Contemporary Mathematics
Volume 86, 1989

ON THE NUMERICAL LOCAL LANGLANDS CONJECTURE

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The purpose of this talk was to state the numerical local Langlands conjecture (NLLC) and to describe a few reductions. The material presented here is due to H. Koch and E.W. Zink and is based on the articles [63], [65]. These notes contain only one proof which simplifies Koch's original argument somewhat. In the last talk of this conference, G. Henniart announced and outlined his solution of the NLLC which does in fact use the reductions described in the following.

Notations and Conventions. Throughout, F will denote a local field, G_F the Galois group of a fixed separable closure F^{sep} of F, and $\hat{G}_F$ the set of equivalence classes of finite irreducible complex representations of G_F. Moreover, n will be a fixed positive integer, D will be a division F-algebra with $[D:F] = n^2$, and $\hat{D}^*$ will denote the set of finite irreducible complex representations of the multiplicative group D^* of D. Sets of representations will be denoted by upper case script letters, and the corresponding non-script letter will denote the cardinality of this set (e. g., card $\mathcal{R}_{n,j} = R_{n,j}$, etc.).

§ 1. Statement of the numerical local Langlands conjecture

The conjectural Langlands bijection

$$\Phi:\ \mathcal{R}_n := \{\Pi \in \hat{G}_F \mid \dim \Pi | n\} \xrightarrow{1-1} \hat{D}^*$$

should have the following properties (amongst others which are irrelevant for our purposes here): For any $\Pi \in \mathcal{R}_n$, one should have

(1) The Swan conductor of Π, $sw(\Pi)$, and the index of $\Phi(\Pi)$, $j(\Phi(\Pi))$, are related by

$$n_\Pi \cdot sw(\Pi) = j(\Phi(\Pi)),$$

where we have set $n_\Pi := n/\dim \Pi$.

Moreover, viewing 1-dimensional representations χ of G_F as characters of F^* via class field theory, the following should hold:

(2) $(\det \Pi)^{n_\Pi}$ is the central character of $\Phi(\Pi)$.

(3) $\Phi(\chi) = \chi \circ \mathrm{Nrd}_{D/F}$ ($\mathrm{Nrd}_{D/F}$ = reduced norm), and

$$\Phi(\Pi \otimes \chi) = \Phi(\Pi) \otimes \Phi(\chi) .$$

It follows from (1) and (2) that Φ will in particular yield bijections, for all $j \geq 0$,

$$\mathcal{R}_{n,j} := \{\Pi \in \mathcal{R}_n \mid n_\Pi \cdot sw(\Pi) \leq j \ , \ (\det \Pi)(\pi_F^{n_\Pi}) = 1\}$$

$$\xrightarrow{1-1} \ \mathcal{S}_{n,j} := \{\Sigma \in \hat{D}^* \mid j(\Sigma) \leq j \ , \ \Sigma(\pi_F) = 1\} ,$$

where π_F is a fixed prime element of F . The NLLC can now be stated as follows:

For all $j \geq 0$, one has $R_{n,j} = S_{n,j}$.

The right hand side of this equality is explicitely known ([61]; cf. Prof. Geyer's talk during the conference):

$$S_{n,j} = \sum_{f|n} \frac{n}{f^2} \sum_{d|f} \mu(\tfrac{f}{d})(q^d-1)q^{d[j/f]} ,$$

where $q = p^*$ is the cardinality of the residue field k of F and [ ] is the usual Gauss bracket.

Actually, a somewhat refined version of the above equality is usually considered. To explain this, let $G_{F,o} \subseteq G_F$ denote the inertia subgroup of G_F , let $U_D \subseteq D^*$ be the unit group of D^* , and put, for $f|n$,

$$\mathcal{R}_{n,j,f} := \{\Pi \in \mathcal{R}_{n,j} \mid \Pi|_{G_{F,o}} \text{ has (composition) length } f\}$$

and

$$\mathcal{S}_{n,j,f} := \{\Sigma \in \mathcal{S}_{n,j} \mid \Sigma|_{U_D} \text{ has length } f\} .$$

A slight variation of the arguments in [2, proof of Satz 19] and very little additional work yields the following formula [ ]:

$$S_{n,j,f} = \frac{n}{f^2} \sum_{d|f} \mu(\frac{f}{d})(q^d-1)q^{d[j/f]} ,$$

which of course implies the above formula for $S_{n,j}$. Furthermore, using fairly standard arguments from Clifford-Mackey theory for cyclic group extensions one easily establishes the following

Proposition. Suppose that $\Psi: \mathcal{R}_n \to \hat{D}^*$ is injective and satisfies (3). Then, for all $\Pi \in \mathcal{R}_n$, length $(\Pi|_{G_{F,o}})$ = length $(\Psi(\Pi)\,_{U_D})$.

In particular, the conjectural Langlands bijection Φ should map $\mathcal{R}_{n,j,f}$ onto $\mathcal{S}_{n,j,f}$. So one is lead to consider the following refined version of the NLLC (for all F, n, j, f as above):

(*) $$\boxed{R_{n,j,f} = \frac{n}{f^2} \sum_{d|f} \mu(\frac{f}{d})(q^d-1)q^{d[j/f]}} .$$

For f = 1 (the case where $\Pi|_{G_{F,o}}$ is irreducible), this specializes to

(**) $$\boxed{R_{n,j,1} = n\cdot(q-1)\cdot q^j} .$$

The main result to be explained below reduces the proof of (*) to the proof of (**), where one can also assume that n is a power of p = char k .

Examples. (a) The case j = 0 : Letting $G_{F,1} \subseteq G_{F,o}$ denote the wild ramification group of G_F , one has

$$\mathcal{R}_{n,o,1} = \{\Pi \in \mathcal{R}_n \mid \Pi|_{G_{F,o}} \text{ is irreducible, } \Pi|_{G_{F,1}} \text{ is trivial,}$$

$$(\det \Pi)(\pi_F^{n_\Pi}) = 1\} .$$

Such Π's must be 1-dimensional, since $G_{F,o}/G_{F,1}$ is topologically cyclic, and so class field theory identifies $\mathcal{R}_{n,o,1}$ with $F^*/\langle U_F^1, \pi_F^n\rangle \simeq \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ (U_F^1 = one-unit group of F^*). Thus $R_{n,o,1} = n\cdot(q-1)$, as required.

(b) The case $n = 1$: Again by class field theory, one can identify $\mathcal{R}_{1,j,1}$ with the set of characters χ of F^* such that $\chi|_{U_F^{j+1}}$ is trivial and $\chi(\pi_F) = 1$. Thus $\mathcal{R}_{1,j,1} \xleftrightarrow{1-1} F^*/< U_F^{j+1}, \pi_F >$ and so $R_{1,j,1} = (q-1)q^j$.

Remarks. (a) Put $\mathcal{C}_n := \{\sigma \in \hat{G}_F \mid \sigma$ is trivial on $G_{F,o}$ and $\pi_F^n\}$. Then $\mathcal{C}_n$ is a cyclic group of order n which operates on $\mathcal{R}_{n,j,1}$ via $\Pi \longmapsto \Pi \otimes \sigma$. All orbits in $\mathcal{R}_{n,j,1}$ have length n , and restriction of representations from G_F to $G_{F,o}$ yields a bijection of $\mathcal{R}_{n,j,1}/\mathcal{C}_n$ onto

$$\mathcal{T}_{n,j} := \{\Gamma \in \hat{G}_{F,o} \mid \dim \Gamma | n \ , \ \Gamma \text{ is } G_F\text{-invariant}, \ \Gamma|_{G_F^{(j/n)+\varepsilon}} \text{ is trivial for all } \varepsilon > 0\}$$

Here, as usual, G_F^j denotes the j-th ramification group of G_F . In particular, $R_{n,j,1} = n \cdot T_{n,j}$ and (**) is equivalent with $T_{n,j} = (q-1) \cdot q^j$.

(b) Similarly, if $\mathcal{D}$ denotes the set of G_F-invariant characters σ of $G_{F,o}$ which are trivial on $G_{F,1}$, then $\mathcal{D}$ is a cyclic group of order $q-1$ which acts on $\mathcal{T}_{n,j}$ via $\Gamma \longmapsto \Gamma \otimes \sigma$. In the case where n is a p-power (the crucial case) all orbits have length $q-1$ and restriction from $G_{F,o}$ to $G_{F,1}$ identifies $\mathcal{T}_{n,j}/\mathcal{D}$ with

$$\mathcal{U}_{n,j} := \{\Delta \in \hat{G}_{F,1} \mid \dim \Delta | n \ , \ \Delta \text{ is } G_F\text{-invariant}, \ \Delta|_{G_F^{(j/n)+\epsilon}} \text{ is trivial for all } \varepsilon > 0\} .$$

Therefore, $R_{n,j,1} = n \cdot (q-1) \cdot U_{n,j}$ and (**) becomes $U_{n,j} = q^j$.

§ 2. Finiteness and Some Reductions

Here we describe the main results of [63], which in particular reduce the proof of (*) to the proof of (**) for p-powers n .

For a given $\alpha \in F^*$ with $v_F(\alpha) \nmid 0$ and a root of unity $\zeta \in \mathbb{C}^*$, put

$$R_{n,j}(\alpha,\zeta) = \{\Pi \in R_n \mid n_\Pi\cdot sw(\Pi) \le j\ ,\quad \det \Pi(\alpha^{n_\Pi}) = \zeta\}\ ,$$

$$R_{n,j,f}(\alpha,\zeta) = \{\Pi \in R_{n,j}(\alpha,\zeta) \mid \Pi|_{G_{F,o}} \text{ has length } f\}\ .$$

Thus $R_{n,j} = R_{n,j}(\pi_F,1)$ and similarly for $R_{n,j,f}$. Part (b) of the following result ensures the independence of $R_{n,j,f}$ of the choice of the uniformizer $\pi_F \in F$.

Theorem 1. (a) $R_{n,j}(\alpha,\zeta)$ is finite.

(b) $R_{n,j,f}(\alpha,\zeta) = |v_F(\alpha)|\cdot R_{n,j,f}$.

The proof of (b) is relatively straightforward and, of course, part (a) is now clear in view of Henniart's solution of the NLLC. Nevertheless, we will sketch an easy proof of (a) in § 3. In the statement of the next result, due to Koch, our previous notations will be extended in an obvious manner so as to make the ground field in question explicit.

Theorem 2. (a) $R_{n,j,f}(F) = \frac{1}{f} \sum_{d|f} \mu(\frac{f}{d})\cdot R_{n/f,[j/f],1}(F_d)$,

where $F_d/F \subseteq F^{sep}/F$ is the unramified extension of degree d.

(b) Write $n = n_p\cdot m$ with n_p a p-power and $(p,m) = 1$. Then

$$R_{n,j,1}(F) = \sum_L R_{n_p,j,1}(L) \quad (= m\cdot R_{n_p,j,1}(F))\ ,$$

where L runs over the totally ramified extensions of degree m of F (inside F^{sep}).

Part (a) reduces (*) to (**), and (b) reduces the general case of (**) to the case where n is a p-power (there are m such L). The proofs use Clifford-Mackey theory for the group extensions

$$1 \to G_{F,o} \to G_F \to G_F/G_{F,o} \approx \hat{\mathbb{Z}} \to 0 \quad \text{(part (a))}$$

$$1 \to G_{F,1} \to G_F \to G_F/G_{F,1} \to 1 \quad \text{(part (b))}\ .$$

Since $R_{1,j,1}$ is of the desired form, by Example (b), Theorem 2 proves the NLLC for all n with $p \nmid n$.

§ 3. A Proof of Theorem 1(a).

We first note some standard facts (e. g., [64]). For any finite field extension M/L with $F \subseteq L \subseteq M \subseteq F^{sep}$, put

$$sw(M/L) := d(M/L) - e(M/L) + 1 ,$$

where $d(\cdot)$ denotes the exponent of the different. Then $sw(M/L) \geq 0$, and equality holds precisely when M/L is tamely ramified.
The following formulas will be implicitely used below:

(a) For $\Pi \in \hat{G}_M$ and $\alpha \in L^*$,

$$sw(\mathrm{Ind}_{G_M}^{G_L} \Pi) = f(M/L)\cdot(sw(M/L)\cdot \dim \Pi + sw(\Pi)) ,$$

$$\det(\mathrm{Ind}_{G_M}^{G_L} \Pi)(\alpha) = \det(\mathrm{Ind}_{G_M}^{G_L} 1)(\alpha)^{\dim \Pi}\cdot \det \Pi(\alpha) .$$

(b) For $\Pi \in \hat{G}_L$ and M/L tamely ramified,

$$sw(\Pi|_{G_M}) = e(M/L)\cdot sw(\Pi) ,$$

$$\det(\Pi|_{G_M})(\beta) = \det \Pi(N_{M/L}\, \beta) \qquad (\beta \in M^*) .$$

Now let n, j, α and ζ be given, as above. The proof of Theorem 1(a) proceeds by induction on $\dim \Pi$, for $\Pi \in \mathcal{R}_{n,j}(\alpha,\zeta)$. Those Π with $\dim \Pi = 1$ correspond, by class field theory, to characters of F^* which factor over $F^*/< U_F^{[j/n]+1}, \alpha^{n\cdot t} >$, where t denotes the order of ζ. As $v_F(\alpha) \neq 0$, the latter group is finite and so there are only finitely many such Π's. Suppose now that Π is induced, say $\Pi = \mathrm{Ind}_{G_L}^{G_F} \Pi'$ for $\Pi' \in \hat{G}_L$, L/F a finite subextension of F^{sep}/F. Then $[L:F] \leq n$ and $sw(L/F) \leq j$, whence only finitely many such L do exist (Krasner, Serre [100]). Moreover,

$$\Pi' \in \mathcal{R}_{n/[L:F],j}(\alpha, \pm \zeta)(L) .$$

If $L \neq F$, then our induction hypothesis implies that there are only finitely many possibilities for Π', hence for Π. In particular, we see that $\mathcal{R}_{n,j}(\alpha,\zeta)$ contains only finitely many monomial representations, and we can concentrate on the case where Π is primitive.

Put $G := \Pi(G_F) \subseteq GL_n(\mathbb{C})$. Then G is a finite linear group of the form $G = G_{E/F}$ for E/F finite Galois. We let $G_o \subseteq G$ denote the inertia group of G and $G_1 \subseteq G_o$ the wild ramification group, a finite p-group. Then G_1 is contained in the Fitting subgroup $N := \mathrm{Fitt}(G)$ of G, and so G/N is metacyclic and N corresponds to a tame Galois extension H/F. There are upper bounds for the index $[G: \mathrm{Fitt}(G)]$ in terms of n (for example, Jordan's theorem yields a rough bound). So, again by the result of Krasner-Serre quoted above, there are only finitely many possibilities for H. Now we use the following generalization of Blichfeldt's theorem (cf.[53 , Theorem 6.22] for an even more general result):

Lemma. Let G be a finite group and let U be a normal subgroup of G such that G/U is supersolvable. Then, for each $\Pi \in \hat{G}$, there exists a subgroup $H \leq G$ with $U \subseteq H$ and a $\Sigma \in \hat{H}$ with $\Sigma|_U$ irreducible such that $\Pi = \mathrm{Ind}_H^G \Sigma$.

Inasmuch as Π is primitive, we conclude from the lemma that $(\Pi|_N = \Pi|_{G_H})$ is irreducible. Hence the same lemma implies that $\Pi|_N = \Pi|_{G_H}$ is monomial. Moreover, $\Pi|_{G_H} \in R_{n,j\ e(H/F)}(\alpha, \zeta^{[H:F]})(H)$ and so, by our above remark about monomial representations, there are only finitely many possibilities for $\Pi|_{G_H}$. Since the different Π's corresponding to one $\Pi|_{G_H}$ only differ by a character of $G_{H/F}$, the number of possible Π's is bounded. This completes the proof.

References

[53] I.M. Isaacs: Character Theory of Finite Groups. Academic Press, New York 1976.

[61] H.Koch: Eisensteinsche Polynomfolgen und Arithmetik in Divisionsalgebren über lokalen Körpern; Math. Nachr. 104 (1981), 239 - 251.

[63] H. Koch: Bemerkungen zur numerischen lokalen Langlands-Vermutung; Trudy Math. Inst. Steklov 163 (1984), 108 - 114.

[64] H.Koch, E.-W. Zink: Zur Korrespondenz von Darstellungen der Galoisgruppen und der zentralen Divisionsalgebren über lokalen Körpern (der zahme Fall); Math. Nachr. 98 (1980), 83 - 119.

[100] J.P. Serre: Une "formule de masse" pour les extensions totalement ramifieés de degreé donné d'un corps local; C.R. Acad. Sci. Paris, Serie A, 286 (1978), 1031 - 1036.