K_0 OF SKEW GROUP RINGS AND SIMPLE NOETHERIAN RINGS WITHOUT IDEMPOTENTS

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ABSTRACT

We construct simple Noetherian rings S of characteristic p, for any prime p, such that S has zero divisors but no non-trivial idempotents, and S is not Morita equivalent to a domain. Our methods are non-computational and rely on a description of K_0 for certain skew group rings.

Introduction

In **[11]**, Zalesskii and Neroslavskii constructed a simple Noetherian ring S of Goldie rank 2 which does not contain any non-trivial idempotents, thereby answering a question of Faith **[5]** in the negative. The computations carried out by Zalesskii and Neroslavskii to verify the non-existence of idempotents are quite difficult, and our goal here is to present a number of general results on skew group rings which allow as an application the painless construction of many further examples of simple Noetherian rings with zero divisors but without non-trivial idempotents. These examples will also have the property of not being Morita equivalent to a domain (cf. **[10]**).

The first section contains some results on the structure of K_0 for skew group rings $S = R * G$ of a finite p-group G over a ring R with $pR = \{0\}$ (with p any prime). Our basic technical tool here is a Morita context for group actions which was introduced in a commutative setting by Chase, Harrison, and Rosenberg **[2]** and has recently been studied for general rings by Cohen **[4]**. These results are then applied in the second section to construct simple Noetherian rings of the desired kind. The actual examples exhibited here are similar to (and include) the original Zalesskii–Neroslavskii construction and involve central localizations of group rings of finitely generated nilpotent groups. Essential in our method is the use of trace functions for the rings under consideration.

Notation and conventions

The following notation will be kept throughout this article:

R is a ring with 1,

G is a finite group with a homomorphism $G \to \mathrm{Aut}(R)$, written as $x \mapsto (\cdot)^x$,

$S = R * G$ is the corresponding skew group ring, with elements $s = \sum_{x \in G} s_x x (s_x \in R)$ and multiplication $(s_x x) \cdot (s_y y) = s_x s_y^{x^{-1}} xy$,

$U = R^G$ is the fixed subring of R under the action of G,

$t_G : R \to U$, $r \mapsto \sum_{x \in G} r^x$, is the trace map.

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Furthermore, we set $t = \sum_{x \in G} x \in S$, and $T = t_G(R)$, an ideal of U. Finally, $K_0(R)$ denotes the Grothendieck group of all finitely generated projective right R-modules, and $G_0(R)$ the Grothendieck group of all finitely generated right R-modules. The element of $K_0(R)$ corresponding to the finitely generated projective R-module P will be written as $[P]$; similar notation will be used for $G_0(R)$.

1. K_0 *of some skew group rings*

(1.1). *The Morita context* (cf. [4]). We view R as (S, U)-bimodule and as (U, S)-bimodule via the obvious isomorphisms $R \simeq St = Rt$ and $R \simeq tS = tR$. Thus, in ${}_SR_U$ any $x \in G$ acts on R as $(\cdot)^{x^{-1}}$ while in ${}_UR_S$ any $x \in G$ acts as $(\cdot)^x$. One has bimodule homomorphisms

$$f\colon {}_SR \underset{U}{\otimes} R_S \longrightarrow {}_SS_S, \qquad r_1 \otimes r_2 \longmapsto r_1 t r_2$$

and

$$g\colon {}_UR \underset{S}{\otimes} R_U \longrightarrow {}_UU_U, \qquad r_1 \otimes r_2 \longmapsto t_G(r_1 r_2).$$

These maps satisfy the associativity conditions

$$r_1 \cdot f(r_2 \otimes r_3) = g(r_1 \otimes r_2) \cdot r_3, \qquad f(r_1 \otimes r_2) \cdot r_3 = r_1 \cdot g(r_2 \otimes r_3)$$

for $r_1, r_2, r_3 \in R$. Thus $(S, U, {}_SR_U, {}_UR_S, f, g)$ is a Morita context or a set of pre-equivalence data, in the terminology of [**1**, pp. 61ff]. Note that if S is a simple ring, then f must be surjective. Therefore, in the following, we shall concentrate on the case where f is surjective, that is,

$$S = StS \ (= RtR).$$

In our later applications, g will in general not be surjective, that is, $T = t_G(R)$ will be a proper ideal of U. (If both f and g are surjective, then S and U are Morita-equivalent [**1**, pp. 62–65].)

(1.2) LEMMA. *Suppose that $StS = S$. Then*

(i) *R_U and ${}_UR$ are finitely generated projective,*

(ii) $S \simeq \operatorname{End}_U(R)$,

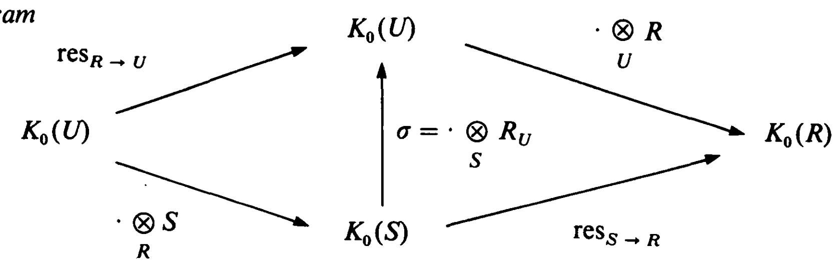
(iii) ${}_SR_U \simeq \operatorname{Hom}_U({}_UR_S, {}_UU)$, ${}_UR_S \simeq \operatorname{Hom}_U({}_SR_U, U_U)$,

(iv) *$f\colon {}_SR \otimes_U R_S \to {}_SS_S$ is an (S, S)-bimodule isomorphism.*

Proof. All assertions follow from [**1**, Theorem (3.4), p. 62].

(1.3) LEMMA. *Suppose that $StS = S$.*

(i) *The tensor product $\cdot \otimes_S R_U$ defines a map $\sigma\colon K_0(S) \to K_0(U)$ which makes the diagram*



commute. (Note that R_U and S_R are finitely generated projective so that the restrictions are defined.) The image $\sigma K_0(S)$ is contained in the kernel $K_0(T)$ of the map $\otimes_U U/T: K_0(U) \to K_0(U/T)$;

(ii) $\cdot \otimes_U R_S$ *defines a map $\tau: G_0(U) \to G_0(S)$ such that the composition $K_0(S) \underset{\sigma}{\to} K_0(U) \underset{\text{Cartan}}{\longrightarrow} G_0(U) \underset{\tau}{\to} G_0(S)$ is the Cartan map $K_0(S) \to G_0(S)$ (cf.* [**1**, p. 453]).

Proof. (i) By Lemma (1.2), R_U is finitely generated projective and $R \otimes_U R \simeq S$ as (S, S)-bimodules. Therefore, σ defines a map $K_0(S) \to K_0(U)$ such that, for all $[Q] \in K_0(S)$,

$$((\cdot \underset{U}{\otimes} R) \circ \sigma)[Q] = [Q \underset{S}{\otimes} R \underset{U}{\otimes} R_R] = [Q \underset{S}{\otimes} S_R] = \text{res}_{S \to R}[Q].$$

Furthermore, for $[P] \in K_0(R)$, we have

$$(\sigma \circ (\cdot \underset{R}{\otimes} S))[P] = [P \underset{R}{\otimes} S \underset{S}{\otimes} R_U] = [P \underset{R}{\otimes} R_U] = \text{res}_{R \to U}[P].$$

Thus the above diagram is commutative. From $R = S \cdot R = StS \cdot R = S \cdot T = RT$ (where $\cdot$ denotes the left S-module action on R) we obtain $R \otimes_U U/T = 0$ and hence, for $[Q] \in K_0(S)$,

$$((\cdot \underset{U}{\otimes} U/T) \circ \sigma)[Q] = [Q \underset{S}{\otimes} R \underset{U}{\otimes} U/T] = 0.$$

This proves (i).

(ii) As ${}_U R$ is projective, and hence is flat, τ defines a map $G_0(U) \to G_0(S)$. For $[Q] \in K_0(S)$, we have

$$(\tau \circ \text{Cartan} \circ \sigma)[Q] = [Q \underset{S}{\otimes} R \underset{U}{\otimes} R_S] = [Q \underset{S}{\otimes} S_S] = [Q] \in G_0(S),$$

which completes the proof of the lemma.

(1.4) *Operation of G on $K_0(R)$.* The operation of G on R induces an operation on $K_0(R)$ which can be described as follows. For any finitely generated projective right R-module P, we have $P \otimes_R S_R = \bigoplus_{x \in G} P \otimes x$, where each $P \otimes x$ is an R-module direct summand of $P \otimes_R S_R$ and as such is finitely generated projective. The operation of G is now given by

$$[P]^x = [P \otimes x], \quad [P] \in K_0(R),\ x \in G.$$

The operation on the subgroup $\langle [R] \rangle$ of $K_0(R)$ is trivial, because the map $R \otimes x \to R$, $r \otimes x \mapsto r^x$, is an isomorphism of R-modules. Set

$$\begin{aligned} K_0(R)_G &= K_0(R)/\langle [P]^x - [P] \mid [P] \in K_0(R),\ x \in G \rangle \\ &= H_0(G, K_0(R)) \end{aligned}$$

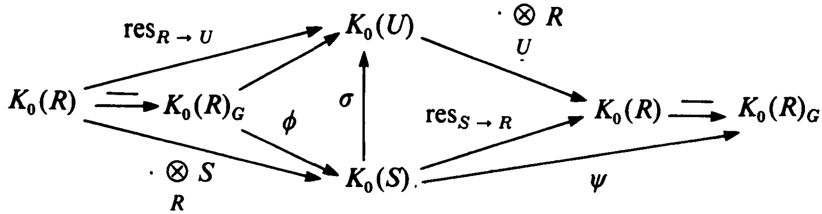
and denote the canonical surjection $K_0(R) \to K_0(R)_G$ by $\bar{\ }$. The maps $\cdot \otimes_R S: K_0(R) \to K_0(S)$ and $\text{res}_{R \to U}: K_0(R) \to K_0(U)$ factor through $K_0(R)_G$ so that, in particular, we have a map

$$\phi: K_0(R)_G \longrightarrow K_0(S), \quad \overline{[P]} \longmapsto [P \underset{R}{\otimes} S].$$

Furthermore, define

$$\psi: K_0(S) \longrightarrow K_0(R)_G, \quad [Q] \longmapsto \overline{[Q_R]}.$$

Then, when $S = StS$, Lemma (1.3) yields the following commutative diagram:



(1.5) LEMMA. (i) *$\psi \circ \phi: K_0(R)_G \to K_0(S) \to K_0(R)_G$ is multiplication by $|G|$ on $K_0(R)_G$.*

(ii) *Suppose that* (1) *for some prime p one has $pR = \{0\}$ and G is a finite p-group, and* (2) *$StS = S$. Then*

$$\phi \circ \psi: K_0(S) \longrightarrow K_0(R)_G \longrightarrow K_0(S)$$

is multiplication by $|G|$ on $K_0(S)$.

Proof. (i) For $[P] \in K_0(R)$, we have

$$(\psi \circ \phi)\overline{[P]} = \overline{[P \otimes_R S_R]} = \overline{\sum_{x \in G} [P]^x} = |G| \cdot \overline{[P]}.$$

(ii) First note that S has a series

$$0 = S_0 \subset S_1 \subset \ldots \subset S_{|G|} = S$$

of (U, S)-subbimodules with $S_i/S_{i-1} \simeq {}_U R_S$ for all i. To see this, consider the group ring $\mathbb{F}_p[G] \subset S$, where $\mathbb{F}_p$ is the field with p elements. By (1), $\mathbb{F}_p[G]$ has a series of right ideals $0 = W_0 \subset W_1 \subset \ldots \subset W_{|G|} = \mathbb{F}_p[G]$ with $W_i/W_{i-1} \simeq \mathbb{F}_p$, the trivial $\mathbb{F}_p[G]$-module. Therefore, setting $S_i = W_i S = W_i R \subset S$ we obtain the required (U, S)-subbimodules with $S_i/S_{i-1} \simeq W_i/W_{i-1} \otimes_{\mathbb{F}_p} R \simeq {}_U R_S$. Since R_U is flat, by Lemma (1.2), we further obtain short exact sequences of (S, S)-bimodules

$$0 \longrightarrow V_{i-1} = R \otimes_U S_{i-1} \longrightarrow V_i = R \otimes_U S_i \longrightarrow R \otimes_U R \simeq S \longrightarrow 0.$$

Thus, for Q finitely generated projective over S, we deduce exact sequences $0 \to Q \otimes_S V_{i-1} \to Q \otimes_S V_i \to Q \otimes_S S \simeq Q \to 0$, whence

$$Q \otimes_S V_i \simeq Q^{(i)} \quad (i = 1, 2, \ldots, |G|).$$

Using the commutative diagram in (1.4), we finally obtain

$$(\phi \circ \psi)[Q] = [(Q \otimes_S R \otimes_U R) \otimes_R S] = [Q \otimes_S (R \otimes_U S)]$$
$$= [Q \otimes_S V_{|G|}] = |G| \cdot [Q],$$

as claimed.

(1.6) COROLLARY. *Suppose that* (1) *for some prime p we have $pR = \{0\}$ and G is a p-group, and* (2) *$StS = S$. Then*

$$K_0(R)_G \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \simeq K_0(S) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p],$$

and

$$K_0(S) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p] \simeq \sigma K_0(S) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p]$$

is a direct summand of $K_0(U) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$.

Proof. By Lemma (1.5), the mappings $\psi \otimes \mathrm{id}$ and $\phi \otimes |G|^{-1}$ yield isomorphisms between $K_0(R)_G \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ and $K_0(S) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ which are inverse to each other. Moreover, by the commutative diagram in (1.4),

$$(\phi \underset{\mathbb{Z}}{\otimes} |G|^{-1}) \circ (\psi \underset{\mathbb{Z}}{\otimes} \mathrm{id}) = \mathrm{id}_{K_0(S) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]}$$

factors through $K_0(U) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ via $\sigma \otimes \mathrm{id}$, which proves the second assertion.

(1.7) *The case* $|G| = 2$. We conclude this section with a few remarks concerning the simplest case, $|G| = 2$ and $2R = \{0\}$. These will not be needed in Section 2, and possibly hold more generally. Thus assume that (1) $|G| = 2$ and $2R = \{0\}$, and (2) $StS = S$.

Recall (Lemma (1.3)) that $\sigma K_0(S) \subset K_0(T) = \mathrm{Ker}(K_0(U) \to K_0(U/T))$. Our goal here is to show that

$$2 \cdot K_0(T) \subset \sigma K_0(S).$$

To see this, first note that

$$0 \longrightarrow U \longrightarrow {}_U R_U \xrightarrow{t_G} U \longrightarrow U/T \longrightarrow 0$$

is an exact sequence of (U, U)-bimodules. Let $a = [V] - [W] \in K_0(T)$, with V and W finitely generated projective over U. Then, for some $s \geqslant 0$,

$$(V \underset{U}{\otimes} U/T) \oplus (U/T)^s \simeq (W \underset{U}{\otimes} U/T) \oplus (U/T)^s$$

and, after replacing V by $V \oplus U^s$ and W by $W \oplus U^s$, we may assume that $s = 0$. Since V_U and W_U are flat, the above exact sequence yields exact sequences of right U-modules

$$0 \longrightarrow V \simeq V \underset{U}{\otimes} U \longrightarrow V \underset{U}{\otimes} R_U \longrightarrow V \simeq V \underset{U}{\otimes} U \longrightarrow V \underset{U}{\otimes} U/T \longrightarrow 0,$$

$$0 \longrightarrow W \simeq W \underset{U}{\otimes} U \longrightarrow W \underset{U}{\otimes} R_U \longrightarrow W \simeq W \underset{U}{\otimes} U \longrightarrow W \underset{U}{\otimes} U/T \longrightarrow 0.$$

As the first three terms in each row are projective, the Schanuel lemma [1, Corollary 6.4, p. 36] yields an isomorphism

$$V \oplus (W \underset{U}{\otimes} R_U) \oplus V \simeq W \oplus (V \underset{U}{\otimes} R_U) \oplus W.$$

Thus, in $K_0(U)$, we have

$$2 \cdot a = (\cdot \underset{U}{\otimes} R_U)(a) \subset \mathrm{Im}(\mathrm{res}_{R \to U}) \subset \sigma K_0(S),$$

where the latter inclusion uses the diagram in Lemma (1.3). This shows that $2 \cdot K_0(T) \subset \sigma K_0(S)$, as we have claimed. As a consequence, we deduce that

$$\sigma K_0(S) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/2] = K_0(T) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/2]$$

and, using Corollary (1.6),

$$K_0(U) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/2] \simeq (K_0(S) \oplus \operatorname{Im}(\cdot \underset{U}{\otimes} U/T)) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/2]$$
$$\simeq (K_0(R)_G \oplus \operatorname{Im}(\cdot \underset{U}{\otimes} U/T)) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/2].$$

(1.8) EXAMPLE. We briefly discuss the original example of Zalesskii and Neroslavskii [11] in a slightly modified form. Let k be a field with $\operatorname{char} k = 2$ containing a non-root of unity $\lambda \in k^*$ and consider the k-algebra $R = B_\lambda = k\{x^{\pm 1}, y^{\pm 1}\}/(xy - \lambda yx)$. Then R is a simple Noetherian domain with $K_0(R) = \langle [R] \rangle \simeq \mathbb{Z}$ [6, Section 1]. Moreover, the automorphism σ of R given by $x^\sigma = x^{-1}$, $y^\sigma = y^{-1}$ is easily seen to be outer so that the skew group ring $S = R * \langle \sigma \rangle$ is simple (cf. [8, Example 2.8]). We claim that $U/T \simeq k$. Indeed, R can be viewed as a twisted group algebra,

$$R \simeq k^t[\Gamma]$$

with $\Gamma = \langle x, y \rangle k^*/k^*$ free abelian of rank 2. Thus each element $a \in R$ has a unique expression as

$$a = \sum_{g \in \Gamma} a_g \bar{g},$$

with $a_g \in k$ (almost all equal to 0). The automorphism σ operates as follows:

$$a^\sigma = \sum_{g \in \Gamma} a_g \overline{g^{-1}} = \sum_{g \in \Gamma} a_{g^{-1}} \bar{g}.$$

It is easy to check that $T = \{a + a^\sigma \mid a \in R\} = \{a \in U \mid a_1 = 0\}$ so that $U = T \oplus k$. Therefore, (1.7) implies that

$$K_0(U) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/2] \simeq (K_0(R)_G \oplus K_0(k)) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/2] \simeq \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2].$$

In particular, U has finitely generated projectives which are not stably free. Finally, in view of Lemma (1.5), the isomorphism $K_0(R) \simeq \mathbb{Z}$ implies that

$$K_0(S) \simeq \mathbb{Z} \oplus (\text{2-torsion}).$$

2. *Simple Noetherian rings without idempotents*

We keep our general notation R, G, $S = R * G$ from the previous section.

(2.1) *Traces.* A trace function of R is an additive map $\operatorname{tr}: R \to A$, where A is some abelian group, such that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ holds for all $a, b \in R$. We shall be interested in traces tr such that $\operatorname{tr}(1) \neq 0$ in A. Such a trace exists if and only if $1 \notin [R, R]$, the additive subgroup of R generated by the Lie commutators $[a, b] = ab - ba$ $(a, b \in R)$. Indeed, if $1 \notin [R, R]$, then the canonical map $R \to R/[R, R] = A$ defines a trace tr of R with $\operatorname{tr}(1) \neq 0$. For the converse just note that any trace of R vanishes on $[R, R]$. Two standard facts that we shall use are as follows.

(a) Any trace $\operatorname{tr}: R \to A$ gives rise to a trace $\operatorname{tr}_n: M_n(R) \to A$ of the matrix ring $M_n(R)$ by setting $\operatorname{tr}_n([r_{ij}]) = \sum_i \operatorname{tr}(r_{ii})$. If $\operatorname{tr}(1) \neq 0$ in A and multiplication by n is injective on A, then $\operatorname{tr}_n(1) = n \cdot \operatorname{tr}(1) \neq 0$.

(b) Let $k^t[\Gamma]$ be a twisted group algebra of the group Γ over the field k. Thus $k^t[\Gamma]$ has a k-basis $\{\bar{g} \mid g \in \Gamma\}$ and multiplication is defined distributively using the fact that $\bar{g} \cdot \bar{h} = t(g, h)\overline{gh}$ $(g, h \in \Gamma)$, where $t: \Gamma \times \Gamma \to k^*$ is a 2-cocycle. In particular, all $\bar{g}$ are units in $k^t[\Gamma]$ and $\bar{1} \in k^*$. The map $\mathrm{tr}: k^t[\Gamma] \to k$, $\Sigma_{g \in \Gamma} a_g \bar{g} \to a_1 \bar{1}$, defines a trace of $k^t[\Gamma]$ with $\mathrm{tr}(1) = 1$. The equality $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in k^t[\Gamma]$ follows from the fact that $t(g, g^{-1}) = t(g^{-1}, g)$ holds for all $g \in \Gamma$.

(2.2) LEMMA. *Assume that R has a trace* tr *with* $\mathrm{tr}(1) \neq 0$ *and* $\mathrm{tr}(a^x) = \mathrm{tr}(a)$ *for all $a \in R$, $x \in G$. Then $1 \notin [S, S]$.*

Proof. Suppose that

$$1 = \sum_i [u_i, v_i]$$

for suitable $u_i, v_i \in S$ and write $u_i = \Sigma_{x \in G} u_{i,x} x$, $v_i = \Sigma_{x \in G} v_{i,x} x$ with $u_{i,x}, v_{i,x} \in R$. Comparing identity coefficients in the above equation we obtain

$$1 = \sum_i \sum_{x \in G} (u_{i,x} v_{i,x^{-1}}^{x^{-1}} - v_{i,x^{-1}} u_{i,x}^x).$$

Applying tr to the right hand side yields 0, since

$$\mathrm{tr}(u_{i,x} v_{i,x^{-1}}^{x^{-1}}) = \mathrm{tr}(u_{i,x}^x v_{i,x^{-1}}) = \mathrm{tr}(v_{i,x^{-1}} u_{i,x}^x)$$

for all i and x. Thus we get $\mathrm{tr}(1) = 0$, which is a contradiction.

(2.3) *Reduced (Goldie-) ranks* (cf. [3]). If R is a semiprime Noetherian ring, then R has a semisimple Artinian classical ring of quotients $Q(R)$. For any finitely generated right R-module V, the reduced rank $\rho(V)$ is then defined by

$$\rho(V) = \text{composition length of } V \otimes_R Q(R) \text{ over } Q(R).$$

Clearly, $\rho(\cdot)$ is additive on direct sums of modules, and hence defines a function on $K_0(R)$. We write $\rho(R)$ for the reduced rank of the regular R-module R_R, often called the Goldie rank of R.

(2.4) THEOREM. *Let p be a prime number and assume that*

(1) *R is a simple Noetherian domain with $pR = \{0\}$, and*

(2) *$G (\neq 1)$ is a finite p-group of outer automorphisms of R.*

Then $S = R * G$ is a simple Noetherian ring with $\rho(S) = |G|$. If in addition,

(3) *$K_0(R) = \langle [R] \rangle$, that is, all finitely generated projectives over R are stably free, and*

(4) *$1 \notin [S, S]$,*

then p divides $\rho(P)$ for all finitely generated projectives P over S. In particular, S is not equivalent to a domain. Moreover, if $|G| = p$, then S has no non-trivial idempotents.

Proof. The first assertion is well known (and independent of the assumptions on $|G|$ and $\mathrm{char}\, R$). Indeed, S is clearly Noetherian, as R is, and (1) and (2) imply that S is a simple ring (cf. [8, Theorem 2.3]). The equality $\rho(S) = |G|$ follows from the fact that the classical ring of quotients $Q(S)$ has the form $Q(S) \simeq M_{|G|}(Q(R)^G)$, where $Q(R)^G = \{q \in Q(R) \mid q^x = q \text{ for all } x \in G\}$ is a division ring ([7, proof of Lemma 1.1iii], for example).

Assume now that (3) and (4) are satisfied, but there exists a finitely generated projective module P over S with $p \nmid \rho(P)$. By Lemma (1.5ii), assumption (3) implies that $|G| \cdot K_0(S) \subset \langle [S] \rangle$ so that, in particular,

$$|G| \cdot [P] = n \cdot [S] \quad \text{for some } n.$$

Taking reduced ranks we see that

$$n = \rho(P).$$

Moreover, the above equality in $K_0(S)$ says that, for some $r \geqslant 0$,

$$P^{|G|} \oplus S^r \simeq S^{n+r}.$$

Here we may assume that $p \mid r$, say $r = p \cdot r'$. Thus setting $V = P^{|G|/p} \oplus S^{r'}$ we have $S^{n+r} \simeq V^p$ and taking endomorphism rings we obtain

$$M_{n+r}(S) \simeq M_p(\text{End } V_S).$$

By (4), the canonical trace tr: $S \to S/[S, S] = A$ does not vanish on 1. Also, our assumption on $\rho(P)$ implies that $n+r$ is non-zero in $\mathbb{F}_p \subset S$ and hence acts injectively on A. Thus, by (2.1 a), we know that $1 \notin [M_{n+r}(S), M_{n+r}(S)]$. On the other hand, using the standard matrices $E_{ij} \in M_p(\text{End } V_S)$, we have

$$1 = (E_{12} + E_{23} + \ldots + E_{p1})^p = E_{12}^p + E_{23}^p + \ldots + E_{p1}^p + X = X,$$

where $X \in [M_p(\text{End } V_S), M_p(\text{End } V_S)]$ (cf. [9, Lemma 2.3.1]). Thus $1 \in [M_p(\text{End } V_S), M_p(\text{End } V_S)]$, contradiction. Therefore, $\rho(P)$ must be divisible by p.

Now assume that S is Morita equivalent to a domain A (necessarily simple Noetherian), with equivalence $F: \text{mod}-A \to \text{mod}-S$. Then $P = F(A)$ is finitely generated projective over S and $\rho_S(P) = \rho_A(A) = 1$, and we have a contradiction.

Finally, if $|G| = p$ and there exists an idempotent $e = e^2 \in S$, $e \neq 0$ or 1, then $P = eS$ satisfies $0 < \rho(P) < \rho(S) = p$, yet p divides $\rho(P)$, which is impossible.

(2.5) Example. As a first application of the above theorem, we discuss the Zalesskii–Neroslavskii example. So let k be a field with char $k = 2$ containing an element $\lambda \in k^*$ of infinite order and let $R = B_\lambda$, $\sigma \in \text{Aut}(R)$ and $S = R * \langle \sigma \rangle$ be as in Example (1.8). Then, as we have seen, assumptions (1), (2), and (3) of Theorem (2.4) are satisfied, with $p = 2$. In particular, S has Goldie rank 2. As to (4), we use the structure of R as a twisted group algebra, $R = k^t[\Gamma]$, and the trace map tr: $R \to k$ sending $a = \sum_{g \in \Gamma} a_g \bar{g} \in R$ to $a_1 \bar{1} \in k$, as in (2.1 b). The expression for a^σ in (1.8) gives $\text{tr}(a^\sigma) = \text{tr}(a)$, and hence Lemma (2.2) implies that $1 \notin [S, S]$. Therefore, S has no non-trivial idempotents and is not Morita equivalent to a domain [**10**, **11**].

(2.6) Lemma. *Let k be a field and let* Γ *be a finitely generated torsion-free nilpotent group with centre Z. Set*

$$R = k[\Gamma]_{k[Z] \setminus \{0\}},$$

the localization of the group algebra $k[\Gamma]$ *at the non-zero elements of* $k[Z]$. *Then R is a simple Noetherian domain with* $K_0(R) = \langle [R] \rangle$. *Let G be a finite group of outer automorphisms of* Γ *such that G acts trivially on Z. Then G acts on R by outer k-algebra automorphisms so that* $S = R * G$ *is a simple ring with* $1 \notin [S, S]$.

Proof. Since Γ is poly-(infinite cyclic), the group algebra $k[\Gamma]$ is a Noetherian domain [**9**, Corollary 10.2.8 and Theorem 13.1.11]. Hence R also is a Noetherian domain. The fact that R is simple is a result due to Zalesskii (see [**9**, Theorem 8.4.10]).

Again, since Γ is poly-(infinite cyclic), the 'twisted Grothendieck theorem' [**9**, Theorem 13.4.9 and Lemma 13.4.3] implies that $K_0(R) = \langle [R] \rangle$.

By Lemma (2.2), in order to show that $1 \notin [S, S]$, it suffices to construct a trace map $\mathrm{tr}: R \to F = Q(k[Z])$, the field of fractions of $k[Z]$, such that $\mathrm{tr}(1) = 1$ and $\mathrm{tr}(a^x) = \mathrm{tr}(a)$ holds for all $a \in R$, $x \in G$. For this, note that R has the structure of a twisted group algebra of Γ/Z over F, $R \simeq F^t[\Gamma/Z]$. Indeed, every element $a \in R$ can be uniquely expressed as

$$a = \sum_{y \in \Gamma/Z} a_y \bar{y},$$

where $a_y \in F$ and $\{\bar{y} \mid y \in \Gamma/Z\}$ is a fixed transversal for Z in Γ. Define $\mathrm{tr}: R \to F$ by $\mathrm{tr}(a) = a_1 \bar{1}$ as in (2.1b). Since G acts trivially on Z, it also acts trivially on $F = F\bar{1}$. Furthermore, G permutes the sets $F^* \bar{y}$ ($y \in \Gamma/Z \backslash \{1\}$) among themselves so that $\mathrm{tr}(a^x) = \mathrm{tr}(a)$ holds for all $a \in R$, $x \in G$.

Finally, since Γ/Z is poly-(infinite cyclic), the units of $R = F^t[\Gamma/Z]$ are all of the form

$$u = fg \qquad (f \in F^*, g \in \Gamma)$$

[**9**, Section 13.1]. Thus if the automorphism of R given by $x \in G$ is conjugation by u, then x acts on Γ by conjugation with g, contradicting the fact that G consists of outer automorphisms of Γ. Therefore, G acts by outer automorphisms on R, and S is simple, by [**8**, Theorem 2.3].

(2.7) Example. We close with a series of explicit examples based on the above lemma. Clearly, many further examples could be constructed along the same lines.

Fix a prime p and let Γ be the group

$$\Gamma = \langle x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_p \mid [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1 \text{ for all } i \neq j,\ [x_1, y_1] = [x_2, y_2] = \ldots = [x_p, y_p] = z \text{ is central} \rangle.$$

Then Γ is finitely generated torsion-free nilpotent of class 2, with centre $Z = \langle z \rangle$. Let σ be the automorphism of Γ which cyclically permutes the x_i and y_i. Then σ has order p, it acts trivially on Z and is outer, since it acts non-trivially on $\Gamma/[\Gamma, \Gamma]$. Thus, if k is a field with $\mathrm{char}\, k = p$ and $R = k[\Gamma]_{k[Z] \backslash \{0\}}$, then we conclude from Lemma (2.6) and Theorem (2.4) that $S = R * \langle \sigma \rangle$ is a simple Noetherian ring of Goldie rank p without non-trivial idempotents, and is not Morita equivalent to a domain.

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