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Primitive Ideals in Crossed Products and Rings with Finite Group Actions

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Introduction

The set Priv(R) of all primitive ideals of the ring $R(\ni 1)$ can be topologized by declaring the subsets of the form $\{P \in \operatorname{Priv}(R) \mid P \supset I\}$, I an ideal of R, to be closed ([9], Chap. 9). The topological space Priv(R) is a T_1-space iff all primitive ideals of R are maximal. This holds trivially in the case of commutative rings. It is also true (but not trivial) if R is a group algebra of a finitely generated nilpotent group or an enveloping algebra of a finite dimensional nilpotent Lie algebra (Zalesskij [14], Dixmier [6]).

For the moment, let $\mathscr{T}_1$ denote the class of rings whose primitive ideals are maximal. $\mathscr{T}_1$ is obviously stable under homomorphisms and is stable under Morita equivalence (cf. f.i. [1], p. 258/259). In general the property $\mathscr{T}_1$ is not inherited by subrings and overrings as easy examples show.

In Section 1 of this note we consider the situation $R \subset R_\alpha^\gamma[G]$, where $R_\alpha^\gamma[G]$ denotes a crossed product of the finite group G over the ring R (1.1). Crossed products can be considered as a generalization of group rings. They are a useful tool for the description of certain factors of group algebras (cf. f.i. [14]) and have been studied in a number of papers, notably by Bovdi (f.i. [5]).

In Theorem (1.7) we prove that the property $\mathscr{T}_1$ is inherited from $R_\alpha^\gamma[G]$ to R and that the converse is true if $|G|^{-1} \in R$ or if G is finite solvable. As a by-product of the proof we obtain some results on the primitivity of R and $R_\alpha^\gamma[G]$.

Section 2 deals with the following situation: R is a ring (associative with 1), G a finite group of automorphisms of R, and R^G the fixed subring of R. We always assume that $|G|^{-1} \in R$. Examining the behaviour of R-modules under restriction to R^G we obtain the analogue of (1.7):

$$R \in \mathscr{T}_1 \quad \text{if and only if} \quad R^G \in \mathscr{T}_1 \quad \text{(Theorem 2.7).}$$

A similar result for prime ideals has been shown by Fisher and Osterburg ([8], Theorems 4.2 and 4.5). The techniques developed to prove Theorem (2.7) can be used to deduce results on the primitivity of R and R^G (2.8). In addition, they easily

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yield Montgomery's theorem [10] on the Jacobson radical of the fixed ring: $J(R^G)=J(R)\cap R^G$ (2.5). The corresponding result also holds for the socle of R^G if R is semiprime: $\mathrm{soc}(R^G)=\mathrm{soc}(R)\cap R^G$ (2.6). Finally, we show that R is a Jacobson ring (this means each prime ideal is an intersection of primitives), if R^G is a Jacobson ring. This result has also been announced by Armendariz [2].

All rings considered in this note contain a unit element. The unspecified word "module" means "unitary right module" and "ideal" stands for "two-sided ideal". Mod$-R$ denotes the category of right R-modules. $U(R)$ is the group of invertible elements of R, and $\mathrm{Int}(R)$ the group of all inner automorphisms of R.

1. Crossed Products

(1.1) For the reader's convenience and in order to fix our notations, we recall the definition of a *crossed product of a group G over a ring R:* Given maps $\alpha: G\to \mathrm{Aut}(R)$ and $\gamma: G\times G\to U(R)$ such that

(i) $\gamma(x,y)\,\gamma(xy,z)=\gamma(y,z)^{\alpha(x)^{-1}}\,\gamma(x,yz)$

and

(ii) $\gamma(x,y)\,r^{\alpha(xy)^{-1}}=r^{\alpha(y)^{-1}\alpha(x)^{-1}}\,\gamma(x,y)$.

for all $x,y,z\in G$, $r\in R$, we define the crossed product $R_\alpha^\gamma[G]$ to be the set of all formal sums of the form $\sum\limits_{x\in G} r_x\bar{x}$ with $r_x\in R$, and $r_x=0$ for almost all $x\in G$. The addition in $R_\alpha^\gamma[G]$ is defined componentwise and the multiplication is given by the rule

$$(r_x\bar{x})(r_y\bar{y})=r_x\,r_y^{\alpha(x)^{-1}}\,\gamma(x,y)\,\overline{xy}.$$

This makes $R_\alpha^\gamma[G]$ an associative ring with unit element $\gamma(1,1)^{-1}\bar{1}$.

(1.2) For later reference we gather some simple properties of the crossed product $R_\alpha^\gamma[G]$, which follow easily from the definitions.

(a) The map $r\mapsto r\gamma(1,1)^{-1}\bar{1}$ is a ring monomorphism of R into $R_\alpha^\gamma[G]$. We therefore consider R as a subring of $R_\alpha^\gamma[G]$.

(b) It follows from (ii) that the composite map $G\xrightarrow{\alpha}\mathrm{Aut}(R)\to\mathrm{Aut}(R)/\mathrm{Int}(R)$ is a homomorphism. Hence the set $\{x\in G\,|\,\alpha(x)\in\mathrm{Int}(R)\}$ is a normal subgroup of G.

(c) If N is a subgroup of G, then the restrictions α' und γ' of α and γ to N (resp. $N\times N$) define a crossed product $R_{\alpha'}^{\gamma'}[N]$. $R_{\alpha'}^{\gamma'}[N]$ is a subring of $R_\alpha^\gamma[G]$ in a canonical way. Furthermore $R_\alpha^\gamma[G]$ is free as a left $R_{\alpha'}^{\gamma'}[N]$-module: a basis is given by $\{\bar{y}_i\,|\,i\in I\}$ where $\{y_i\,|\,i\in I\}$ is a full set of coset representatives for N in G. If N is normal in G, then $R_\alpha^\gamma[G]$ carries the structure of a crossed product of G/N over $R_{\alpha'}^{\gamma'}[N]$.

(d) Let I be an ideal of R such that $I^{\alpha(x)}=I$ for all $x\in G$ and let $\bar{R}:=R/I$. Then α and γ give rise to maps $\bar{\alpha}: G\to\mathrm{Aut}(\bar{R})$ and $\bar{\gamma}: G\times G\to U(\bar{R})$ which obviously satisfy (i) and (ii). The crossed product $\bar{R}_{\bar{\alpha}}^{\bar{\gamma}}[G]$ is isomorphic to $R_\alpha^\gamma[G]/IR_\alpha^\gamma[G]$.

(1.3) The following lemma is an appropriate version of Clifford's classical restriction theorem. The usual proof (see f.i. [11], 7.2.16) works with the obvious notational changes.

Lemma. *Let G be a finite group, R a ring and $U := R_\alpha^\gamma[G]$ a crossed product. If $V \in \mathrm{Mod}-U$ is irreducible, then the restricted module V_R contains an irreducible submodule W and we have*

$$V_R = \sum_{x\in G} W\bar{x}.$$

Let $T := \{x \in G \mid W\bar{x} \cong W\}$ and let V_1 be the sum of all submodules X of V_R such that $X \cong W$. Then T is a subgroup of G and V_1 is an irreducible U'-module ($U' := R_{\alpha'}^{\gamma'}[T]$, (1.2)(c)) such that

$$V \cong V_1 \underset{U'}{\otimes} U.$$

(1.4) A ring S is called *projective relative to the subring R* (or *R-projective*) if the following holds: Each exact sequence $0 \to A \to B \to C \to 0$ of S-modules which splits when considered as a sequence in $\mathrm{Mod}-R$ splits in $\mathrm{Mod}-S$. – The following lemma is based on the well-known Maschke averaging process.

Lemma. *Let G be a finite group, R a ring such that $|G|^{-1} \in R$ and $U := R_\alpha^\gamma[G]$ a crossed product. Then U is R-projective.*

Proof. Let A be a submodule of $B \in \mathrm{Mod}-U$ such that A_R is a direct summand of B_R. This means there is an R-homomorphism $h': B \to A$ such that $h'(a) = a$ for all $a \in A$.

Let

$$h(b) = |G|^{-1} \sum_{x\in G} h'(b\bar{x}^{-1})\,\bar{x}.$$

Then h is a U-homomorphism from B to A such that $h(a) = a$ for all $a \in A$.

(1.5) **Lemma.** *Let R be a ring, G a finite group and T a subgroup of G such that $|T|^{-1} \in R$. Let $U := R_\alpha^\gamma[G]$ be a crossed product and $U' := R_{\alpha'}^{\gamma'}[T]$, where α' and γ' denote the restrictions of α and γ (1.2)(c). If V is an irreducible U-module, then the restricted module $V_{U'}$ is completely reducible of finite length.*

Proof. (1) *If $X \in \mathrm{Mod}-R$ is irreducible, then $X \underset{R}{\otimes} U'$ is completely reducible of finite length:* Let (*) $0 \to A \to X \underset{R}{\otimes} U' \to B \to 0$ be exact in $\mathrm{Mod}-U'$. The sequence (*) splits as an R-sequence since $X \underset{R}{\otimes} U'|_R$ is a finite direct sum of conjugates of X and hence completely reducible. Therefore, by Lemma (1.4), the sequence (*) splits in $\mathrm{Mod}-U'$. $X \underset{R}{\otimes} U'$ is clearly of finite length.

(2) *$V_R \underset{R}{\otimes} U'$ is completely reducible of finite length:* By (1.3), V_R is completely reducible of finite length. Therefore, by step (1), $V_R \underset{R}{\otimes} U'$ is completely reducible of finite length.

(3) The assertion of the lemma now follows from the fact that $V_{U'}$ is a homomorphic image of $V_R \bigotimes_R U'$.

(1.6) The next lemma is well-known (see [3]).

Lemma. *Let R be a simple ring, G a finite group and let* $U := R_\alpha^\gamma[G]$ *be a crossed product such that the automorphisms* $\alpha(x)$, $1 \neq x \in G$, *are outer. Then U is simple.*

(1.7) **Theorem.** *Let R be a ring, G a finite group and* $U := R_\alpha^\gamma[G]$ *a crossed product. If all primitive ideals of U are maximal, then the primitive ideals of R are maximal. If* $|G|^{-1} \in R$ *or G is finite solvable, then the converse holds.*

Proof. The proof of the first assertion is analogous to that of Snider [13], Lemma 2: Suppose we are given ideals P, M of R such that $P \subsetneqq M$ and P is primitive. Then since G is finite, we have $\bar{P} := \bigcap_{x \in G} P^{\alpha(x)} \subsetneqq \bar{M} := \bigcap_{x \in G} M^{\alpha(x)}$.

Fix $V \in \mathrm{Mod}-R$ irreducible such that $P = \mathrm{ann}_R(V)$ and let $W := V \bigotimes_R U$. The module W is obviously of finite length and hence we may fix a finite composition series $W = W_0 \supset W_1 \supset \cdots \supset W_s = (0)$. Let $Q_i := \mathrm{ann}_U(W_{i-1}/W_i)$. Then $Q_1 Q_2 \ldots Q_s \subset \mathrm{ann}_U(W) = \bar{P}U \subsetneqq \bar{M}U$. Since $\bar{M}U$ is a proper two-sided (!) ideal of U, we can choose a maximal ideal N of U containing $\bar{M}U$ and obtain $Q_i \subset N$ for some i. Since $W_R = \sum_{x \in G} \oplus V \otimes \bar{x}$ it follows that W_{i-1}/W_i contains a copy $V \otimes \bar{x}$ for some $x \in G$ and hence being U-invariant contains a copy of $V \otimes \bar{x}$ for each $x \in G$. We conclude that

$$\mathrm{ann}_R(W_{i-1}/W_i) = Q_i \cap R \subset \bigcap_{x \in G} \mathrm{ann}_R(V \otimes \bar{x}) = \bigcap_{x \in G} \mathrm{ann}_R(V)^{\alpha(x)} = \bar{P} \subsetneqq \bar{M} \subset N \cap R$$

and hence $Q_i \subsetneqq N$. Therefore Q_i is a nonmaximal primitive ideal of U.

The proof of the second assertion is by induction on $|G|$. The case $|G| = 1$ being trivial, we assume that $|G| > 1$ and that the assertion is true for groups of smaller order. In particular using (1.2)(c) we may assume that G is simple.

Now let $P \supset Q$ be primitive ideals of U. We have to show that $P = Q$. Let $P = \mathrm{ann}_U(V)$, $Q = \mathrm{ann}_U(W)$, where $V, W \in \mathrm{Mod}-U$ are irreducible. The restricted module V_R contains an irreducible submodule L (1.3). Set $T := \{x \in G \mid L\bar{x} \cong L\}$ and $U' := R_{\alpha'}^{\gamma'}[T]$ ($\subset U$). We treat the two cases (i) $T \subsetneqq G$ and (ii) $T = G$ separately.

Case (i). By (1.3) we have $V \cong V_1 \bigotimes_{U'} U$ where V_1 is an irreducible U'-module. Let $P_1 := \mathrm{ann}_{U'}(V_1)$.

The restriction $W_{U'}$ of W is completely reducible of finite length. In case $|G|^{-1} \in R$ this follows from (1.5) and when G is finite solvable it follows from the fact that in this case $T = \langle 1 \rangle$ and $U' = R$. Let $W_{U'} = W_1 \oplus \cdots \oplus W_s$, $W_i \in \mathrm{Mod}-U'$ irreducible, and let $Q_i := \mathrm{ann}_{U'}(W_i)$. Then

$$\bigcap_{i=1}^{s} Q_i = \mathrm{ann}_{U'}(W) = Q \cap U' \subset P \cap U' \subset P_1.$$

Therefore $Q_i \subset P_1$, say, and by induction ($|T|<|G|$) we conclude that $Q_i = P_1$. From that, we immediately obtain

$$\operatorname{ann}_U(W_i \underset{U'}{\otimes} U) = \operatorname{ann}_U(V_1 \underset{U'}{\otimes} U) = \operatorname{ann}_U(V) = P.$$

The embedding of W_i into W extends to a U-homomorphism $W_i \underset{U'}{\otimes} U \to W$ which has to be epi since W is irreducible. Hence $\operatorname{ann}_U(W_i \underset{U'}{\otimes} U) \subset \operatorname{ann}_U(W) = Q$, that is $P \subset Q$ and we are done in case (i).

Case (ii). Now V_R is isomorphic to a finite direct sum of copies of L (1.3) and hence $P \cap R = \operatorname{ann}_R(L)$ is a maximal ideal of R. Furthermore by (1.3) and by assumption on R we obtain $Q \cap R = \bigcap_{x \in G} M^{\alpha(x)}$ for some maximal ideal M of R. It follows immediately that $Q \cap R = P \cap R$. The right ideal $(P \cap R)U$ of U is actually two-sided and the ring $\bar{U} := U/(P \cap R)U$ carries the structure of a crossed product $\bar{U} \cong \bar{R}_{\bar{\alpha}}^{\bar{\gamma}}[G]$, $\bar{R} := R/P \cap R$ (1.2)(d).

If $\bar{U}$ is simple, we have $P = Q = (P \cap R)U$. Otherwise, by Lemma (1.6), there exists $x \in G$, $x \neq 1$, such that $\bar{\alpha}(x)$ is an inner automorphism of $\bar{R}$. Since G is simple it follows from (1.2)(b) that each $\bar{\alpha}(x)$ is inner. Hence for each $x \in G$ there exists a unit $r_x \in \bar{R}$ such that $z_x := r_x \bar{x} \in \bar{U}$ centralizes $\bar{R}$. Let $C := \operatorname{centre}(\bar{R})$, a field. Since $\bar{U} = \sum_{x \in G} \bar{R} z_x$ we have $C \subset \operatorname{centre}(\bar{U})$ and we may consider $\bar{U}$ as a C-algebra.

We have

$$z_x z_y = (r_x \bar{x})(r_y \bar{y}) = r_x r_y^{\bar{\alpha}(x)^{-1}} \bar{\gamma}(x,y)\, \overline{xy} = r_x r_y^{\bar{\alpha}(x)^{-1}} \bar{\gamma}(x,y) r_{xy}^{-1} z_{xy} = c_{x,y} z_{xy},$$

where $c_{x,y} \in C$. Hence $A := \sum_{x \in G} C z_x$ is a C-subalgebra of $\bar{U}$ and obviously $\bar{U} = \bar{R} \cdot A$. Since A centralizes the central-simple subalgebra $\bar{R}$ the algebra $\bar{U}$ has the structure of a tensor product, $\bar{U} \cong \bar{R} \underset{C}{\otimes} A$, and the ideals of $\bar{U}$ are of the form $\bar{R} \underset{C}{\otimes} I$, where I is an ideal of A ([9], p. 110). Obviously I has to be prime if $\bar{R} \underset{C}{\otimes} I$ is prime. Since A is a finite-dimensional algebra, we conclude that all prime ideals in $\bar{U}$ are maximal. In particular $P = Q$. This finishes the proof of the theorem.

(1.8) **Proposition.** *Let R be a ring, G a finite group and $U := R_\alpha^\gamma[G]$ a crossed product.*

(a) *If U is prime and R is primitive, then U is primitive.*

(b) *If R is prime and U is primitive, then R is primitive.*

Proof (a). Let V be a faithful irreducible R-module. Then $W := V \underset{R}{\otimes} U \in \operatorname{Mod}-U$ has finite length and its annihilator is $(\bigcap_{x \in G} \operatorname{ann}_R(V)^{\alpha(x)})\, U = (0)$. Fix a composition series $W = W_0 \supset W_1 \supset \cdots \supset W_s = (0)$ of W and let $Q_i := \operatorname{ann}_U(W_{i-1}/W_i)$. Then $Q_1 Q_2 \ldots Q_s \subset \operatorname{ann}_U(W) = (0)$. Since U is prime it follows that $Q_i = (0)$, some i. Hence U is primitive.

(b) Let $V \in \operatorname{Mod}-U$ be faithful and irreducible. By (1.3) we have

$V_R \cong \sum_{i=1}^{s} \oplus W_i$, $W_i \in \text{Mod}-R$ irreducible. If Q_i denotes the annihilator of W_i, then $\bigcap_{i=1}^{s} Q_i = \text{ann}_R(V) = (0)$ and since R is prime, $Q_i = (0)$ some i. Hence R is primitive.

(1.9) *Remarks.* The primitivity of R does not in general imply $R_\alpha^\gamma[G]$ to be prime as can be seen for instance by considering the group algebra of a finite group over a field. Conversely, the primitivity of $R_\alpha^\gamma[G]$ does not imply the primeness of R: Let $R = k \oplus k$, k a field. Let $y \in \text{Aut}(R)$ be given by $(\lambda, \mu)^y = (\mu, \lambda)$ $(\lambda, \mu \in k)$ and let $U := R_\alpha[\mathbb{Z}_2]$ be the corresponding crossed product of R over $\mathbb{Z}_2 = \langle y \rangle$. Then U is simple, hence primitive, but R is not prime.

2. Fixed Rings

(2.1) **Notations and Conventions.** In the following R will always mean a ring with unit element, G a finite group of automorphisms of R and R^G the fixed ring, $R^G = \{r \in R \mid r^x = r \text{ for all } x \in G\}$. Furthermore $U := R_\alpha^1[G]$ will denote the crossed product of G over R with trivial factor set $\gamma \equiv 1$ and α given by the embedding of G into $\text{Aut}(R)$.

For convenience, we shall assume $|G|$ is invertible in R throughout the remainder of this paper. An easy consequence of this assumption is the following: Let I be a G-stable ideal of R, let $\bar{R} := R/I$ and let $\bar{G} \subset \text{Aut}(\bar{R})$ be induced by G. Then $\bar{R}^{\bar{G}} = \overline{R^G}$ (see [8], p. 11). We set $e := |G|^{-1} \sum_{x \in G} \bar{x} \in U$. Then e is an idempotent of U.

(2.2) R is a left U-module via the action $(\sum r_x \bar{x})\, r := \sum r_x r^{x^{-1}}$. We have a U-isomorphism ${}_U R \cong Ue$ given by $r \to r \sum_{x \in G} \bar{x}$ and a ring isomorphism $R^G \cong \text{End}_U R$ associating to $r \in R^G$ the right multiplication by r ([7], Lemma 1.2). Therefore $R^G \cong eUe$.

(2.3) **Theorem** (Bergman-Isaacs [4]).

(i) *If R^G is nilpotent, then R is nilpotent.*

(ii) *If R is semiprime, then R^G is semiprime.*

(iii) *If I is a G-invariant right ideal of R, then either I is nilpotent or $I \cap R^G \neq (0)$.*

(Here we do *not* assume that R has a 1.)

(2.4) **Lemma.** (a) *Let $V \in \text{Mod}-R$ be of finite length. Then the restricted module V_{R^G} is of finite length.*

(b) *If V is completely reducible, then V_{R^G} is completely reducible.*

Proof. (1) *Let A be an arbitrary ring, $f = f^2 \in A$ and $W \in \text{Mod}-A$. If W is of finite length, then so is $Wf \in \text{Mod}-fAf$. If W is completely reducible, then the same holds for Wf:*

Let $X \subset Wf$ be an fAf-submodule of Wf. Then $XA = X \oplus XA(1-f)$. Therefore, if $X_1 \subsetneqq X_2$ are fAf-submodules of Wf, we obtain $X_1 A \subsetneqq X_2 A$. The first assertion follows. As to the second, notice that by hypothesis there are A-homomorphisms

$$\pi, \mu\colon XA \underset{\mu}{\overset{\pi}{\leftrightarrows}} W \quad \text{such that } \pi\mu = \mathrm{id}_{XA}.$$

Let $\mu_1 := \mu|_X$ and $\pi_1 := \pi|_{Wf}$. Then $\mu_1\colon X \to Wf$ and $\pi_1\colon Wf \to X$ are fAf-linear and satisfy $\pi_1 \mu_1 = \mathrm{id}_X$.

(2) *One has* $eUe = R^G e$: Let $r \in R$ and $y \in G$. Then

$$e(r\bar{y})e = er(\bar{y}e) = ere = (|G|^{-1} \sum_{x\in G} \bar{x} r)e = |G|^{-1} \sum_{x\in G} r^{x^{-1}} \bar{x} e = (|G|^{-1} \sum_{x\in G} r^{x^{-1}})e.$$

Since $|G|^{-1} \sum_{x\in G} r^{x^{-1}} \in R^G$ we have $eUe \subset R^G e$. As to the other inclusion it suffices to remark that for $r \in R^G$ we have $re = er = ere$.

(3) Consider the module $\bar{V} := (V \otimes_R U)e \in \mathrm{Mod}-eUe$. Direct calculation shows that $\bar{V} = \{\sum_{x\in G} v \otimes \bar{x} \mid v \in V\}$. Hence we have an R^G-isomorphism $\Phi\colon V \to \bar{V}$, $\Phi(v) = \sum_{x\in G} v \otimes \bar{x}$. By step (2), the eUe-submodules of $\bar{V}$ coincide with the R^G-submodules of $\bar{V}$. The assertions of the lemma now follow from step (1), because V of finite length certainly implies $V \otimes_R U \in \mathrm{Mod}-U$ to be of finite length, and if V is completely reducible, then $V \otimes_R U$ is completely reducible, by (1.4).

(2.5) **Corollary** (Montgomery [10]).

$$J(R^G) = J(R) \cap R^G.$$

Proof. The inclusion $J(R) \cap R^G \subset J(R^G)$ follows readily from the fact that $U(R) \cap R^G = U(R^G)$. – Let $x \in J(R^G)$ and let $V \in \mathrm{Mod}-R$ be irreducible. By the lemma, V_{R^G} is completely reducible. Therefore $Vx = (0)$, and $x \in J(R)$.

(2.6) The following result has been obtained independently by Reiter [12]. **Corollary.**

(a) $\mathrm{soc}(R) \cap R^G \subset \mathrm{soc}(R^G)$.

(b) *Equality holds if* R *is semiprime.*

Proof. Let $x \in \mathrm{soc}(R) \cap R^G$. Then the right ideal xR of R is completely reducible. Using the lemma we conclude that $xR^G \subset xR|_{R^G}$ is completely reducible. Therefore $x \in \mathrm{soc}(R^G)$. –

The second assertion follows from Fisher-Osterburg [7], Lemma 1.9.

(2.7) **Theorem.** *All primitive ideals of* R *are maximal if and only if the primitive ideals of* R^G *are maximal.*

Proof. If the primitive ideals of R are maximal, then by Theorem (1.7), the same holds in U. Now $R^G \cong eUe$ (2.2). Therefore (see [9], p. 206) $\mathrm{Priv}(R^G)$ is homeomorphic to a subspace of $\mathrm{Priv}(U)$ and hence inherits the property $\mathcal{T}_1$, which means that the primitive ideals of R^G are maximal.

In order to prove the other implication, fix a primitive ideal P of R and an irreducible R-module V such that $P=\operatorname{ann}_R(V)$.

Let $\bar{P}:=\bigcap_{x\in G} P^x$ and $\bar{R}:=R/\bar{P}$. Then V is an irreducible $\bar{R}$-module such that $\operatorname{ann}_{\bar{R}^G}(V)=P/\bar{P}\cap\bar{R}^G=(0)$. By (2.4), V is completely reducible of finite length when considered as an $\bar{R}^G$-module. Therefore $(0)=\operatorname{ann}_{\bar{R}^G}(V)=\bigcap_{i=1}^{s} P_i$ for some primitive ideals P_i of $\bar{R}^G$. As $\bar{R}^G$ is a homomorphic image of R^G, all primitive ideals of $\bar{R}^G$ are maximal. Using the Chinese remainder theorem we conclude that $\bar{R}^G$ is a finite direct sum of simple rings. By Fisher and Osterburg ([8], Theorem 4.3), $\bar{R}$ is a finite direct sum of simple rings. This forces the prime ideal $P/\bar{P}$ of $\bar{R}$ to be maximal, as was to be shown.

(2.8) **Proposition.**

(a) *Let R be prime. Then R is primitive if R^G is primitive.*

(b) *If R is primitive and R^G is prime, then R^G is primitive.*

(c) *If R^G is primitive and U is prime, then U is primitive.*

(d) *Suppose U is prime. Then the following implications hold:*
R primitive $\Rightarrow$ U primitive $\Rightarrow$ R^G primitive.

Proof. (a) Let $V\in\operatorname{Mod}-R^G$ be faithful and irreducible. We shall construct an irreducible R-module W containing V:

For this write $V\cong R^G/I$, I a maximal right ideal of R^G. Now $IR\neq R$, for otherwise $\sum_l i_l r_l=1$ $(i_l\in I,\ r_l\in R)$ would imply

$$|G|=\sum_{x\in G} 1^x=\sum_{x\in G}\big(\sum_l i_l r_l\big)^x=\sum_{x\in G}\sum_l i_l r_l^x=\sum_l i_l\big(\sum_{x\in G} r_l^x\big)\in I$$

and hence $1\in I$ since $|G|$ is invertible. Therefore we can find a maximal right ideal J of R containing IR. The module $W:=R/J\in\operatorname{Mod}-R$ is irreducible and contains

$$R^G+J/J=R^G/R^G\cap J=R^G/I=V.$$

Let P be its annihilator in R. Then $P\cap R^G\subset\operatorname{ann}_{R^G}(V)$, which is zero, since V is faithful. Therefore, by the Bergman-Isaacs' theorem (2.3), we conclude that $\bigcap_{x\in G} P^x=(0)$. Finally, the primeness of R yields $P=(0)$. Thus R is primitive.

(b) R has a faithful irreducible module V. By (2.4) V_{R^G} is completely reducible of finite length: $V_{R^G}=V_1\oplus\cdots\oplus V_s$, $V_i\in\operatorname{Mod}-R^G$ irreducible. Writing

$$Q_i:=\operatorname{ann}_{R^G}(V_i)\quad\text{we obtain}\quad\bigcap_{i=1}^{s} Q_i=\operatorname{ann}_{R^G}(V)=(0).$$

Using the assumption on R^G we conclude that $Q_i=(0)$, some i, and hence R^G is primitive.

(c) We have $R^G\cong eUe$ (2.2). Since eUe is primitive, there exists a primitive ideal P of U such that $ePe=(0)$ (see [9], p. 206). Again, the primeness of U yields $P=(0)$.

(d) The first implication is contained in (1.8), and the second again follows immediately from the fact that $R^G\cong eUe$ ([9] l.c.)

(2.9) *Remarks.* The following examples (taken from [7], [8]) show that the primeness assumptions in Proposition (2.8) are not superfluous.

Let $R := M_2(k)$, the 2×2-matrix ring over the field k of characteristic $\neq 2$, and let $g \in \mathrm{Aut}(R)$ be given by $g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Then R is certainly primitive, whereas $R^{\langle g\rangle} \cong k \oplus k$ is not.

Let k be a field, $\mathrm{char}(k) \neq 3$, and let $R := k + k\,x + k\,y$, $0 = x^2 = y^2 = x\,y = y\,x$. Let $h \in \mathrm{Aut}(R)$ be given by $h(x) = x + y$, $h(y) = x$, $h(\xi) = \xi$, $\xi \in k$. Then $R^{\langle h\rangle} = k$ is primitive, but R has a nonzero radical $J(R) = k\,x + k\,y$.

(2.10) Recall that the ring R is called a *Jacobson ring* if each of its prime ideals is an intersection of primitive ideals. –

Proposition. *If R^G is a Jacobson ring, then R is a Jacobson ring.*

Proof. Let P be a prime ideal of R. We have to show that the radical $J(R/P)$ is zero. Let $\bar{P} := \bigcap_{x\in G} P^x$ and $\bar{R} := R/\bar{P}$. We first show that $J(\bar{R}) = (0)$:
$\bar{R}$ is a semiprime ring and hence, by the Bergman-Isaacs' theorem, $\bar{R}^G = \overline{R^G}$ is also semiprime. Therefore, by assumption on R^G, we have $J(\overline{R^G}) = (0)$. Using (2.5) we obtain $J(\bar{R}) \cap \bar{R}^G = (0)$. Now since $\bar{R}$ is semiprime and $J(\bar{R})$ is $\bar{G}$-stable, a second application of the Bergman-Isaacs' theorem yields $J(\bar{R}) = (0)$.

Suppose by way of contradiction that $J(R/P) \neq (0)$. Then $J(R/P^x) \neq (0)$ for all $x \in G$. Writing J_x for the inverse image of $J(R/P^x)$ in R we have $J_x \supsetneqq P^x$ for all $x \in G$. From this one deduces easily that $\bigcap_{x\in G} J_x \supsetneqq \bigcap_{x\in G} P^x = \bar{P}$. The required contradiction now follows from the fact that $\bigcap_{x\in G} J_x/\bar{P} = J(\bar{R}) = (0)$.

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Note Added in Proof

Prof. Fisher kindly pointed out to me that from this article it follows easily that $(|G|^{-1} \in R)$ $V \in \mathrm{Mod}-R$ noetherian (artinian) $\Rightarrow V_{RG}$ noetherian (artinian), V has Krull dimension $\Rightarrow V_{RG}$ has Krull dimension and $K \dim V_R = K \dim V_{RG}$.