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ON THE COHOMOLOGY OF POLYCYCLIC-BY-FINITE GROUPS

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Let $k[G]$ be the group algebra of a polycyclic-by-finite group G over a field k. We show that, for any finitely generated $k[G]$-module V, $H^*(G,V)=\bigoplus_{n\geq 0}H^n(G,V)$ is a Noetherian module over the cohomology algebra $H^*(G,k)$. In particular, the power series $P_G(V;t)=\sum_{n\geq 0}\dim_k H^n(G,V)t^n\in\mathbb{Z}[\![t]\!]$ is a rational function in t of the form $f(t)/\prod_{i=1}^{s}(1-t^{k_i})$, where $f(t)\in\mathbb{Z}[t]$. For example, if G has no non-trivial finite normal subgroups and contains a torsion-free normal subgroup of index 2 and char $k=2$, then we show that $P_G(V;-1)$ is always defined and equals $(-1)^h\varrho(V)/\varrho(k[G])$, where $h=h(G)$ is the Hirsch number of G and $\varrho(\cdot)$ denotes the Goldie rank.

Introduction

A celebrated theorem due to Farkas, Snider [9], and Cliff [5] asserts that the group algebra $k[G]$ of a torsion-free polycyclic-by-finite group G over a field k has no zero divisors. The proof heavily uses the fact that, for G torsion-free, $k[G]$ has finite global dimension. In the presence of torsion in G, however, $k[G]$ has infinite global dimension if char k divides the order of an element of $G\setminus\{1\}$. The goal of this note is to establish a cohomological finiteness result which holds for general polycyclic-by-finite G and arbitrary coefficient fields. We also briefly discuss the relations between the cohomology of a finitely generated $k[G]$-module and its so-called Goldie rank in the case when $k[G]$ is prime.

Specifically, let R be a commutative Noetherian ring and let V be a finitely generated (left) module over the group ring $R[G]$, where G is polycyclic-by-finite. Then we will show in Section 1 that $H^*(G,V)=\bigoplus_{n\geq 0}H^n(G,V)$ is a Noetherian module over the cohomology ring $H^*(G,R)=\bigoplus_{n\geq 0}H^n(G,R)$ under the action given by the cup product $\cup: H^*(G,R)\times H^*(G,V)\to H^*(G,V)$. As a standard consequence, it follows that for any additive $\mathbb{Z}$-valued function λ on the class of all finitely generated R-modules, the power series

$$P(V;t)=\sum_{n\geq 0}\lambda(H^n(G,V))t^n\in\mathbb{Z}[\![t]\!]$$

is a rational function in t of the form $f(t)/\prod_{i=1}^{s}(1-t^{k_i})$ with $f(t)\in\mathbb{Z}[t]$. For finite groups, these results are due to Evens [7]. Now each polycyclic-by-finite group G contains a normal subgroup N of finite index which is poly-(infinite cyclic), and hence of finite cohomological dimension. Thus our strategy is to *use* Evens's theorem and extend it by means of the Lyndon-Hochschild-Serre (LHS) spectral sequence for $1\to N\to G\to G/N\to 1$. The arguments here are similar to the ones used by Evens.

In Section 2, we consider finitely generated modules V over the group algebra $k[G]$ of G over a field k, under the assumption that $k[G]$ is prime. (This holds if and only if G has no finite normal subgroups $\neq\langle 1\rangle$ [14, Theorem 4.2.10].) We discuss the relations between the Poincaré series $P(V;t)=\sum_{n\geq 0}\dim_k H^n(G,V)t^n$ and the Goldie rank $\varrho(V)$ of V. Here, by definition, $\varrho(V)$ is the composition length of $Q(k[G])\otimes_{k[G]}V$ over $Q(k[G])$, the simple Artinian quotient ring of $k[G]$. In the special case where G has a torsion-free normal subgroup of index 2 and char $k=2$ it is shown that, for any finitely generated $k[G]$-module V, $P_G(V;-1)=(-1)^h\varrho(V)/\varrho(k[G])$ with $h=h(G)$, the Hirsch number of G.

The crucial facts about polycyclic-by-finite groups that will be used in this article are the following:

(1) group rings of polycyclic-by-finite groups over Noetherian rings are Noetherian [14, 10.2.7], and

(2) polycyclic-by-finite groups are virtual Poincaré duality groups [4, Ch. VIII, §§10, 11], [2].

A good deal of the material in Section 1 holds for more general classes of groups, provided one assumes the coefficient modules V in $H^*(G,V)$ to be finitely generated over the ground ring R (see Remark 3). Since we are interested in applications to infinite-dimensional modules, we will concentrate on polycyclic-by-finite groups.

Notations and conventions

Throughout, R will be a commutative Noetherian ring, k will denote a commutative field, and G will be a polycyclic-by-finite groups. All modules over the group rings $R[G]$ and $k[G]$ will be left modules. In general, our notation follows [14] and [6].

1. The finiteness theorem

The proof of the first lemma has been shown to us by R. Bieri.

Lemma 1. *Let V be a finitely generated $R[G]$-module. Then for all $n\geq 0$, $H^n(G,V)$ and $H_n(G,V)$ are Noetherian as R-modules.*

Proof. Since $R[G]$ is Noetherian, we can choose a projective resolution $\mathbf{P}=(P_n)_{n\geq 0}$

of V over $R[G]$ with each P_n finitely generated over $R[G]$. Now $H_n(G,V)\cong \mathrm{Tor}_n^{R[G]}(R,V)\cong H_n(R\otimes_{R[G]}\mathbf{P})$, and $R\otimes_{R[G]}\mathbf{P}$ is a complex of finitely generated R-modules. Since R is Noetherian, each $H_n(R\otimes_{R[G]}\mathbf{P})$ is Noetherian over R, which proves the assertion for $H_n(G,V)$.

Now let N be a torsion-free polycyclic normal subgroup of G having finite index in G. Then, by [2, Satz 3.1.2 and Bemerkung 1], there exists R-isomorphisms

$$H^n(N,V)\cong H_{\mathrm{cd}\,N-n}(N,\tilde{R}\otimes_R V),$$

where cd N denotes the cohomological dimension of N and $\tilde{R}$ is an $R[N]$-module such that $\tilde{R}\cong R$ as R-modules but each $x\in N$ acts as Id or $-$Id. Clearly, $\tilde{R}\otimes_R V$ is finitely generated over $R[N]$, as V is, and so the foregoing implies that $H^n(N,V)$ is a Noetherian R-module. To prove the assertion for $H^n(G,V)$ consider the LHS-spectral sequence for $1\to N\to G\to G/N\to 1$. Its E_2-term $E_2^{p,q}(V)=H^p(G/N,H^q(N,V))$ is finitely generated over R. Hence $E_\infty^{p,q}(V)$ is Noetherian over R, being a subquotient of $E_2^{p,q}(V)$, and we conclude that $H^{p+q}(G,V)$ is Noetherian over R. $\square$

Now consider $H^*(G,R)=\bigoplus_{n\geq 0}H^n(G,R)$, with the trivial action of G on R, and $H^*(G,V)=\bigoplus_{n\geq 0}H^n(G,V)$. The multiplication of R gives rise to a cup product $\cup : H^p(G,R)\times H^q(G,R)\to H^{p+q}(G,R)$ which makes $H^*(G,R)$ an R-algebra and, similarly, the action of R on V yields an action of $H^*(G,R)$ on $H^*(G,V)$ via cup products. The following result is the promised extension of Evens's theorem [7, Theorem 6.1].

Theorem 2. *Let V be a finitely generated $R[G]$-module. Then $H^*(G,V)$ is a Noetherian module over $H^*(G,R)$.*

Proof. Let N be a torsion-free polycyclic normal subgroup of G having finite index in G. Then N has finite cohomological dimension [10, §8.8, Lemma 8] and so Lemma 1 implies that $H^*(N,V)=\bigoplus_{n=0}^{\mathrm{cd}\,N}H^n(N,V)$ is Noetherian over R. By Evens's theorem, we conclude that $H^*(G/N,H^*(N,V))$ is Noetherian over $H^*(G/N,R)$.

Now consider the LHS-spectral sequence for $1\to N\to G\to G/N\to 1$, with E_2-term $E_2^{p,q}(\cdot)\cong H^p(G/N,H^q(N,\cdot))$. For each $r\geq 2$, there is a canonical pairing

$$\cdot : E_r^{p,q}(R)\times E_r^{s,t}(V)\to E_r^{p+s,q+t}(V)$$

induced by the action of R on V. If d_r denotes the differential of $E_r(\cdot)$, then one has the following product rule

$$d_r(a\cdot b)=(d_ra)\cdot b+(-1)^{p+q}a\cdot(d_rb) \qquad (a\in E_r^{p,q}(R),\, b\in E_r^{s,t}(V)).$$

Moreover, since $E_r^{p,q}(\cdot)=E_\infty^{p,q}(\cdot)$ for $r>\max\{p,q+1\}$, we get an analogous pairing for the E_∞-terms. For $r=2$, the above pairing coincides with the cup product pairing

$$\cup : H^p(G/N,H^q(N,R))\times H^s(G/N,H^t(N,V))\to H^{p+s}(G/N,H^{q+t}(N,V)),$$

except for a $\pm$ sign [12, Section II.5]. Specifically, letting Φ_V denote the isomorphism $E_2(V) \xrightarrow{\sim} H^*(G/N, H^*(N, V))$ and similarly for R, we have

$$\Phi_V(a \cdot b) = (-1)^{pt} \Phi_R(a) \cup \Phi_V(b) \qquad (a \in E_2^{p,q}(R), b \in E_2^{s,t}(V)).$$

In particular, $E_2^{*,0}(R)$ becomes a graded ring, isomorphic to the cohomology ring $H^*(G/N, R)$, and $E_2(V) = \bigoplus_{t=0}^{\mathrm{cd}\, N} E^{*,t}(V)$ is a direct sum of graded modules over $E_2^{*,0}(R)$. Altering the action by a $\pm$ sign as above doesn't affect the lattice of graded submodules and so we conclude from the first paragraph of the proof that $E_2(V)$ has the ascending chain condition for graded $E_2^{*,0}(R)$-submodules. Hence, by [13, Ch. A, Theorem II.3.5], $E_2(V)$ is Noetherian over $E_2^{*,0}(R)$.

Let $h_r : \ker d_r \to E_{r+1}(\cdot)$ be the canonical epimorphism and set

$$E_{2,\infty}(V) = \{ b \in E_2(V) \mid h_{r-1} \cdot \ldots \cdot h_2(b) \in \ker d_r \text{ for all } r \geq 2 \},$$

and similarly for $E_{2,\infty}(R)$. Note that $E_2^{*,0}(R) \subseteq E_{2,\infty}(R)$, since the spectral sequence lies in the first quadrant. The product rule for the differentials d_r implies that $E_{2,\infty}(R) \cdot E_{2,\infty}(V) \subseteq E_{2,\infty}(V)$. In particular, $E_{2,\infty}(V)$ is an $E_2^{*,0}(R)$-submodule of $E_2(V)$ and as such is Noetherian. Under the projection maps

$$E_{2,\infty}(V) \twoheadrightarrow E_\infty(V) \quad \text{and} \quad E_2^{*,0}(R) \twoheadrightarrow E_\infty^{*,0}(R),$$

the action of $E_2^{*,0}(R)$ on $E_{2,\infty}(V)$ induces the pairing $\cdot : E_\infty^{*,0}(R) \times E_\infty(V) \to E_\infty(V)$ mentioned above. Thus $E_\infty(V)$ is Noetherian as a module over $E_\infty^{*,0}(R)$ and, a fortiori, over $E_\infty(R)$.

Finally, $E_\infty(\cdot)$ is the associated graded module of $H^*(G, \cdot)$ with respect to a suitable filtration of $H^*(G, \cdot)$ which makes $H^*(G, R)$ a filtered ring and $H^*(G, V)$ a filtered module over $H^*(G, R)$ [12, Section II.1]. Therefore, $H^*(G, V)$ is Noetherian over $H^*(G, R)$ [13, Ch. D, Corollary IV.4], and the theorem is proved. □

Remark 3. If V in the above theorem is assumed to be finitely generated over R, then the same conclusion holds for much wider classes of groups G. Namely, it certainly suffices to assume that G has a subgroup N of finite index such that the trivial $R[N]$-module R has a finite resolution by finitely generated projectives over $R[N]$ (so, in the terminology of [4, Chap. VIII], G is of type VFP over R). Indeed, the above proof can be copied literally, with Lemma 1 becoming superfluous, as $H^*(N, V)$ is clearly Noetherian over R. Therefore, part (i) of the following corollary holds more generally and so does part (ii) provided the coefficient module V is finitely generated over R.

Corollary 4. (i) *$H^*(G, R)$ is a Noetherian ring and a finitely generated R-algebra* (Quillen [15, Proposition 14.5]).

(ii) *Let V be a finitely generated $R[G]$-module. Then for any additive $\mathbb{Z}_{\geq 0}$-valued function λ on the class of all finitely generated R-modules, the power series* (*cf. Lemma* 1)

$$P_G(V;t)=\sum_{n\geq 0}\lambda(H^n(G,V))t^n\in\mathbb{Z}[\![t]\!]$$

is a rational function in t of the form $f(t)/\prod_{i=1}^{s}(1-t^{k_i})$ *with* $f(t)\in\mathbb{Z}[t]$. *Moreover, if* $\mathrm{cr}_G(V)$ *denotes the order of the pole of* $P_G(V;t)$ *at* $t=1$, *then*

$$\mathrm{cr}_G(V)=\inf\{r\in\mathbb{R}\mid \lambda(H^n(G,V))\leq c\,n^{r-1}\ \textit{for some}\ c>0\ \textit{and all}\ n\gg 0\}.$$

Furthermore, $\mathrm{cr}_G(V)\leq\mathrm{cr}_G(R)$ *where R is the trivial* $R[G]$*-module.*

Proof. (i) Theorem 2 implies that $H^*(G,R)$ is a Noetherian ring. In particular, the ideal $H^+(G,R)=\bigoplus_{n>0}H^n(G,R)$ is finitely generated as a left ideal, say by the elements $x_i\in H^{k_i}(G,R)$ $(i=1,2,\ldots,s)$. It is then easy to see that $x_1,x_2,\ldots,x_s$ together with 1 generate $H^*(G,R)$ as an R-algebra (cf. [1, Proposition 10.7]).

(ii) In view of part (i) and Theorem 2, the assertion concerning the rationality of $P_G(V;t)$ follows from the Hilbert–Serre theorem [1, Theorem 11.1]. Indeed, s and the exponents k_i in $f(t)/\prod_{i=1}^{s}(1-t^{k_i})$ can be chosen as in part (i). (The proof given in [1] works for $H^*(G,R)$, since $H^*(G,R)$ satisfies $xy=(-1)^{nm}yx$ for $x\in H^n(G,R)$, $y\in H^m(G,R)$.)

As a consequence of the specific form of the rational function $P_G(V;t)$ it follows that, for large enough n, the n-th coefficient $d_n=\lambda(H^n(G,V))$ can be written as

$$d_n=\sum_{j=1}^{r}P_j(n)\alpha_j^n.$$

Here, the α_j's are roots of $\prod_{i=1}^{s}(1-t^{k_i})$ and each $P_j(n)$ is a polynomial in n with rational coefficients [11, Section 3.1]. Set $k=lcm\{k_i\mid i=1,2,\ldots,s\}$ and $\varrho=\inf\{r\in\mathbb{R}\mid\lambda(H^n(G,V))\leq c\,n^{r-1}$ for some $c>0$ and all $n\gg 0\}$. Then $\alpha_j^k=1$ for all j and there are functions $c_l:\mathbb{Z}/k\mathbb{Z}\to\mathbb{Q}$ $(l=0,1,\ldots,h)$ with $c_h\neq 0$ and

$$d_n=\sum_{l=0}^{h}c_l(n+k\mathbb{Z})n^l\quad\text{for all } n\gg 0.$$

Clearly, $\varrho-1=h$. Now consider

$$Q(t)=P_G(V;t)/(1-t)=\sum_{n\geq 0}D_nt^n$$

with

$$D_n=\sum_{m\leq n}d_n=\sum_{l=0}^{h+1}C_l(n+k\mathbb{Z})\,n^l\qquad(n\gg 0)$$

for suitable functions $C_l:\mathbb{Z}/k\mathbb{Z}\to\mathbb{Q}$. Since $D_{n+1}\geq D_n\geq 0$ for all n, it follows that C_{h+1} is constant >0.

Write $Q(t)=Q_1(t)+Q_2(t)$, where $Q_1(t)$ has coefficients of absolute value $<D\,n^h$ for a suitable constant $D>0$ and $Q_2(t)=C_{h+1}\sum_{n\geq 0}n^{h+1}t^n$. Then, for $0<x<1$, we have

$$|Q_1(x)|(1-x)^{h+1} \leq D\left(\sum_{n\geq 0} n^h x^n\right)(1-x)^{h+1}$$
$$\leq D\left(\frac{\mathrm{d}^h}{\mathrm{d}x^h}\frac{1}{1-x}\right)(1-x)^{h+1} = D\,h!$$

Therefore, $Q_1(t)$ has a pole of order at most $h+1$ at $t=1$. Similarly, $Q_2(t)$ has a pole of order exactly $h+2$ at $t=1$ and so the same holds for $Q(t)$. This shows that $P_G(V;t)$ has a pole of order $h+1$ at $t=1$, whence $\mathrm{cr}_G(V)=h+1=\varrho$, as we have claimed.

Finally, if $\varrho(R)$ is defined in analogy with $\varrho=\varrho(V)$ above, then Theorem 2 and the foregoing together imply that $\mathrm{cr}_G(V)=\varrho(V)\leq\varrho(R)=\mathrm{cr}_G(R)$. This completes the proof. □

Example 5. Suppose G does not contain any subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ and let the coefficient ring R be of characteristic p. Then, by [4, Ch. X, §6], there exists a positive integer d such that $H^n(G,V)\cong H^{n+d}(G,V)$ holds for any $R[G]$-module V and all $n>h(G)$, the Hirsch number of G. In particular, if V is finitely generated over $R[G]$ and $P_G(V;t)$ is as in the above corollary, then

$$P_G(V;t)=\frac{f(t)}{1-t^d}$$

for some polynomial $f(t)\in\mathbb{Z}[t]$ of degree $\leq h(G)+d$. For concrete computations see Proposition 11.

We conclude this section with a brief discussion of the case where $R=k$ is a commutative field. There are several notions of dimension for $H^*(G,k)$ at our disposal. Namely, (i) the Gelfand–Kirillov dimension over k (see [3]), (ii) the prime length or classical Krull dimension of $H^*(G,k)$, and (iii) the Gabriel–Rentschler Krull dimension [17]. However, using the fact that $H^*(G,k)$ is a finitely generated module over the central subalgebra $\bigoplus_{n\geq 0}H^{2n}(G,k)$ which in turn is a finitely generated k-algebra, by Corollary 4(i) and the Artin–Tate lemma, it is easy to show that (i), (ii), and (iii) all coincide with $\mathrm{cr}_G(k)$. The following result of Quillen's (again valid for G of type VFP over k) determines this number [15, Theorem 14.1].

Proposition 6 (Quillen). *If* char $k=p$, *then* $\mathrm{cr}_G(k)$ *equals the p-rank of G, i.e. the largest integer r such that G contains an elementary abelian subgroup of order* p^r.

2. Relations with Goldie ranks

Throughout this section, *G will be a polycyclic-by-finite group without finite normal subgroups* $\neq\langle 1\rangle$. Thus, for any field k, the group algebra $k[G]$ is prime [14, Theorem 4.2.10]. If V is a finitely generated $k[G]$-module, then we let $\varrho(V)$ denote its *Goldie rank*, i.e.,

$$\varrho(V)=\varrho_G(V)=\text{composition length of } Q(k[G])\otimes_{k[G]} V,$$

where $Q(k[G])$ is the classical simple Artinian ring of quotients of $k[G]$. Furthermore, we define the *normalized Goldie rank* $\chi(V)$ of V to be

$$\chi(V)=\chi_G(V)=\varrho(V)/\varrho(k[G]).$$

Since $Q(k[G])$ is flat over $k[G]$, ϱ is additive on short exact sequences and hence χ is also additive. The following lemma describes some further properties of χ. Part (ii) is due to Rosset [16, Proposition 4] and will not be needed in the sequel.

Lemma 7. (i) *Let V be a finitely generated $k[G]$-module and let $H\le G$ be a subgroup of finite index. Then*

$$\chi_G(V)=[G:H]^{-1}\chi_H(V).$$

(ii) *Let $H\le G$ be a subgroup of G with $k[H]$ prime and let W be a finitely generated $k[H]$-module. Then*

$$\chi_G(k[G]\otimes_{k[H]} W)=\chi_H(W).$$

Proof of (i). Note that H has no finite normal subgroups. Choose a torsion-free normal subgroup N of G with $N\le H$ and $[G:N]$ finite. We can clearly assume that $H=N$ so that H is normal in G and torsion-free.

Let $\mathscr{C}$ denote the set of nonzero elements of $k[H]$. Since $k[H]$ is a Noetherian domain [9, 5], $\mathscr{C}$ is an Ore set of regular elements in $k[H]$, and in $k[G]$ [14, proof of Lemma 13.3.5(ii)]. Moreover, $Q(k[G])=\mathscr{C}^{-1}k[G]\supseteq\mathscr{C}^{-1}k[H]=Q(k[H])$, and $Q=Q(k[G])$ is simple Artinian and has (right and left) dimension $[G:H]$ over the division subring $D=Q(k[H])$. If V is a finitely generated $k[G]$-module, then $Q\otimes_{k[G]} V$ is finitely generated over Q and hence of the form $Q\otimes_{k[G]} V\cong B^{(r)}$, where B is a simple left ideal of Q and $r=\varrho(V)$. In particular, $Q\cong B^{(s)}$ with $s=\varrho(k[G])$ and so we have

$$\dim_D B=\frac{1}{s}\dim_D Q=[G:H]/s.$$

Therefore, $\chi_G(V)=r/s=[G:H]^{-1}\dim_D(Q\otimes_{k[G]} V)$. To complete the proof, note that $Q\otimes_{k[G]} V\cong D\otimes_{k[H]} V$ as D-modules, since $Q=\mathscr{C}^{-1}k[G]\cong D\otimes_{k[H]} k[G]$, and so $\dim_D(Q\otimes_{k[G]} V)=\varrho_H(V)=\chi_H(V)$. □

The following proposition gives a cohomological description of $\chi_G(V)$ in certain situations where V has finite projective dimension.

Proposition 8. *Let V be a finitely generated $k[G]$-module and set $h=h(G)$, the Hirsch number of G. Assume that either*

(i) *V has a finite free resolution, or*
(ii) *G is torsion-free.*
Then

$$\chi_G(V) = \sum_{i\geq 0} (-1)^i \dim_k H_i(G,V)$$
$$= (-1)^h \sum_{i\geq 0} (-1)^i \dim_k H^i(G,V).$$

Proof. For any finitely generated $k[G]$-module W with $H_i(G,W)=0$ and $H^i(G,W)=0$ for $i \gg 0$ set

$$\phi_G(W) = \sum_{i\geq 0} (-1)^i \dim_k H_i(G,W)$$

and

$$\psi_G(W) = \sum_{i\geq 0} (-1)^i \dim_k H^i(G,W).$$

Note that ϕ_G and ψ_G are additive on short exact sequences of finitely generated $k[G]$-modules with finite homology and cohomology.

(i) Since χ_G, ϕ_G, and ψ_G are additive, it suffices to consider the case where $V=k[G]$. Then

$$\chi_G(k[G]) = 1 = \dim_k H_0(G,k[G]) = \phi_G(k[G]).$$

Moreover, if $H \leq G$ is a torsion-free subgroup of finite index in G, then, by [2, Satz 3.1.2] and [4, Ch. VIII, Theorem 10.1],

$$H^i(G,k[G]) \simeq H^i(H,k[H]) = \begin{cases} 0 & \text{for } i \neq h, \\ k & \text{for } i = h. \end{cases}$$

Therefore, $(-1)^h \psi_G(k[G]) = \dim_k H^h(G,k[G]) = 1 = \phi_G(k[G]) = \chi_G(k[G])$, and (i) follows.

(ii) Since $k[G]$ has finite global dimension, we may assume that V is finitely generated projective. Fix an orientable poly-(infinite cyclic) normal subgroup $H \leq G$ of finite index in G. Then, by the theorem of Cliff, Farkas and Snider ([14, proof of Theorem 13.4.18] and [5, proof of Theorem 1]), we have

$$\phi_H(V) = \dim_k V/\omega(k[H]) \cdot V = [G:H] \dim_k V/\omega(k[G]) \cdot V$$
$$= [G:H]\, \phi_G(V).$$

Here, $\omega(\cdot)$ denotes the augmentation ideal. Furthermore, by [2, Satz 3.1.2] again, we have, with the usual twisting $\tilde{k} = H^h(G,k[G])$ of k,

$$H^i(G,V) \cong H_{h-i}(G, \tilde{k} \otimes_k V) = 0 \quad \text{for } i \neq h,$$

because $\tilde{k} \otimes_k V$ is also projective over $k[G]$. Thus

$$(-1)^h \psi_G(V) = \dim_k H^h(G,V) = \phi_G(\tilde{k} \otimes_k V),$$

and the foregoing implies that

$$(-1)^h[G:H]\,\psi_G(V)=[G:H]\,\phi_G(\tilde{k}\otimes_k V)=\phi_H(\tilde{k}\otimes_k V)$$
$$=(-1)^h\psi_H(V).$$

Finally, by the 'twisted Grothendieck theorem' [14, Theorem 13.4.9], $V|_{k[H]}$ is stably free, and so part (i) implies that

$$\chi_H(V)=\phi_H(V)=(-1)^h\psi_H(V).$$

As $\chi_H(V)=[G:H]\,\chi_G(V)$, by Lemma 7(i), part (ii) is proved. □

Corollary 9. *Let V be a finitely generated $k[G]$-module and let $h=h(G)$ denote the Hirsch number of G. Then, for any torsion-free subgroup $H\leq G$ of finite index in G, we have*

$$\chi_G(V)=[G:H]^{-1}\sum_{i\geq 0}(-1)^i\dim_k H_i(H,V)$$
$$=(-1)^h[G:H]^{-1}\sum_{i\geq 0}(-1)^i\dim_k H^i(H,V).$$

Proof. This follows from Lemma 7(i) and Proposition 8(ii). □

Using the notation $P_G(V;t)=\sum_{i\geq 0}\dim_k H^i(G,V)\,t^i$ introduced in Corollary 4, the second equality in the above corollary can be expressed as

$$\chi_G(V)=(-1)^h[G:H]^{-1}P_H(V;-1).$$

Similarly, Proposition 8(i) states that

$$\chi_G(V)=(-1)^hP_G(V;-1)$$

in case V has a finite free resolution over $k[G]$.

Lemma 10. *Let $0\to U\to F\to V\to 0$ be an exact sequence of finitely generated $k[G]$-modules and assume that $H^i(G,F)=0$ ($i\gg 0$). Then $P_G(V;t)$ is defined at $t=-1$ if and only if this holds for $P_G(U;t)$ and, in this case,*

$$P_G(V;-1)+P_G(U;-1)=P_G(F;-1).$$

Proof. Let $M_i\subset H^i(G,U)$ denote the kernel of $H^i(G,U)\to H^i(G,F)$. Then the cohomology sequence yields

$$\dim_k H^i(G,U)+\dim_k H^i(G,V)=\dim_k H^i(G,F)+\dim_k M_i+\dim_k M_{i+1}$$

for all $i\geq 0$. Therefore, in $\mathbb{Z}[\![t]\!]$, we have

$$P_G(U;t)+P_G(V;t)=P_G(F;t)+\sum_{i\geq 0}(\dim_k M_i+\dim_k M_{i+1})t^i$$
$$=P_G(F;t)+M(t)(t+1),$$

where we have set $M(t)=\sum_{i\geq 0}\dim_k M_{i+1}t^i\in\mathbb{Z}[t]$. If $H^i(G,F)=0$ for $i>n$, say, then $M_{i+1}=H^{i+1}(G,U)\simeq H^i(G,V)$ for $i>n$ and so $t\cdot P_G(V;t)=t\cdot M(t)+f=P_G(U;t)+g$ for suitable polynomials $f,g\in\mathbb{Z}[t]$. This shows that if one of $P_G(V;t)$, $P_G(U;t)$, or $M(t)$ is defined at $t=-1$, then so are the other two and, substituting $t=-1$ in the above equation, we get $P_G(U;-1)+P_G(V;-1)=P_G(F;-1)$. □

As an application of the foregoing we offer the following result which deals with $k[G]$-modules of infinite projective dimension. Recall our general convention in this section: G has no non-trivial finite normal subgroups.

Proposition 11. *Suppose G as a torsion-free normal subgroup H such that G/H is finite cyclic and let $h=h(G)$ denote the Hirsch number of G.*

(i) *Let V be a finitely generated $k[G]$-module such that $V|_{k[H]}$ is projective. Set $\tilde{V}=\tilde{k}\otimes_k V$, where $\tilde{k}=H^h(G,k[G])$ as usual, and $\bar{V}=\tilde{V}/\omega(k[H])\cdot\tilde{V}$. Then*

$$P_G(V;t)=t^h\left(\dim_k \bar{V}^{G/H}+\frac{t}{1-t^2}(\dim_k H^1(G/H,\bar{V})+\dim_k H^2(G/H,\bar{V})t)\right).$$

(ii) *Suppose* char $k=2$ *and G/H has order 2. Then $P_G(V;-1)$ is defined for all finitely generated $k[G]$-modules V, and $P_G(V;-1)=(-1)^h\chi_G(V)$.*

Proof. (i) We have $H^i(H,V)\simeq H_{h-i}(H,\tilde{V})=0$ for $i\neq h$, since $\tilde{V}$ is projective over $k[H]$. The LHS-spectral sequence for $1\to H\to G\to G/H\to 1$ implies that $H^i(G,V)=0$ for $i<h$ and $H^i(G,V)\simeq H^{i-h}(G/H,H^h(H,V))\simeq H^{i-h}(G/H,\bar{V})$ for $i\geq h$. Using the fact that G/H is cyclic, we obtain

$$P_G(V;t)=t^h\left(\dim_k \bar{V}^{G/H}+\sum_{i\geq 1}\dim_k H^i(G/H,\bar{V})t^i\right)$$

$$=t^h\left(\dim_k \bar{V}^{G/H}+\frac{t}{1-t^2}(\dim_k H^1(G/G,\bar{V})+\dim_k H^2(G/h,\bar{V})t)\right),$$

as we have claimed.

(ii) Let $0\to U\to F\to V\to 0$ be an exact sequence of $k[G]$-modules with F finitely generated free. Then $P_G(F;-1)=(-1)^h\chi_G(F)$, by Proposition 8(i), and Lemma 10 implies that $P_G(V;-1)$ is defined at $t=-1$ and equals $(-1)^h\chi_G(V)$ if and only if the corresponding fact holds for $P_G(U;-1)$. Thus, arguing by induction on the projective dimension of $V|_{k[H]}$, we may assume that $V|_{k[H]}$ is projective. But then part (i) implies that

$$P_G(V;-1)=t^h\left(\dim_k \bar{V}^{G/H}+\frac{t}{1-t}\dim_k H^1(G/H,\bar{V})\right),$$

where we have used the notation of (i) and the fact that, for G/H of order 2 and char $k=2$, $H^1(G/H,W)\simeq H^2(G/H,W)$ holds for all $k[G/H]$-modules W.

Therefore, $P_G(V;-1)$ is defined and, moreover,

$$P_G(V;-1)=(-1)^h(\dim_k \bar{V}^{G/H}-\tfrac{1}{2}\dim_k \bar{V}^{G/H}/\omega(k[G/H])\cdot \bar{V})$$
$$=(-1)^h\tfrac{1}{2}\dim_k \bar{V}.$$

To see the latter equality, note that $\bar{V}$ has the form $\bar{V}\simeq k[G/H]^r\oplus k^s$ for suitable $r,s\geq 0$, where k denotes the trivial module. Thus $\dim_k \bar{V}^{G/H}=r+s$, $\dim_k \bar{V}^{G/H}/\omega(k[G/H])\cdot \bar{V}=s$ and so the expression in brackets equals $r+\frac{1}{2}s=\frac{1}{2}\dim_k \bar{V}$. Finally, by Corollary 9,

$$\chi_G(\tilde{V})=\tfrac{1}{2}\dim_k H_0(H,\tilde{V})=\tfrac{1}{2}\dim_k \bar{V}$$

and so it suffices to show that $\chi_G(\tilde{V})=\chi_G(V)$. But if N denotes the kernel of the action of H on $\tilde{k}$, then N has index at most 2 in H and the restrictions of $\tilde{V}$ and V to $k[N]$ are isomorphic. Lemma 7(i) now implies that $\chi_G(\tilde{V})=\chi_G(V)$, as required. □

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