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INDUCED RESOLUTIONS AND GROTHENDIECK GROUPS OF POLYCYCLIC-BY-FINITE GROUPS

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Let Γ be a polycyclic-by-finite group, R a commutative Noetherian ring, $G_0(R\Gamma)$ the Grothendieck group of finitely generated $R\Gamma$-modules, and $G_0(R\Gamma, \mathbf{F})$ the subgroup generated by the classes of modules induced from finite subgroups of Γ. It is expected that $G_0(R\Gamma) = G_0(R\Gamma, \mathbf{F})$. As partial evidence for this, we show that $G_0(R\Gamma)/G_0(R\Gamma, \mathbf{F})$ is torsion, with an explicit bound for the exponent, in the case where Γ is abelian-by-finite and R a regular Noetherian Hilbert ring of finite global dimension.

1. Introduction

Let Γ be a group, let R be a commutative Noetherian ring, and let $G_0(R\Gamma)$ denote the Grothendieck group of finitely generated $R\Gamma$-modules. Let $\mathbf{X}$ be a class of groups, and let $G_0(R\Gamma, \mathbf{X})$ denote the subgroup of $G_0(R\Gamma)$ generated by the classes of modules of the form $M \otimes_{RH} R\Gamma$, where H is an $\mathbf{X}$-subgroup of Γ and M is a finitely generated RH-module. Let $\mathbf{F}$ be the class of finite groups.

Suppose Γ is torsion-free polycyclic-by-finite. Then $\{1\}$ is the only $\mathbf{F}$-*subgroup of* Γ, so $G_0(R\Gamma, \mathbf{F})$ is the image of the induction map $G_0(R) \to G_0(R\Gamma)$. When $R = \mathbb{Z}$, the Cartan homomorphisms $K_0(\mathbb{Z}) \to G_0(\mathbb{Z})$, $K_0(\mathbb{Z}\Gamma) \to G_0(\mathbb{Z}\Gamma)$ are isomorphisms, since $\mathbb{Z}\Gamma$ has finite global dimension. A result of Farrell and Hsiang [5] asserts that $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}\Gamma)$ is also an isomorphism. Hence $G_0(\mathbb{Z}\Gamma) = G_0(\mathbb{Z}\Gamma, \mathbf{F})$ in this case.

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The situation when Γ has torsion is somewhat more complicated. However, we shall prove the following result (a commutative Noetherian ring is *regular* if all its finitely generated modules have projective resolutions of finite length, and is *Hilbert* if each of its prime ideals is an intersection of maximal ideals):

Theorem A. *Let Γ be a finitely generated group with an abelian normal subgroup of finite index a. Let h be the Hirsch number of Γ. Let R be a commutative Noetherian regular Hilbert ring of finite Krull dimension d. Then $G_0(R\Gamma)/G_0(R\Gamma, \mathbf{F})$ is periodic, of exponent dividing a^{h+d}.*

We know of no example where $G_0(R\Gamma, \mathbf{F}) \subsetneq G_0(R\Gamma)$[1]. The restriction to abelian-by-finite groups is essential for our proof of Theorem A, but most of our preliminary results hold for a polycyclic-by-finite group Γ. We have stated these results in their most general form.

When Γ is abelian-by-finite, some insight into the structure of $G_0(\mathbb{Z}\Gamma)$ may be obtained from the action of crystallographic groups on euclidean space. In fact, something along the same lines is true for polycyclic-by-finite groups in general.

Theorem B. *Let Γ be a polycyclic-by-finite group with Hirsch number h. Then Γ acts smoothly and simplicially on some smooth triangulation of euclidean space $\mathbb{R}^h$, with compact quotient and finite isotropy groups.*

There is nothing essentially new in Theorem B. The ingredients are readily available in the literature. Indeed, something akin to Theorem B seems to be implicit in [17]. Nevertheless, it seems to be worthwhile to include a proof here, in view of the following interesting algebraic consequence, which, when Γ is torsion-free, is just the well-known fact that $\mathbb{Z}$ has a finite free $\mathbb{Z}\Gamma$-resolution of length h:

Corollary C. *Let Γ be as in Theorem* B. *Then there exists an exact sequence*

$$(*) \qquad 0 \to Q_h \to \cdots \to Q_1 \to Q_0 \to \mathbb{Z} \to 0$$

of right $\mathbb{Z}\Gamma$-modules, where each Q_i is a finite direct sum of modules of the form

$$\mathbb{Z} \otimes_{ZH} \mathbb{Z}\Gamma$$

for various finite subgroups H of Γ.

The paper is organised as follows. In Section 2 we prove Theorem B and Corollary C, and in Section 3 a stronger form of the latter is deduced (Theorem 3.2). This result provides the starting point in Section 4 for an inductive proof of Theorem A. A result on the uniform dimension of prime factor rings, which is need-

[1] Since this paper was first submitted, our attention has been drawn to an announcement by Moody [14] that, in fact $G_0(R\Gamma, \mathbf{F}) = G_0(R\Gamma)$ for any polycyclic-by-finite group Γ and any regular domain R.

ed in the proof of Theorem A and which may have some independent interest, is proved in Section 5.

As applications of Theorem A, we offer

Corollary D. *Let R and Γ be as in Theorem* A. *Suppose R is a Dedekind ring for which the Jordan–Zassenhaus Theorem holds.*

(i) *$G_0(R\Gamma)=T\times F$, where T is the torsion subgroup, and F is free abelian of finite rank, t say. Further, T contains a finite subgroup T_0 such that T/T_0 has exponent dividing a^{h+d}. If R is a field, then*

$$t\leq \sum_{H\in \mathbf{H}} |\mathrm{irr}(RH)|,$$

where $\mathbf{H}$ is a set of representatives of the conjugacy classes of maximal finite subgroups of Γ, and $|\mathrm{irr}(RH)|$ is the number of isomorphism classes of irreducible RH-modules.

(ii) *Suppose R is a field of characteristic p. The cokernel of the Cartan homomorphism of $K_0(R\Gamma)$ into $G_0(R\Gamma)$ is torsion of exponent dividing $p^r a^h$, where p^r is the maximal order of a p-subgroup of Γ.*

These follow immediately from Theorem A and [21, proof of Theorem 3.8] and [4, Theorem 21.22] respectively.

2. Topology

Throughout this section Γ is an arbitrary polycyclic-by-finite group of Hirsch number h. The proof of Theorem B follows that of [1, Theorem 1] in constructing a smooth action of Γ on $\mathbb{R}^h$ with finite isotropy groups and compact quotient. (The only difference being that, in [1], Γ is assumed to be torsion-free, so that the quotient $\mathbb{R}^h/\Gamma$ is a $K(\Gamma, 1)$-space.)

Explicitly, there exists a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow \| & & \\
1 & \longrightarrow & D(\Delta) & \longrightarrow & \Gamma D(\Delta) & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

with exact rows and vertical monomorphisms, where Δ is a torsion-free subgroup of finite index in Γ, and $D(\Delta)$ is a soluble Lie group containing Δ as a discrete cocompact subgroup. As in [1], let K be a maximal compact subgroup of $\Gamma D(\Delta)$. Then $K\setminus \Gamma D(\Delta)$ is diffeomorphic to $\mathbb{R}^h$, and Δ acts freely and smoothly on the right, so that the quotient space M is a smooth manifold. Moreover, the finite group $G=\Gamma/\Delta$ acts smoothly on M, so by [9] there exists a smooth G-equivariant triangulation T_0

of M. This lifts to a smooth Γ-equivariant triangulation T of $\mathbb{R}^h$, and the proof of Theorem B is complete.

For the proof of Corollary C, take ($*$) to be the simplicial chain complex of T. Thus each Q_i is a free abelian group, with basis the i-simplices of T. As a $\mathbb{Z}\Gamma$-module, Q_i is a permutation module (since Γ permutes simplices), with finite stabilizers (since Γ has finite isotropy groups), and finitely generated (since T/Γ is a finite complex). But such a module has precisely the form stated in Corollary C.

3. Modules

We shall need the following version of Frobenius reciprocity [21, Theorem 2.2].

Lemma 3.1. *Let R be a commutative ring and let H be a subgroup of a group G. Let W be a finitely generated RH-module and let X be a finitely generated RG-module. Then, as RG-modules,*

$$(W \otimes_{RH} RG) \otimes_R X \cong (W \otimes_R X|_H) \otimes_{RH} RG.$$

where each tensor product over R is equipped with the diagonal group action.

Proof. (See also [20, Lemma 1.1].) It is routine to check that the map $(w \otimes g) \otimes x \mapsto (w \otimes xg^{-1}) \otimes g$ is well defined and gives an isomorphism of RG-modules. □

Theorem 3.2. *Let R be a commutative Noetherian ring. Let Γ be a polycyclic-by-finite group, and let V be an $R\Gamma$-module which is finitely generated as an R-module. Then $[V] \in G_0(R\Gamma, F)$.*

Proof. By Corollary C there is an exact sequence

$$0 \to Q_h \to \cdots \to Q_1 \to Q_0 \to \mathbb{Z} \to 0 \tag{1}$$

of $\mathbb{Z}\Gamma$-modules, with each Q_i a finite direct sum of modules

$$\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}\Gamma$$

for various finite subgroups H of Γ.

Apply the functor $(-) \otimes_{\mathbb{Z}} R$ to (1) to get a sequence

$$0 \to P_h \to \cdots \to P_1 \to P_0 \to R \to 0 \tag{2}$$

of $R\Gamma$-modules. Then (2) is exact, since (1) consists of free $\mathbb{Z}$-modules. Furthermore each P_i is a finite direct sum of modules

$$(\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}} R \cong R \otimes_{RH} R\Gamma$$

for various finite subgroups H of Γ.

Finally, apply the functor $(-) \otimes_R V$ to (2) (where each term is given the $R\Gamma$-

module structure with diagonal Γ-action). The resulting sequence is exact, since the modules in (2) are free R-modules. By Lemma 3.1, its terms are finite direct sums of modules

$$(R \otimes_{RH} R\Gamma) \otimes_R V \cong V|_H \otimes_{RH} R\Gamma.$$

The sequence obtained is a sequence of $R\Gamma$-modules, since if $f_i : P_i \to P_{i-1}$, then $(f_i \otimes 1) : P_i \otimes V \to P_{i-1} \otimes V$ takes $(\pi \otimes v)g = \pi g \otimes vg$ to

$$f_i(\pi g) \otimes vg = f_i(\pi)g \otimes vg = (f_i(\pi) \otimes v)g = [(f_i \otimes 1)(\pi \otimes v)]g.$$

The result now follows from the fact that V is finitely generated as an R-module. □

4. Abelian-by-finite groups

We begin this section by recalling some well-known facts and definitions concerning a Noetherian ring S. Let M be an S-module. We say that M is *uniform* if it is non-zero and, if X and Y are any two non-zero submodules of M, then $X \cap Y \neq 0$. The *uniform dimension* of M, u-dim(M) is 0 if $M = 0$, t if M contains an essential direct sum of t uniform submodules, and ∞ if no such finite direct sum exists; if M is finitely generated, then u-dim(M) $< \infty$. See [15, Chapter 10, §4] for details. An element m of M is *torsion* if $mc = 0$ for some regular element c of S; M is *torsion free* if it contains no non-zero torsion elements.

Let P be a prime ideal of S. By Goldie's theorem [15, Theorem 10.4.10], S/P has a simple Artinian quotient ring Q; Q is a ring of $t \times t$ matrices over a division ring, where t is the uniform dimension of S/P (as right or left module). Let U and V be uniform right ideals of S/P. Thus $U \otimes_{S/P} Q$ and $V \otimes_{S/P} Q$ are both irreducible right Q-modules, and so isomorphic. It follows easily that U and V are subisomorphic as S-modules; in other words each is isomorphic to a submodule of the other. More generally, if X and Y are finitely generated torsion free S/P-modules of the same uniform dimension, then X embeds in Y and the cokernel is torsion as an S/P-module. These facts are clear when S/P is a finite module over its centre (the only case we require here). For the general case, one may consult [10, Lemma 2.2.13], for example.

Let M be a finitely generated S-module. Choose a uniform submodule U_1 of M whose annihilator P_1 is maximal among annihilators of non-zero submodules of M. It is easy to see that P_1 is prime. Repeat this process for M/U_1, and so on; we get a chain (finite, since M is Noetherian) $0 \subset U_1 \subset U_2 \cdots \subset U_n = M$ of submodules whose factors U_i/U_{i-1} are uniform, every non-zero submodule of U_i/U_{i-1} having annihilator P_i. Suppose now that S is finitely generated as a module over its centre. Let U be a finitely generated uniform S-module all of whose non-zero submodules have prime annihilator P. We can form the quotient ring Q of S/P by inverting the non-zero elements of the centre of S/P (since the resulting partial quotient ring is a finite-dimensional algebra over a field and hence Artinian). Thus, if $c+P$ is a regular element of S/P, $(cS+P)/P$ must have non-zero intersection with the centre of S/P. It follows that U is a torsion free S/P-module, and so, by the previous paragraph, U is (isomorphic to) a uniform, right ideal of S/P. To sum up:

Proposition 4.1. *Let S be a Noetherian ring which is a finitely generated module over its centre.*

(i) *Let M be a finitely generated S-module. Then M has a finite series of submodules with successive factors isomorphic to uniform right ideals of prime factor rings of S.*

(ii) *Let P be a prime ideal of S, with* u-dim$(S/P)=t$. *Let U be a uniform right ideal of S/P. Each of* $U^{(t)}$ *and S/P embeds in the other, the cokernel having annihilator strictly containing P.* □

We shall use induction arguments involving the Krull dimension, k-dim(M), of an S-module M. Details may be found in [6], but it is almost enough to know that, for S and M as in Proposition 4.1,

(K1) k-dim(S) is the supremum of the lengths of descending chains of prime ideals of S;

(K2) If $0\to A\to B\to C\to 0$ is an exact sequence of S-modules, then k-dim$(B)=\max\{$k-dim(A), k-dim$(C)\}$;

(K3) [6, Theorem 9.2 and 19, Lemma 8]. If R is a commutative Noetherian ring of finite Krull dimension d and Γ is a finitely generated abelian-by-finite group of Hirsch number h, then k-dim$(R\Gamma)=h+d$.

The next result contains the crux of the inductive step in the proof of Theorem A. Let R be a commutative Noetherian ring and let G be a polycyclic-by-finite group. For a non-negative integer n, define subgroups of $G_0(RG)$ as follows:

$$G_0(RG)_n=\langle [V]\colon \text{k-dim}(V)\leq n\rangle,$$

$$G_0(RG)_{n-}=\langle [V]\colon \text{k-dim}(V)<n\rangle.$$

Proposition 4.2. *With the above notation, let P be a prime ideal of RG and set* $n=$ k-dim(RG/P). *Then, in* $G_0(RG)$,

$$[RG/P]\in G_0(RG,\mathbf{F})+G_0(RG)_{n-}.$$

Proof. Let H and N be subgroups of G, with N normal. If M is a finitely generated RH-module with $[M]\in G_0(RH,\mathbf{F})$, then $[M\otimes_{RH}RG]\in G_0(RG,\mathbf{F})$. Hence, in view of Theorem 3.2, inflation from $R(G/N)$- to RG-modules maps $G_0(R(G/N),\mathbf{F})$ to $G_0(RG,\mathbf{F})$. Moreover, under this map $G_0(R(G/N))_{n-}$ is sent to $G_0(RG)_{n-}$. Thus, in proving the proposition we may assume that $\{g\in G\colon (g-1)\in P\}=1$.

In the notation of [18], set $H=\text{nio}(G)$, an orbitally sound normal subgroup of finite index in G. We argue by induction on $|G:H|$. Suppose first that

$G=H$, or more generally that $P\cap RH=Q$ is a prime ideal of RH. Then $\{g\in G: (g-1)\in Q\}=1$, so $Q=(Q\cap R\Delta)RH$, by [18, Theorem C1], where Δ is the FC-subgroup (see [15, §4.1] for definition) of H and of G. By [13, Lemma 2.1], $QRG=(Q\cap R\Delta)RG$ is a prime ideal of RG which is contained in P by construction. Since $QRG\cap RH=P\cap RH=Q$, we conclude that $QRG=P$ by incomparability [12, Theorem 1.2]. Now, if Δ_0 denotes the (finite) torsion subgroup of Δ, then Δ/Δ_0 is free abelian of finite rank and so $R\Delta$ is an iterated skew Laurent extension of $R\Delta_0$. The so-called twisted Grothendieck Theorem [16, Exercise following Theorem 8, §6], or [22, Proposition 4.1(2)] therefore implies that $G_0(R\Delta)=G_0(R\Delta,\Delta_0)=G_0(R\Delta,\mathbf{F})$. Consequently, $[RG/P]=[(R\Delta/Q\cap R\Delta)\otimes_{R\Delta}RG]\in G_0(RG,\mathbf{F})$.

We may therefore assume that $P\cap RH=\bigcap_{x\in G}Q^x$ for some prime ideal Q of RH with $B=\{g\in G: Q^g=Q\}$ a proper subgroup of G. Let $J=\{x_1,\dots,x_t\}$ be a right transversal to B in G. By [13, Theorem 1.7], there is a unique prime ideal L of RB with $L\cap RH=Q$ and $\bigcap_{i=1}^t L_iRG=P$, where L_i denotes L^{x_i} for $i=1,\dots,t$. Let θ denote the natural embedding of RG/P in $\sum_i^{\otimes}(RG/L_iRG)$ as right RG-modules, with cokernel Y. We claim that

$$\text{k-dim}(Y)<n. \tag{3}$$

Let us complete the proof, assuming that (3) is true. By definition, $[Y]\in G_0(RG)_{n-}$, and so, setting $B_i=B^{x_i}$,

$$[RG/P]=\sum_i[(RB_i/L_i)\otimes_{RB_i}RG]-[Y].$$

By [3, Lemma 4.2], $\text{k-dim}(RB_i/L_i)=n$ for all i. By induction on $|G:H|$, $[RB_i/L_i]\in G_0(RB_i,\mathbf{F})+G_0(RB_i)_{n-}$ for all i. By [19, Lemma 8] and the first paragraph of the proof,

$$[(RB_i/L_i)\otimes RG]\in G_0(RG,\mathbf{F})+G_0(RG)_{n-}$$

for all i. Thus (3) shows that $[RG/P]$ is in the required subgroup of $G_0(RG)$. It remains therefore to prove (3).

For $j=1,\dots,t$, let $\pi_j:\sum^{\otimes}RG/L_iRG\to RG/L_jRG$ be the projection map. Since $Q_i:=Q^{x_i}=L_i\cap RH$,

$$0\neq\pi_j\circ\theta\left(\left(\bigcap_{\substack{i\\ i\neq j}}L_iRG\cap RH\right)RB_j+P/P\right)$$

$$:=X\subseteq RB_j+L_jRG/L_jRG\cong RB_j/L_j.$$

Further, $\bigcap_{i,\,i\neq j}(L_iRG\cap RH)$ is the annihilator in RH of $Q_j/(P\cap RH)$, and as such is invariant under conjugation by B_j. Hence, X is a non-zero two-sided ideal of RB_j/L_j. Since L_j is a prime ideal, $\text{k-dim}((RB_j/L_j)/X)<\text{k-dim}(RB_j/L_j)$, by [6], (or by (K1) if we assume G is abelian-by-finite). Therefore, identifying RB_j/L_j with RB_j+L_jRG/L_jRG,

$$\text{k-dim}_{RB_j}((RB_j/L_j)/\text{im}(\pi_j\circ\theta)\cap(RB_j/L_j))<\text{k-dim}_{RB_j}(RB_j/L_j)=n,$$

and so, by [19, Lemma 8],

$$\text{k-dim}_{RB_j}((RG/L_jRG)/\text{im}(\pi_j \circ \theta)) < \text{k-dim}_{RB_j}(RG/L_jRG) = n.$$

By [19, Lemma 8] once more,

$$\text{k-dim}_{RG}((RG/L_jRG)/\text{im}(\pi_j \circ \theta)) < \text{k-dim}_{RG}(RG/L_jRG) = n. \tag{4}$$

Since (4) holds for all $j = 1, \dots, t$, (3) is proved. □

Let p be a prime. A finite group is *p-hyperelementary* if it has the form $\langle x \rangle \rtimes P$, with x an element of order prime to p, and P a p-group. Let G be a finite group. We denote by $\mathbf{H}$ the class of finite groups which are p-hyperelementary for some prime p. We need a version of the Brauer–Berman–Witt induction theorem:

Theorem 4.3. *Let R be a commutative Noetherian ring. Let G be a finite group. Then $G_0(RG, \mathbf{H}) = G_0(RG)$.*

For a proof, see [20, Corollary 4.2(c)].

A commutative Noetherian ring R is *regular* if every finitely generated R-module has a finite resolution by projective R-modules. Let G be a finite group. The abelian group with generators $[M]$, where M is a finitely generated RG-module which is R-projective, and relations given by short exact sequences, is denoted by $G_0^R(RG)$. If R is regular, $G_0^R(RG) = G_0(RG)$ [21, Theorem 1.2]. The group $G_0^R(RG)$ can be given a ring structure by setting $[M][N] = [M \otimes_R N]$, [21, Theorem 1.5]. Note that $[R]$ is the identity element of this ring.

Theorem A. *Let R be a commutative Noetherian regular Hilbert ring of finite Krull dimension d, and let Γ be a finitely generated abelian-by-finite group with Hirsch number h. Let A be a maximal abelian normal subgroup of Γ, and set $G = \Gamma/A$; put $|G| = a$. Then $G_0(R\Gamma)/G_0(R\Gamma, \mathbf{F})$ is periodic, with exponent dividing a^{h+d}.*

Proof. The ring $R\Gamma$ is Noetherian and is a finite module over its centre. [15, Corollary 10.2.8 and proof of Lemma 4.1.10], so we can make use of the facts and concepts given at the start of Section 4.

Step 1. *Reduction to the case where G is hyper-elementary.* Let $\hat{\mathbf{H}}(\Gamma)$ be the set of inverse images in Γ of the $\mathbf{H}$-groups in G. By Theorem 4.3,

$$G_0(RG, \mathbf{H}) = G_0(RG). \tag{5}$$

As pointed out above, the hypotheses on R ensure that $G_0(RG)$ is a ring with identity element. Viewing $G_0(R\Gamma)$ as a $G_0(RG)$-module via inflation and $- \otimes_R -$,

$$G_0(R\Gamma) \cdot G_0(RG, \mathbf{H}) \subseteq G_0(R\Gamma, \hat{\mathbf{H}}(\Gamma)), \tag{6}$$

by Lemma 3.1. By (5) and (6),

$$G_0(R\Gamma) = G_0(R\Gamma, \hat{\mathbf{H}}(\Gamma)). \tag{7}$$

It follows from (7) that there is a surjection induced by induction.

$$\sum_{X \in \hat{\mathbf{H}}(\Gamma)}^{\otimes} G_0(RX)/G_0(RX, \mathbf{F}) \to G_0(R\Gamma)/G_0(R\Gamma, \mathbf{F}). \tag{8}$$

Since $|X/A|^{h+d}$ divides a^{h+d} for all $X \in \hat{\mathbf{H}}(\Gamma)$, (8) shows that we may replace Γ by one of the groups X in $\hat{\mathbf{H}}(\Gamma)$ in proving the theorem.

Step 2. *The induction set-up.* We shall deduce the theorem from the following more precise set of statements:

> *Let P be a prime ideal of $R\Gamma$, with*
>
> k-dim$(R\Gamma/P) = m$. *Let* $\omega = \max\{1, \alpha^{m-1}\}$. (9;$m$)
>
> *Then* $\omega \cdot [R\Gamma/P] \in G_0(R\Gamma, \mathbf{F})$.

Since R is Hilbert, every finitely generated Artinian $R\Gamma$-module is finitely generated as an R-module [11, Theorem 31]. Thus Theorem 3.2 shows that (9;0) is true.

We claim:

> *If* $m \geq 0$ *and if* (9;i) *is true for all* $i \leq m$,
>
> *then* $a^m \cdot (G_0(R\Gamma)_m) \subseteq G_0(R\Gamma, \mathbf{F})$. (10;$m$)

Again, Theorem 3.2 allows us to assume that $m > 0$ and that (10;l) is true for all $l < m$. By Proposition 4.1 and (K2), $G_0(R\Gamma)_m$ is generated by $[M]$, where M is a uniform right ideal of $R\Gamma/P$ and P ranges over the set of all prime ideals for which k-dim$(R\Gamma/P) \leq m$. Let M be one such right ideal, of $R\Gamma/P$, say. Let $t =$ u-dim$(R\Gamma/P)$. By Proposition 4.1(ii) there is an exact sequence

$$0 \to R\Gamma/P \to M^{(t)} \to X \to 0. \tag{11}$$

By Proposition 4.1(ii) and (K1), (K2), k-dim$(X) = l < m$. Hence, by (10;l), $a^{m-1} \cdot [X] \in G_0(R\Gamma, \mathbf{F})$. By (9;$m$) and (11), $ta^{m-1} \cdot [M] \in G_0(R\Gamma, \mathbf{F})$. Thus (10;$m$) follows from this, Proposition 5.2(ii), and Step 1.

Proposition 4.2 shows that the statements (10;i), for $i = 0, \ldots, m-1$, together imply (9;m). Thus the proof is complete. □

5. Uniform dimension of prime factors

Our aim here is to prove Proposition 5.2, part of which was used in the proof of Theorem A.

Let H be a subgroup of a group G and let R be a commutative ring. Let Q be an ideal of RH. Then Q^G denotes the biggest ideal of RG inside QRG, so $Q^G = \bigcap_{g \in G} (QRG)^g$; see [13].

Lemma 5.1. *Let R be a commutative ring, and let G be a polycyclic-by-finite group containing a subgroup H of finite index.*

(i) *Let P be a prime ideal of RG and let $Q_1, \dots, Q_r$ be the prime ideals of RH minimal over $P \cap RH$. Then there exist positive integers $z_1, \dots, z_r$ such that* $\text{u-dim}(RG/P) = \sum_i z_i \cdot \text{u-dim}(RH/Q_i)$.

(ii) *Let Q be a prime ideal of RH and let $P_1, \dots, P_s$ be the primes of RG minimal over Q^G. Then there exist positive integers $w_1, \dots, w_s$ such that* $\sum_j w_j \cdot \text{u-dim}(RG/P_j) = |G:H| \cdot \text{u-dim}(RH/Q)$.

Proof. By factoring by $P \cap R$ in (i) and by $Q \cap R$ in (ii), and then inverting the non-zero elements of R, we reduce to the case where R is a field. Thus all rings involved here are Noetherian. In particular there are indeed finite sets of primes lying over P and Q in (i) and (ii) respectively.

(i) This simply expresses the fact that the inclusion of rings $RH \subset RG$ satisfies the additivity principle [23, Corollary 2 and preceding remarks].

(ii) Put $I = Q^G$ and $V = RG/QRG$. Thus V is an $(RH\text{–}RG)$-bimodule with right annihilator I, and

$$_{RH}| V \cong \sum^{\oplus} (RH/Q)\bar{x} \cong (RH/Q)^{(t)},$$

where $T = \{x_1, \dots, x_t\}$ is a right transversal for H in G, and $\bar{x}$ denotes the image of x in V. Thus

$$A := RG/I \subseteq B := \text{End}_{RH}(V) \cong M_t(RH/Q),$$

and (ii) will follow if we can show that the inclusion $A \subseteq B$ satisfies the additivity principle. By [23, Corollary 2], it suffices to show that B is finitely generated as a right and as a left A-module.

For this, fix a normal subgroup N of G with $N \subseteq H$ and $|G:N| < \infty$, and set $A_0 = RN/I \cap RN \subseteq A$. Under the embedding $A \subseteq M_t(RH/Q)$, A_0 corresponds to the subring

$$D = \left\{ \begin{bmatrix} \overline{r^{x_1}} & & 0 \\ & \ddots & \\ 0 & & \overline{r^{x_t}} \end{bmatrix} : r \in A_0 \right\}$$

of $M_t(\bar{A}_0)$, where $^-$ denotes images in RH/Q. Clearly the elementary matrices $\{E_{ij} : 1 \le i, j \le t\}$ generate $M_t(\bar{A}_0)$ as a left and as a right D-module; and $M_t(RH/Q)$ is finitely generated as a left and as a right module over $M_t(\bar{A}_0)$, since this holds for RH/Q over $\bar{A}_0$. Therefore B is finitely generated on both sides over A_0, and hence over A, as required. □

For any ring S, set

$$u(S) := \sup\{\text{u-dim}(S/P) \mid P \text{ a prime ideal of } S\},$$

a positive integer or ∞. Then, in the situation of Lemma 5.1, we have

$$u(RH) \le u(RG) \le [G:H] \cdot u(RH).$$

To derive this from Lemma 5.1 one uses the fact that each prime ideal of RH is minimal over $P \cap RH$ for a suitable prime ideal P of RG, and each prime ideal of RG is minimal over Q^G for some prime Q of RH. The details are fairly routine and are left to the reader.

We can now state and prove the main result of this section.

Proposition 5.2. *Let N be a normal subgroup of finite index a in a polycyclic-by-finite group Γ. Let R be a commutative ring, and let P be a prime ideal of $R\Gamma$. Let p be the characteristic of $R/P \cap R$. Let Q be a prime ideal of RN minimal over $P \cap RN$, with $P \cap RN = \bigcap_{\gamma \in \Gamma} Q^\gamma$. Then*

(i) $\quad$ $\text{u-dim}(RN/Q) \mid \text{u-dim}(R\Gamma/P)$.

(ii) *Let $\hat{C}$ denote the algebraic closure of the centre C of the simple Artinian ring of quotients F of RN/Q. Let $\hat{C}N$ denote the subring of $\hat{C} \otimes_C F$ generated by $\hat{C}$ and RN/Q. If either $p \nmid a$ or Γ/N is p-soluble, then*

$$\text{u-dim}(R\Gamma/P) \mid a \cdot \text{u-dim}(\hat{C}N/\hat{Q}),$$

where $\hat{Q}$ is a suitable prime ideal of $\hat{C}N$ lying over the zero ideal of RN/Q.

In particular if N is abelian, and either $p \nmid a$ or Γ/N is p-soluble, then

$$\text{u-dim}(R\Gamma/P) \mid a.$$

Proof. As in Lemma 5.1 we reduce at once to the case where R is a field. By [18, Lemma 5] we have $P \cap RN = \bigcap_{\gamma \in \Gamma} Q^\gamma$. Since $\text{u-dim}(RN/Q) = \text{u-dim}(RN/Q^\gamma)$ for all $\gamma \in \Gamma$, (i) is a special case of Lemma 5.1(i).

(ii) We argue by induction on a. For $a = 1$, the claim is that $\text{u-dim}(R\Gamma/P) \mid \text{u-dim}(\hat{C}\Gamma/\hat{P})$ for some prime ideal $\hat{P}$ of $\hat{C}\Gamma$ with $\hat{P} \cap R\Gamma = P$. But, for *any* such $\hat{P}$, $R\Gamma/P \subseteq \hat{C}\Gamma/\hat{P}$ is a centralizing extension of prime Noetherian rings, so the assertion follows from [23, Theorem 3], for example. Thus we may assume that (ii) is true for all proper subgroups H of Γ with $N \subseteq H$.

Let M be a proper normal subgroup of Γ with $N \subseteq M$ and such that $P \cap RM$ is *not* prime. Then $P = P_1^\Gamma$ for some prime ideal P_1 of RM_1, where $M \subseteq M_1 \subset \Gamma$, by [13, Theorem 1.7]. By Lemma 5.1(ii),

$$\text{u-dim}(R\Gamma/P) \mid \text{u-dim}(RM_1/P_1) \cdot |G : M_1|,$$

so the inductive hypothesis applied to M_1 yields the result. (Note that the primes of RN minimal over $P_1 \cap RN$ are minimal over $P \cap RN$, and hence Γ-conjugate to Q.) Thus we may assume that, for all normal subgroups M of Γ with $N \subseteq M$, $P \cap RM$ is prime.

In particular, $Q = P \cap RN$ is prime. Standard arguments along the lines of [13, Lemma 1.5] show that the set $\mathbf{C}$ of regular elements of RN/Q forms an Ore set of regular elements in $R\Gamma/QR\Gamma$ and in $R\Gamma/P$. By localising at $\mathbf{C}$ we obtain the classical rings of quotients of the rings under consideration:

$$\begin{array}{ccc} A := (RN/Q)\mathbf{C}^{-1} = Q(RN/Q) & \subseteq & B := (R\Gamma/P)\mathbf{C}^{-1} = Q(R\Gamma/P) \\ \cap\downarrow & \nearrow & \\ (R\Gamma/QR\Gamma)\mathbf{C}^{-1} = Q(R\Gamma/QR\Gamma) & & \end{array}$$

(Here we are abusing notation by writing $\mathbf{C}$ for its image in $R\Gamma/QR\Gamma$ and $R\Gamma/P$.) Note that B has the structure of a crossed product over A. $B \cong A * G$ with $G = \Gamma/N$. Let Γ_{inn} be the normal subgroup of Γ consisting of those elements acting by inner automorphisms on the simple Artinian ring A, and set $G_{\text{inn}} = \Gamma_{\text{inn}}/N \subseteq G$. By our assumption, $P \cap R\Gamma_{\text{inn}}$ is prime. Let $^-$ denote images modulo $QR\Gamma$. Thus $T := (\overline{P \cap R\Gamma_{\text{inn}}})\mathbf{C}^{-1}$ is a prime ideal of $A * G_{\text{inn}} \subseteq B$, and, by [12, Theorem 2.5(i)], $P' := \bar{P}\mathbf{C}^{-1} = T \cdot B$. Lifted back to $R\Gamma$, this yields $P = (P \cap R\Gamma_{\text{inn}})R\Gamma$, and so if $\Gamma_{\text{inn}} \neq \Gamma$ the induction hypothesis and Lemma 5.1(ii) again give the result. We may therefore assume that $\Gamma = \Gamma_{\text{inn}}$.

Let E denote the centraliser of A in B, and let C be the centre of A. Then $E \cong C^t G$ is a twisted group algebra of G over the field C, with $B \cong A \otimes_C E$, and moreover, $P' = (P' \cap E)B$; see [12, §2]. Let $\hat{C}$ denote the algebraic closure of C and choose a prime ideal P'' of $\hat{E} := \hat{C} \otimes_C E \cong \hat{C}^t G$ with $P'' \cap E = P' \cap E$. (Simply take P'' to be maximal among ideals I of $\hat{E}$ with $I \cap E = P' \cap E$.) Then

$$\hat{E}/P'' \cong M_\upsilon(\hat{C}) \tag{12}$$

where

$$\upsilon \mid a, \tag{13}$$

by Lemma 5.3 below,

$$S := B/P' \cong A \otimes_C (E/P' \cap E) \subseteq A \otimes_C (\hat{E}/P'') \cong M_\upsilon(A \otimes_C \hat{C}),$$

it follows from the additivity principle [23, Lemma 1] that the composition length of S divides u-dim$(M_\upsilon(A \otimes_C \hat{C}))$; that is, that

$$\text{u-dim}(R\Gamma/P) \mid \upsilon \cdot \text{u-dim}(A \otimes_C \hat{C}). \tag{14}$$

Now $A \otimes_C \hat{C}$ is a simple ring, and the map from RN to $A \otimes_C \hat{C}$ yields a map from $\hat{C}N$ with $\hat{Q} \cap RN = Q$ and u-dim$(\hat{C}N/\hat{Q}) =$ u-dim$(A \otimes_C \hat{C})$. With (13) and (14), this completes the proof of the proposition, except that we still have to establish

Lemma 5.3. *Let G be a finite group of order a, let K be an algebraically closed field of characteristic p, and let $K^t G$ be a twisted group algebra of G over K. Assume that either*

(i) *$p \nmid a$ or*

(ii) *G is p-soluble.*

Then for any simple $K^t G$-module V, $\dim_K(V) \mid a$.

Proof. Case (i) is essentially covered by [4, Proposition 11.44], where we can replace the hypothesis that char $K=0$ by condition (i), by using the generalized form of Ito's Theorem [7, Satz V. 12.11] at the appropriate point in the proof.

For (ii), note that there is a finite central extension H of G such that V is a simple KH-module [4, Theorem 11.40(i)]. Since H is also p-soluble, the Fong–Swan–Rukolaine theorem [4, Theorem 22.1] ensures that V can be 'lifted to characteristic zero'. Hence, by Ito's theorem [4, Theorem 11.33], $\dim_K(V)$ divides $|H/Z(H)|$, (where $Z(H)$ denotes the centre of H). Thus, $\dim_K(V)$ divides $|G|$, as required. □□

Lemmas 5.3, (and so also Proposition 5.2(ii)) is false without the hypothesis (i) or (ii), even for ordinary group algebras. For example, if K is algebraically closed with char $K=7$, then $G=\mathrm{SL}(2,7)$ has a 5-dimensional simple module over K, and $5\nmid 336=|G|$ (cf. [8, p. 41]). Also $\hat{C}N/\hat{Q}$ cannot in general be replaced by RN/Q in the situation of Proposition 5.2(ii). An explicit counterexample is as follows. Take $G=Q_8\times C_3$, the direct product of the quaternion group of order 8 and the cyclic group of order 3, and let $R=\mathbb{R}$ be the field of real numbers. Viewing Q_8 and C_3 as multiplicative subgroups of the quaternions $\mathbf{H}$ and the complex numbers $\mathbb{C}$ respectively, we obtain surjections $\phi:\mathbb{R}[Q_8]\to\mathbf{H}$ and $\psi:\mathbb{R}[G]\to\mathbf{H}\otimes_{\mathbb{R}}\mathbb{C}\cong M_2(\mathbb{C})$. Thus, with $Q=\operatorname{Ker}\phi$, $P=\operatorname{Ker}\psi$ and $N=Q_8$, the hypotheses of Proposition 5.2(ii) are satisfied, yet u-dim$(\mathbb{R}[G]/P)=2$ does not divide $[G:N]\cdot$ u-dim$(\mathbb{R}[N]/Q)=3$.

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