ON THE COHOMOLOGY OF POLYCYCLIC-BY-FINITE GROUPS

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Let \( k[G] \) be the group algebra of a polycyclic-by-finite group \( G \) over a field \( k \). We show that, for any finitely generated \( k[G] \)-module \( V \), \( H^*(G, V) = \bigoplus_{n \geq 0} H^n(G, V) \) is a Noetherian module over the cohomology algebra \( H^*(G, k) \). In particular, the power series \( P(G; t) = \sum_{n \geq 0} \dim_k H^n(G, V) t^n \in \mathbb{Z}[t] \) is a rational function in \( t \) of the form \( f(t)/\prod_{i=1}^r (1 - t^{k_i}) \), where \( f(t) \in \mathbb{Z}[t] \). For example, if \( G \) has no non-trivial finite normal subgroups and contains a torsion-free normal subgroup of index 2 and char \( k = 2 \), then we show that \( P(G; -1) \) is always defined and equals \((-1)^h \varphi(V)/\varphi(k[G])\), where \( h = h(G) \) is the Hirsch number of \( G \) and \( \varphi(\cdot) \) denotes the Goldie rank.

Introduction

A celebrated theorem due to Farkas, Snider [9], and Cliff [5] asserts that the group algebra \( k[G] \) of a torsion-free polycyclic-by-finite group \( G \) over a field \( k \) has no zero divisors. The proof heavily uses the fact that, for \( G \) torsion-free, \( k[G] \) has finite global dimension. In the presence of torsion in \( G \), however, \( k[G] \) has infinite global dimension if char \( k \) divides the order of an element of \( G \setminus \{1\} \). The goal of this note is to establish a cohomological finiteness result which holds for general polycyclic-by-finite \( G \) and arbitrary coefficient fields. We also briefly discuss the relations between the cohomology of a finitely generated \( k[G] \)-module and its so-called Goldie rank in the case when \( k[G] \) is prime.

Specifically, let \( R \) be a commutative Noetherian ring and let \( V \) be a finitely generated (left) module over the group ring \( R[G] \), where \( G \) is polycyclic-by-finite. Then we will show in Section 1 that \( H^*(G, V) = \bigoplus_{n \geq 0} H^n(G, V) \) is a Noetherian module over the cohomology ring \( H^*(G, R) = \bigoplus_{n \geq 0} H^n(G, R) \) under the action given by the cup product \( \cup : H^*(G, R) \times H^*(G, V) \to H^*(G, V) \). As a standard consequence, it follows that for any additive \( \mathbb{Z} \)-valued function \( \lambda \) on the class of all finitely generated \( R \)-modules, the power series

\[
P(V; t) = \sum_{n \geq 0} \lambda(H^n(G, V)) t^n \in \mathbb{Z}[t]
\]
is a rational function in $t$ of the form $f(t)/\prod_{i=1}^{j} (1 - t^{k_i})$ with $f(t) \in \mathbb{Z}[t]$. For finite groups, these results are due to Evens [7]. Now each polycyclic-by-finite group $G$ contains a normal subgroup $N$ of finite index which is poly-(infinite cyclic), and hence of finite cohomological dimension. Thus our strategy is to use Evens’s theorem and extend it by means of the Lyndon–Hochschild–Serre (LHS) spectral sequence for $1 \to N \to G \to G/N \to 1$. The arguments here are similar to the ones used by Evens.

In Section 2, we consider finitely generated modules $V$ over the group algebra $k[G]$ of $G$ over a field $k$, under the assumption that $k[G]$ is prime. (This holds if and only if $G$ has no finite normal subgroups $\neq \langle 1 \rangle$ [14, Theorem 4.2.10].) We discuss the relations between the Poincaré series $P(V; t) = \sum_{n \geq 0} \dim_k H^n(G, V)t^n$ and the Goldie rank $\varrho(V)$ of $V$. Here, by definition, $\varrho(V)$ is the composition length of $Q(k[G]) \otimes_{k[G]} V$ over $Q(k[G])$, the simple Artinian quotient ring of $k[G]$. In the special case where $G$ has a torsion-free normal subgroup of index 2 and char $k = 2$ it is shown that, for any finitely generated $k[G]$-module $V$, $P_G(V; -1) = (-1)^h \varrho(V)/\varrho(k[G])$ with $h = h(G)$, the Hirsch number of $G$.

The crucial facts about polycyclic-by-finite groups that will be used in this article are the following:

1. group rings of polycyclic-by-finite groups over Noetherian rings are Noetherian [14, 10.2.7], and
2. polycyclic-by-finite groups are virtual Poincaré duality groups [4, Ch. VIII, §§10, 11], [2].

A good deal of the material in Section 1 holds for more general classes of groups, provided one assumes the coefficient modules $V$ in $H^*(G, V)$ to be finitely generated over the ground ring $R$ (see Remark 3). Since we are interested in applications to infinite-dimensional modules, we will concentrate on polycyclic-by-finite groups.

**Notations and conventions**

Throughout, $R$ will be a commutative Noetherian ring, $k$ will denote a commutative field, and $G$ will be a polycyclic-by-finite groups. All modules over the group rings $R[G]$ and $k[G]$ will be left modules. In general, our notation follows [14] and [6].

**1. The finiteness theorem**

The proof of the first lemma has been shown to us by R. Bieri.

**Lemma 1.** Let $V$ be a finitely generated $R[G]$-module. Then for all $n \geq 0$, $H^n(G, V)$ and $H_n(G, V)$ are Noetherian as $R$-modules.

**Proof.** Since $R[G]$ is Noetherian, we can choose a projective resolution $P = (P_n)_{n \geq 0}$
of $V$ over $R[G]$ with each $P_n$ finitely generated over $R[G]$. Now $H_n(G, V) \cong \text{Tor}_n^{R[G]}(R, V) \cong H_n(R \otimes_{R[G]} P)$, and $R \otimes_{R[G]} P$ is a complex of finitely generated $R$-modules. Since $R$ is Noetherian, each $H_n(R \otimes_{R[G]} P)$ is Noetherian over $R$, which proves the assertion for $H_n(G, V)$.

Now let $N$ be a torsion-free polycyclic normal subgroup of $G$ having finite index in $G$. Then, by [2, Satz 3.1.2 and Bemerkung 1], there exists $R$-isomorphisms

$$H^n(N, V) \cong H_{cdN-n}(N, \bar{R} \otimes_R V),$$

where $cd N$ denotes the cohomological dimension of $N$ and $\bar{R}$ is an $R[N]$-module such that $\bar{R} \cong R$ as $R$-modules but each $x \in N$ acts as $1d$ or $-1d$. Clearly, $\bar{R} \otimes_R V$ is finitely generated over $R[N]$, as $V$ is, and so the foregoing implies that $H^n(N, V)$ is a Noetherian $R$-module. To prove the assertion for $H^n(G, V)$ consider the LHS-spectral sequence for $1 \to N \to G \to G/N \to 1$. Its $E_2$-term $E_2^{0,q}(V) = H^p(G/N, H^q(N, V))$ is finitely generated over $R$. Hence $E_2^{0,q}(V)$ is Noetherian over $R$, being a sub-quotient of $E_2^{0,q}(V)$, and we conclude that $H^{p+q}(G, V)$ is Noetherian over $R$. \]

Now consider $H^*(G, R) = \bigoplus_{n \geq 0} H^n(G, R)$, with the trivial action of $G$ on $R$, and $H^*(G, V) = \bigoplus_{n \geq 0} H^n(G, V)$. The multiplication of $R$ gives rise to a cup product $\cup : H^p(G, R) \times H^q(G, R) \to H^{p+q}(G, R)$ which makes $H^*(G, R)$ an $R$-algebra and, similarly, the action of $R$ on $V$ yields an action of $H^*(G, R)$ on $H^*(G, V)$ via cup products. The following result is the promised extension of Evens's theorem [7, Theorem 6.1].

**Theorem 2.** Let $V$ be a finitely generated $R[G]$-module. Then $H^*(G, V)$ is a Noetherian module over $H^*(G, R)$.

**Proof.** Let $N$ be a torsion-free polycyclic normal subgroup of $G$ having finite index in $G$. Then $N$ has finite cohomological dimension [10, §8.8, Lemma 8] and so Lemma 1 implies that $H^*(N, V) = \bigoplus_{n \geq 0} H^n(N, V)$ is Noetherian over $R$. By Evens's theorem, we conclude that $H^*(G/N, H^*(N, V))$ is Noetherian over $H^*(G/N, R)$.

Now consider the LHS-spectral sequence for $1 \to N \to G \to G/N \to 1$, with $E_2$-term $E_2^{p,q}(\cdot) \cong H^p(G/N, H^q(N, \cdot))$. For each $r \geq 2$, there is a canonical pairing

$$\cdot : E_r^{p,q}(R) \times E_r^{s,t}(V) \to E_r^{p+q,s+t}(V)$$

induced by the action of $R$ on $V$. If $d_r$ denotes the differential of $E_r(\cdot)$, then one has the following product rule

$$d_r(a \cdot b) = (d_r a) \cdot b + (-1)^{p+q} a \cdot (d_r b) \quad (a \in E_r^{p,q}(R), b \in E_r^{s,t}(V)).$$

Moreover, since $E_r^{p,q}(\cdot) = E_\infty^{p,q}(\cdot)$ for $r > \max\{ p, q + 1 \}$, we get an analogous pairing for the $E_\infty$-terms. For $r = 2$, the above pairing coincides with the cup product pairing

$$\cup : H^p(G/N, H^q(N, R)) \times H^s(G/N, H^t(N, V)) \to H^{p+s}(G/N, H^{q+t}(N, V)),$$
except for a ± sign [12, Section II.5]. Specifically, letting $\Phi_V$ denote the isomorphism $E_2(V) \cong H^*(G/N, H^*(N, V))$ and similarly for $R$, we have

$$\Phi_V(a \cdot b) = (-1)^{pr} \Phi_R(a) \cup \Phi_V(b) \quad (a \in E_2^{p,q}(R), b \in E_2^{r,s}(V)).$$

In particular, $E_2^{*,0}(R)$ becomes a graded ring, isomorphic to the cohomology ring $H^*(G/N, R)$, and $E_2(V) = \bigoplus_{i=0}^{n} E_2^{*,i}(V)$ is a direct sum of graded modules over $E_2^{*,0}(R)$. Altering the action by a ± sign as above doesn’t affect the lattice of graded submodules and so we conclude from the first paragraph of the proof that $E_2(V)$ has the ascending chain condition for graded $E_2^{*,0}(R)$-submodules. Hence, by [13, Ch. A, Theorem II.3.5], $E_2(V)$ is Noetherian over $E_2^{*,0}(R)$.

Let $h_r : \ker d_r \to E_{r+1}(\cdot)$ be the canonical epimorphism and set

$$E_{\infty}(V) = \{ b \in E_2(V) \mid h_{r-1}(\ldots h_2(b) \in \ker d, \text{ for all } r \geq 2 \},$$

and similarly for $E_{\infty}(R)$. Note that $E_2^{*,0}(R) \subseteq E_{\infty}(R)$, since the spectral sequence lies in the first quadrant. The product rule for the differentials $d_r$ implies that $E_{\infty}(R) \cdot E_{\infty}(V) \subseteq E_{\infty}(V)$. In particular, $E_{\infty}(V)$ is an $E_2^{*,0}(R)$-submodule of $E_2(V)$ and as such is Noetherian. Under the projection maps

$$E_{\infty}(V) \to E_\infty(V) \quad \text{and} \quad E_2^{*,0}(R) \to E_{\infty}(0)(R),$$

the action of $E_2^{*,0}(R)$ on $E_\infty(V)$ induces the pairing $\cdot : E_{\infty}(0)(R) \times E_\infty(V) \to E_\infty(V)$ mentioned above. Thus $E_\infty(V)$ is Noetherian as a module over $E_{\infty}(0)(R)$ and, a fortiori, over $E_{\infty}(R)$.

Finally, $E_{\infty}(\cdot)$ is the associated graded module of $H^*(G, \cdot)$ with respect to a suitable filtration of $H^*(G, \cdot)$ which makes $H^*(G, R)$ a filtered ring and $H^*(G, V)$ a filtered module over $H^*(G, R)$ [12, Section II.1]. Therefore, $H^*(G, V)$ is Noetherian over $H^*(G, R)$ [13, Ch. D, Corollary IV.4], and the theorem is proved.

**Remark 3.** If $V$ in the above theorem is assumed to be finitely generated over $R$, then the same conclusion holds for much wider classes of groups $G$. Namely, it certainly suffices to assume that $G$ has a subgroup $N$ of finite index such that the trivial $R[N]$-module $R$ has a finite resolution by finitely generated projectives over $R[N]$ (so, in the terminology of [4, Chap. VIII], $G$ is of type VFP over $R$). Indeed, the above proof can be copied literally, with Lemma 1 becoming superfluous, as $H^*(N, V)$ is clearly Noetherian over $R$. Therefore, part (i) of the following corollary holds more generally and so does part (ii) provided the coefficient module $V$ is finitely generated over $R$.

**Corollary 4.** (i) $H^*(G, R)$ is a Noetherian ring and a finitely generated $R$-algebra (Quillen [15, Proposition 14.5]).

(ii) Let $V$ be a finitely generated $R[G]$-module. Then for any additive $\mathbb{Z}_{\geq 0}$-valued function $\lambda$ on the class of all finitely generated $R$-modules, the power series (cf. Lemma 1)
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\[ P_G(V; t) = \sum_{n \geq 0} \lambda(H^n(G, V))t^n \in \mathbb{Z}[t] \]

is a rational function in \( t \) of the form \( f(t)/\prod_{i=1}^s (1-t^{k_i}) \) with \( f(t) \in \mathbb{Z}[t] \). Moreover, if \( \text{cr}_G(V) \) denotes the order of the pole of \( P_G(V; t) \) at \( t = 1 \), then

\[ \text{cr}_G(V) = \inf \{ r \in \mathbb{R} \mid \lambda(H^n(G, V)) \leq cn^{r-1} \text{ for some } c > 0 \text{ and all } n \gg 0 \} \]

Furthermore, \( \text{cr}_G(V) \leq \text{cr}_G(R) \) where \( R \) is the trivial \( R[G] \)-module.

Proof. (i) Theorem 2 implies that \( H^*(G, R) \) is a Noetherian ring. In particular, the ideal \( H^+(G, R) = \bigoplus_{n > 0} H^n(G, R) \) is finitely generated as a left ideal, say by the elements \( x_i \in H^{k_i}(G, R) \) \( (i = 1, 2, \ldots, s) \). It is then easy to see that \( x_1, x_2, \ldots, x_s \) together with 1 generate \( H^*(G, R) \) as an \( R \)-algebra (cf. [1, Proposition 10.7]).

(ii) In view of part (i) and Theorem 2, the assertion concerning the rationality of \( P_G(V; t) \) follows from the Hilbert–Serre theorem [1, Theorem 11.1]. Indeed, \( s \) and the exponents \( k_i \) in \( f(t)/\prod_{i=1}^s (1-t^{k_i}) \) can be chosen as in part (i). (The proof given in [1] works for \( H^*(G, R) \), since \( H^*(G, R) \) satisfies \( xy = (-1)^{m+n}yx \) for \( x \in H^n(G, R), y \in H^m(G, R) \).)

As a consequence of the specific form of the rational function \( P_G(V; t) \) it follows that, for large enough \( n \), the \( n \)-th coefficient \( d_n = \lambda(H^n(G, V)) \) can be written as

\[ d_n = \sum_{j=1}^r P_j(n) \alpha_j^n. \]

Here, the \( \alpha_j \)'s are roots of \( \prod_{i=1}^s (1-t^{k_i}) \) and each \( P_j(n) \) is a polynomial in \( n \) with rational coefficients [11, Section 3.1]. Set \( k = \text{lcm}\{k_i \mid i = 1, 2, \ldots, s \} \) and \( \varrho = \inf \{ r \in \mathbb{R} \mid \lambda(H^n(G, V)) \leq cn^{r-1} \text{ for some } c > 0 \text{ and all } n \gg 0 \} \). Then \( \alpha_j^k = 1 \) for all \( j \) and there are functions \( c_l : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Q} \) \( (l = 0, 1, \ldots, h) \) with \( c_h \neq 0 \) and

\[ d_n = \sum_{i=0}^h c_i(n + k\mathbb{Z})n^l \quad \text{for all } n \gg 0. \]

Clearly, \( \varrho - 1 = h \). Now consider

\[ Q(t) = P_G(V; t)/(1-t) = \sum_{n \geq 0} D_nt^n \]

with

\[ D_n = \sum_{m \leq n} d_n = \sum_{i=0}^{h+1} C_i(n + k\mathbb{Z})n^l \quad (n \gg 0) \]

for suitable functions \( C_i : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Q} \). Since \( D_{n+1} \geq D_n \geq 0 \) for all \( n \), it follows that \( C_{h+1} \) is constant \( > 0 \).

Write \( Q(t) = Q_1(t) + Q_2(t) \), where \( Q_1(t) \) has coefficients of absolute value \( < Dn^h \) for a suitable constant \( D > 0 \) and

\[ Q_2(t) = C_{h+1} \sum_{n \geq 0} n^{h+1}t^n. \]

Then, for \( 0 < x < 1 \), we have
Therefore, \( Q_1(t) \) has a pole of order at most \( h + 1 \) at \( t = 1 \). Similarly, \( Q_2(t) \) has a pole of order exactly \( h + 2 \) at \( t = 1 \) and so the same holds for \( Q(t) \). This shows that \( P_G(V; t) \) has a pole of order \( h + 1 \) at \( t = 1 \), whence \( cr_G(V) = h + 1 = q \), as we have claimed.

Finally, if \( q(R) \) is defined in analogy with \( q = q(V) \) above, then Theorem 2 and the foregoing together imply that \( cr_G(V) = q(V) \leq q(R) = cr_G(R) \). This completes the proof. 

**Example 5.** Suppose \( G \) does not contain any subgroup isomorphic to \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \) and let the coefficient ring \( R \) be of characteristic \( p \). Then, by [4, Ch. X, §6], there exists a positive integer \( d \) such that \( H^n(G, V) \cong H^{n+d}(G, V) \) holds for any \( R[G] \)-module \( V \) and all \( n > h(G) \), the Hirsch number of \( G \). In particular, if \( V \) is finitely generated over \( R[G] \) and \( P_G(V; t) \) is as in the above corollary, then

\[
P_G(V; t) = \frac{f(t)}{1 - t^d}
\]

for some polynomial \( f(t) \in \mathbb{Z}[t] \) of degree \( \leq h(G) + d \). For concrete computations see Proposition 11.

We conclude this section with a brief discussion of the case where \( R = k \) is a commutative field. There are several notions of dimension for \( H^*(G, k) \) at our disposal. Namely, (i) the Gelfand-Kirillov dimension over \( k \) (see [3]), (ii) the prime length or classical Krull dimension of \( H^*(G, k) \), and (iii) the Gabriel-Rentschler Krull dimension [17]. However, using the fact that \( H^*(G, k) \) is a finitely generated module over the central subalgebra \( \bigoplus_{n=0}^\infty H^{2n}(G, k) \) which in turn is a finitely generated \( k \)-algebra, by Corollary 4(i) and the Artin-Tate lemma, it is easy to show that (i), (ii), and (iii) all coincide with \( cr_G(k) \). The following result of Quillen’s (again valid for \( G \) of type VFP over \( k \)) determines this number [15, Theorem 14.1].

**Proposition 6 (Quillen).** If \( \text{char} \ k = p \), then \( cr_G(k) \) equals the \( p \)-rank of \( G \), i.e. the largest integer \( r \) such that \( G \) contains an elementary abelian subgroup of order \( p^r \).

2. Relations with Goldie ranks

Throughout this section, \( G \) will be a polycyclic-by-finite group without finite normal subgroups \( \neq \langle 1 \rangle \). Thus, for any field \( k \), the group algebra \( k[G] \) is prime [14, Theorem 4.2.10]. If \( V \) is a finitely generated \( k[G] \)-module, then we let \( q(V) \) denote
its Goldie rank, i.e.,
$$
\varrho(V) = \varrho_G(V) = \text{composition length of } Q(k[G]) \otimes_{k[G]} V,
$$
where $Q(k[G])$ is the classical simple Artinian ring of quotients of $k[G]$. Furthermore, we define the normalized Goldie rank $\chi(V)$ of $V$ to be
$$
\chi(V) = \chi_G(V) = \varrho(V) / \varrho(k[G]).
$$
Since $Q(k[G])$ is flat over $k[G]$, $\varrho$ is additive on short exact sequences and hence $\chi$ is also additive. The following lemma describes some further properties of $\chi$. Part (ii) is due to Rosset [16, Proposition 4] and will not be needed in the sequel.

**Lemma 7.** (i) Let $V$ be a finitely generated $k[G]$-module and let $H \leq G$ be a subgroup of finite index. Then
$$
\chi_G(V) = [G : H]^{-1} \chi_H(V).
$$
(ii) Let $H \leq G$ be a subgroup of $G$ with $k[H]$ prime and let $W$ be a finitely generated $k[H]$-module. Then
$$
\chi_G(k[G] \otimes_{k[H]} W) = \chi_H(W).
$$

**Proof of (i).** Note that $H$ has no finite normal subgroups. Choose a torsion-free normal subgroup $N$ of $G$ with $N \leq H$ and $[G : N]$ finite. We can clearly assume that $H = N$ so that $H$ is normal in $G$ and torsion-free.

Let $\mathcal{E}$ denote the set of nonzero elements of $k[H]$. Since $k[H]$ is a Noetherian domain [9, 5], $\mathcal{E}$ is an Ore set of regular elements in $k[H]$, and in $k[G]$ [14, proof of Lemma 13.3.5(ii)]. Moreover, $Q(k[G]) = \mathcal{E}^{-1}k[G] \supset \mathcal{E}^{-1}k[H] = Q(k[H])$, and $Q = Q(k[G])$ is simple Artinian and has (right and left) dimension $[G : H]$ over the division subring $D = Q(k[H])$. If $V$ is a finitely generated $k[G]$-module, then $Q \otimes_{k[G]} V$ is finitely generated over $Q$ and hence of the form $Q \otimes_{k[G]} V \cong B(r)$, where $B$ is a simple left ideal of $Q$ and $r = \varrho(V)$. In particular, $Q \cong B(s)$ with $s = \varrho(k[G])$ and so we have
$$
\dim_D B = \frac{1}{s} \dim_D Q = [G : H] / s.
$$
Therefore, $\chi_G(V) = r / s = [G : H]^{-1} \dim_D (Q \otimes_{k[G]} V)$. To complete the proof, note that $Q \otimes_{k[G]} V \cong D \otimes_{k[H]} V$ as $D$-modules, since $Q = \mathcal{E}^{-1}k[G] \cong D \otimes_{k[H]} k[G]$, and so $\dim_D (Q \otimes_{k[G]} V) = \varrho_H(V) = \chi_H(V)$. \[\square\]

The following proposition gives a cohomological description of $\chi_G(V)$ in certain situations where $V$ has finite projective dimension.

**Proposition 8.** Let $V$ be a finitely generated $k[G]$-module and set $h = h(G)$, the Hirsch number of $G$. Assume that either
(i) $V$ has a finite free resolution, or
(ii) $G$ is torsion-free.

Then
\[ \chi_G(V) = \sum_{i \geq 0} (-1)^i \dim_k H_i(G, V) \]
\[ = (-1)^{\text{dim}_k H^h(G, V)} \]

Proof. For any finitely generated $k[G]$-module $W$ with $H_i(G, W) = 0$ and $H^i(G, W) = 0$ for $i > 0$ set
\[ \phi_G(W) = \sum_{i \geq 0} (-1)^i \dim_k H_i(G, W) \]
and
\[ \psi_G(W) = \sum_{i \geq 0} (-1)^i \dim_k H^i(G, W). \]

Note that $\phi_G$ and $\psi_G$ are additive on short exact sequences of finitely generated $k[G]$-modules with finite homology and cohomology.

(i) Since $\chi_G$, $\phi_G$, and $\psi_G$ are additive, it suffices to consider the case where $V = k[G]$. Then
\[ \chi_G(k[G]) = 1 = \dim_k H_0(G, k[G]) = \phi_G(k[G]). \]

Moreover, if $H \leq G$ is a torsion-free subgroup of finite index in $G$, then, by [2, Satz 3.1.2] and [4, Ch. VIII, Theorem 10.1],
\[ H^i(G, k[G]) = H^i(H, k[H]) = \begin{cases} 0 & \text{for } i \neq h, \\ k & \text{for } i = h. \end{cases} \]

Therefore, \((-1)^h \psi_G(k[G]) = \dim_k H^h(G, k[G]) = 1 = \phi_G(k[G]) = \chi_G(k[G])\), and (i) follows.

(ii) Since $k[G]$ has finite global dimension, we may assume that $V$ is finitely generated projective. Fix an orientable poly-(inf'mite cyclic) normal subgroup $H \leq G$ of finite index in $G$. Then, by the theorem of Cliff, Farkas and Snider ([14, proof of Theorem 13.4.18] and [5, proof of Theorem 1]), we have
\[ \phi_H(V) = \dim_k V/\omega(k[H]) \cdot V = [G : H] \dim_k V/\omega(k[G]) \cdot V \]
\[ = [G : H] \phi_G(V). \]

Here, $\omega(\cdot)$ denotes the augmentation ideal. Furthermore, by [2, Satz 3.1.2] again, we have, with the usual twisting $\tilde{k} = H^h(G, k[G])$ of $k$,
\[ H^i(G, V) \cong H_{n-i}(G, \tilde{k} \otimes_k V) = 0 \quad \text{for } i \neq h, \]
because $\tilde{k} \otimes_k V$ is also projective over $k[G]$. Thus
\[ (-1)^h \psi_G(V) = \dim_k H^h(G, V) = \phi_G(\tilde{k} \otimes_k V), \]
and the foregoing implies that
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\[ (-1)^h [G : H] \psi_G(V) = [G : H] \phi_G(\bar{k} \otimes_k V) = \phi_H(\bar{k} \otimes_k V) = (-1)^h \psi_H(V). \]

Finally, by the 'twisted Grothendieck theorem' [14, Theorem 13.4.9], \( V|_{k[H]} \) is stably free, and so part (i) implies that

\[ \chi_H(V) = \phi_H(V) = (-1)^h \psi_H(V). \]

As \( \chi_H(V) = [G : H] \chi_G(V) \), by Lemma 7(i), part (ii) is proved. \( \square \)

**Corollary 9.** Let \( V \) be a finitely generated \( k[G] \)-module and let \( h = h(G) \) denote the Hirsch number of \( G \). Then, for any torsion-free subgroup \( H \leq G \) of finite index in \( G \), we have

\[
\begin{align*}
\chi_G(V) &= [G : H]^{-1} \sum_{i \geq 0} (-1)^i \dim_k H_i(H, V) \\
&= (-1)^h [G : H]^{-1} \sum_{i \geq 0} (-1)^i \dim_k H^i(H, V).
\end{align*}
\]

**Proof.** This follows from Lemma 7(i) and Proposition 8(ii). \( \square \)

Using the notation \( P_G(V; t) = \sum_{i \geq 0} \dim_k H^i(G, V) t^i \) introduced in Corollary 4, the second equality in the above corollary can be expressed as

\[ \chi_G(V) = (-1)^h [G : H]^{-1} P_H(V; -1). \]

Similarly, Proposition 8(i) states that

\[ \chi_G(V) = (-1)^h P_G(V; -1) \]

in case \( V \) has a finite free resolution over \( k[G] \).

**Lemma 10.** Let \( 0 \rightarrow U \rightarrow F \rightarrow V \rightarrow 0 \) be an exact sequence of finitely generated \( k[G] \)-modules and assume that \( H^i(G, F) = 0 \) \( (i \gg 0) \). Then \( P_G(V; t) \) is defined at \( t = -1 \) if and only if this holds for \( P_G(U; t) \) and, in this case,

\[ P_G(V; -1) + P_G(U; -1) = P_G(F; -1). \]

**Proof.** Let \( M_i \subset H^i(G, U) \) denote the kernel of \( H^i(G, U) \rightarrow H^i(G, F) \). Then the cohomology sequence yields

\[ \dim_k H^i(G, U) + \dim_k H^i(G, V) = \dim_k H^i(G, F) + \dim_k M_i + \dim_k M_{i+1} \]

for all \( i \geq 0 \). Therefore, in \( \mathbb{Z}[t] \), we have

\[ P_G(U; t) + P_G(V; t) = P_G(F; t) + \sum_{i \geq 0} (\dim_k M_i + \dim_k M_{i+1}) t^i \]

\[ = P_G(F; t) + M(t)(t + 1), \]
where we have set $M(t) = \sum_{i \geq 0} \dim_k M_{i+1} t^i \in \mathbb{Z}[t]$. If $H^i(G, F) = 0$ for $i > n$, say, then $M_{i+1} = H^{i+1}(G, U) = H^i(G, V)$ for $i > n$ and so $t \cdot P_G(V; t) = t \cdot M(t) + f = P_G(U; t) + g$ for suitable polynomials $f, g \in \mathbb{Z}[t]$. This shows that if one of $P_G(V; t)$, $P_G(U; t)$, or $M(t)$ is defined at $t = -1$, then so are the other two and, substituting $t = -1$ in the above equation, we get $P_G(U; -1) + P_G(V; -1) = P_G(F; -1)$. □

As an application of the foregoing we offer the following result which deals with $k[G]$-modules of infinite projective dimension. Recall our general convention in this section: $G$ has no non-trivial finite normal subgroups.

**Proposition 11.** Suppose $G$ as a torsion-free normal subgroup $H$ such that $G/H$ is finite cyclic and let $h = h(G)$ denote the Hirsch number of $G$.

(i) Let $V$ be a finitely generated $k[G]$-module such that $V|_{k[H]}$ is projective. Set $\bar{V} = \bar{k} \otimes_k V$, where $\bar{k} = H^h(G, k[G])$ as usual, and $\bar{V} = \bar{V}/\omega(k[H]) \cdot \bar{V}$. Then

$$P_G(V; t) = t^h \left( \dim_k \bar{V}^{G/H} + \frac{t}{1 - t^2} (\dim_k H^1(G/H, \bar{V}) + \dim_k H^2(G/H, \bar{V}) t) \right).$$

(ii) Suppose $\text{char } k = 2$ and $G/H$ has order 2. Then $P_G(V; -1)$ is defined for all finitely generated $k[G]$-modules $V$, and $P_G(V; -1) = (-1)^h \chi_G(V)$.

**Proof.** (i) We have $H^i(H, V) = H_{h-i}(H, \bar{V}) = 0$ for $i \neq h$, since $\bar{V}$ is projective over $k[H]$. The LHS-spectral sequence for $1 \to H \to G \to G/H \to 1$ implies that $H^i(G, V) = 0$ for $i < h$ and $H^i(G, V) = H^{i-h}(G/H, H^h(H, V)) = H^{i-h}(G/H, \bar{V})$ for $i \geq h$. Using the fact that $G/H$ is cyclic, we obtain

$$P_G(V; t) = t^h \left( \dim_k \bar{V}^{G/H} + \sum_{i \geq 1} \dim_k H^i(G/H, \bar{V}) t^i \right) = t^h \left( \dim_k \bar{V}^{G/H} + \frac{t}{1 - t^2} (\dim_k H^1(G/H, \bar{V}) + \dim_k H^2(G/H, \bar{V}) t) \right),$$

as we have claimed.

(ii) Let $0 \to U \to F \to V \to 0$ be an exact sequence of $k[G]$-modules with $F$ finitely generated free. Then $P_G(F; -1) = (-1)^h \chi_G(F)$, by Proposition 8(i), and Lemma 10 implies that $P_G(V; -1)$ is defined at $t = -1$ and equals $(-1)^h \chi_G(V)$ if and only if the corresponding fact holds for $P_G(U; -1)$. Thus, arguing by induction on the projective dimension of $V|_{k[H]}$, we may assume that $V|_{k[H]}$ is projective. But then part (i) implies that

$$P_G(V; -1) = t^h \left( \dim_k \bar{V}^{G/H} + \frac{t}{1 - t} \dim_k H^1(G/H, \bar{V}) \right),$$

where we have used the notation of (i) and the fact that, for $G/H$ of order 2 and $\text{char } k = 2$, $H^1(G/H, W) = H^2(G/H, W)$ holds for all $k[G/H]$-modules $W$. 


Therefore, $P_G(V; -1)$ is defined and, moreover,

$$P_G(V; -1) = (-1)^h \dim_k \tilde{V}^G/H - \frac{1}{2} \dim_k \tilde{V}^G/H/\omega(k[G/H], \tilde{V})$$

$$= (-1)^{\frac{h}{2}} \dim_k \tilde{V}.$$

To see the latter equality, note that $\tilde{V}$ has the form $\tilde{V} = k[G/H]^r \oplus k^s$ for suitable $r, s \geq 0$, where $k$ denotes the trivial module. Thus $\dim_k \tilde{V}^G/H = r + s$, $\dim_k \tilde{V}^G/H/\omega(k[G/H]) \cdot \tilde{V} = s$ and so the expression in brackets equals $r + \frac{1}{2}s = \frac{1}{2} \dim_k \tilde{V}$. Finally, by Corollary 9,

$$\chi_G(\tilde{V}) = \frac{1}{2} \dim_k H_0(H, \tilde{V}) = \frac{1}{2} \dim_k \tilde{V}$$

and so it suffices to show that $\chi_G(\tilde{V}) = \chi_G(V)$. But if $N$ denotes the kernel of the action of $H$ on $\tilde{K}$, then $N$ has index at most 2 in $H$ and the restrictions of $\tilde{V}$ and $V$ to $k[N]$ are isomorphic. Lemma 7(i) now implies that $\chi_G(\tilde{V}) = \chi_G(V)$, as required. $\Box$

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