R] = n \cdot [R]$ for some n . After replacing P by $P \oplus S^m$ for a suitable m , we may assume that $n \geq 0$. (Use Lemma 2. Actually, $n \geq 0$ is automatic, since $M_t(R)$ is right Noetherian and hence directly finite for all t , cf. [4, Prop. 15.3].) In view of assumption (a), Lemma 1 yields the following equalities in $K_0(S)$:

$$|G|\cdot[P] = [P \otimes_R S] = n\cdot[S] \quad .$$

Applying ρ (or $\rho \circ c$ rather) and using the fact that $|G|$ divides $\rho(S)$, by Lemma 2, we see that n divides $\rho(P)$, so that $p \nmid n$. The equality $|G| \cdot [P] = n\cdot[S]$ in $K_0(S)$ says that, for some $r \geq 0$,

$$P^{|G|} \oplus S^r \cong S^{n+r} \quad .$$

We may clearly assume that $p \mid r$, say $r = pr'$. Thus, setting $V = P^{|G|/p} \oplus S^{r'}$, we have $S^{n+r} \cong V^p$ and, taking endomorphism rings, we obtain a ring isomorphism

$$M_{n+r}(S) \cong M_p(\mathrm{End}\, V_S) \quad .$$

By (c), the universal trace $tr : S \longrightarrow S/[S,S] =: A$ does not vanish on 1. Defining, as usual, $tr_{n+r} : M_{n+r}(S) \longrightarrow A$ by $tr_{n+r}([s_{ij}]) = \sum_i tr(s_{ii})$ we obtain a trace function for $M_{n+r}(S)$ with $tr_{n+r}(1_{n+r}) = (n+r)\cdot tr(1) \neq 0$. Here we have used the fact that $p \nmid n+r$ so that $n+r$ acts injectively on the k-space A. Therefore, $1_{n+r} \notin [M_{n+r}(S), M_{n+r}(S)]$.

On the other hand, in $M_p(k) \subset M_p(\mathrm{End}\, V_S)$, the identity is a Lie commutator: $1 = [A,B]$ for $A = \sum_{i=1}^{p-1} iE_{i,i+1}$ and $B = \sum_{i=1}^{p-1} E_{i+1,i}$. This is a contradiction, whence $p \mid \rho(P)$, as asserted.

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5. SOME REMARKS.

(a) It is not enough to merely assume that $p \mid |G|$ in the above theorem. For example, if $G = S_4$ is the symmetric group on four letters and $S = kG$ is the group algebra of G over $k = \overline{\mathbb{F}_3} = R$ (so $p = 3$), then S has two simple projective modules (cf. [8, p.166]). Thus the theorem fails for the composition length function ρ.

(b) Examples of rings which satisfy hypothesis (b) of the theorem include local rings and iterated polynomial rings over fields or, more generally, enveloping algebras of finite-dimensional Lie-algebras ([7, p.122]). Moreover, by the "twisted Grothendieck theorem" [3, Thm. 27], if R is right Noetherian of finite global dimension with (b), then (b) also holds for any skew polynomial or skew Laurent extension of R. It follows from a much more general recent theorem of J. Moody [6] that group rings of torsion-free polycyclic-by-finite groups over Noetherian domains of finite global dimension with (b) also satisfy (b).

(c) Viewing R as a subring of $S = R * G$ via $r \longmapsto r \cdot 1$ (1 = neutral element of G = identity of S), a trace function of R with values in some abelian group A, $tr : R \longrightarrow A$, extends to a trace $Tr : S \longrightarrow A$ exactly if tr is G-invariant, that is $tr(r^x) = tr(r)$ holds for all $r \in R, x \in G$. Indeed, if Tr exists then $tr(r) = Tr(rx \cdot x^{-1}) = Tr(x^{-1} \cdot rx) = tr(r^x)$. Conversely, if tr is G-invariant, then setting $Tr(\Sigma_{x \in G} r_x x) = tr(r_1)$ gives the desired extension. *Thus hypothesis (c) in the theorem is satisfied precisely if R has a G-invariant trace function which is nonzero for $1 \in R$.*

(d) Any (Morita-) equivalence of module categories $\text{mod} - S \xrightarrow{\sim} \text{mod} - T$, T any ring, induces a commutative diagram

$$\begin{array}{ccc} G_0(S) & \xrightarrow{\sim} & G_0(T) \\ \uparrow & & \uparrow \\ K_0(S) & \xrightarrow{\sim} & K_0(T) \end{array}$$

where the vertical maps are the Cartan maps. *Therefore, if S satisfies hypotheses (a) - (c) of the theorem, then the conclusion of the theorem also holds for rings T which are Morita equivalent to S.* In particular, if $\rho : G_0(T) \longrightarrow \mathbb{Z}$ is Goldie's, reduced rank function for such a ring T, then we must have $p \mid \rho(T)$ which certainly rules out the case where T is a domain.

ACKNOWLEDGEMENT. The author is supported by the Deutsche Forschungsgemeinschaft / Heisenberg Programm (Lo 261/2-2).

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