JOURNAL OF ALGEBRA 85, 368–381 (1983)

Division Algebras Generated by Finitely Generated Nilpotent Groups

MARTIN LORENZ*

Max-Planck-Institut für Mathematik, Gottfried-Claren Strasse 26, D-5300 Bonn 3, West Germany

Communicated by I. N. Herstein

Received May 15, 1982

Division algebras D generated by some finitely generated nilpotent subgroup G of the multiplicative group D^* of D are studied and the question to what extent G is determined by D is considered. Trivial examples show that D does not determine G up to isomorphism. However, it is proved that if F denotes the center of D, then the F-subalgebra of D generated by G is in fact determined up to isomorphism by D. Using the structure of this subalgebra it is further concluded that D does at least determine (i) the group G/Δ, where Δ is the FC-center of G, (ii) the division subalgebra $K(\Delta)$ of D generated by Δ, and (iii) the subgroup $K(\Delta)^*G$ of D^*. The principal technical tools are the so-called (crossed) Hilbert–Neumann rings of ordered groups over rings which are also studied here in their own right.

INTRODUCTION

Let D be a division algebra over a commutative field K (not necessarily finite-dimensional over its center) and suppose D is generated, as division K-algebra, by some finitely generated nilpotent subgroup G of the multiplicative group D^* of D. Trivial examples show that D does not determine G up to isomorphism, but our goal here is to show that it almost does. More precisely, if F denotes the center of D then we will prove that the F-subalgebra of D generated by G is in fact determined up to isomorphism by D (Corollary 4.2). Analyzing the structure of this subalgebra, in particular its group of units, we obtain that D determines the following data up to isomorphism: (i) the group G/Δ, where $\Delta = \{x \in G \mid x \text{ has a finite } G\text{-conjugacy class}\}$ is the FC-center of G, (ii) the division subalgebra $K(\Delta)$ of D generated by Δ, and (iii) the subgroup $K(\Delta)^*G$ of D^*.

* While this article was written, the author was supported by Deutsche Forschungsgemeinschaft Grant Lo 261/2-1. Part of the work was done while he was enjoying the hospitality of the Mathematics Department at the University of Warwick during the Ring Theory Symposium 1981–1982.

0021-8693/83 $3.00

We briefly outline our approach to the question at hand. Note that, in the given situation, there exists an obvious homomorphism of the group algebra $K[G]$ into D. Its kernel, P say, is a faithful completely prime ideal of $K[G]$. (Here, an ideal I of $K[G]$ is called *faithful* if G, under the canonical map, embeds into $K[G]/I$ and I is said to be *completely prime* if $K[G]/I$ has no zero divisors.) Moreover, since G generates D, it follows that D is isomorphic to the classical ring of quotients $Q(K[G]/P)$ of $K[G]/P$. We recall that $K[G]$ is a Noetherian ring [11, 10.2.8] so $Q(K[G]/P)$ does exist. By the work of Zalesskii [13], the structure of $K[G]/P$ is well under control, even for primes P which are not necessarily completely prime. Namely, the center Z of $K[G]/P$ is contained in the subalgebra $K[\Delta]/P \cap K[\Delta]$ of $K[G]/P$ and the central localization $(K[G]/P)_{Z^*}$ has the structure of a crossed product of $H = G/\Delta$ over $R = (K[\Delta]/P \cap K[\Delta])_{Z^*}$. In short, $(K[G]/P)_{Z^*} \cong R * H$. The crucial fact that we will use is that H is torsion-free nilpotent and hence is an *ordered group*. Hence, using a construction originally due to Levi-Civita [6] and Hilbert [5] and later brought to its present form by Neumann [10] and Malcev [9] (cf. [1, 10] for a historical account), we can embed $(K[G]/P)_{Z^*} \cong R * H$ into a suitable larger ring $R * ((H))$ which is in fact simple Artinian, and this embedding extends to an embedding of $Q(K[G]/P)$ into $R * ((H))$. Thus, by studying $R * ((H))$, we can derive information about $Q(K[G]/P)$. The advantage in using $R * ((H))$ lies in the fact that its elements "look very much like" elements of the crossed product $R * H$ so crossed product and group algebra techniques do apply quite nicely.

In Section 1, we study $R * ((H))$ in its own right, for any ordered group H and any ring R, and establish a number of extensions of the well-known Malcev–Neumann theorem which asserts that $R * ((H))$ is a division ring if R is. For example, we show that if R is H-prime Artinian, then $R * ((H))$ is simple Artinian and its composition length equals the length of R (Proposition 1.2). Strictly speaking, the material of Section 1 and much of Section 2 is not really necessary for the later applications to division algebras, but it seems worthwhile to study $R * ((H))$ in a broader context, especially since the structure theory of prime ideals in $K[G]$ does in fact cover arbitrary primes.

In Section 2, we briefly review the relevant facts about primes P in $K[G]$, all due to Zalesskii [13], and apply the results of Section 1 to $(K[G]/P)_{Z^*} = R * H$ to obtain what we will call the *Hilbert–Neumann ring associated with* P, written as $S(P) = R * ((H))$. Elementary properties of $S(P)$ show that if $\operatorname{char} K = 0$, then $S(P)$ does not contain a Weyl algebra (Lemma 2.3). Hence neither does $Q(K[G]/P)$, which in particular shows that the division algebras under consideration are different from the division algebras generated by finite-dimensional noncommutative nilpotent Lie algebras over fields of characteristic 0 [2, 8.3].

Section 3 is devoted to the case of completely prime ideals P. We show that any subalgebra of $S(P)=R*((H))$ which is algebraic over the center F of $S(P)$ can be embedded into R. In case char $K=0$ (or under some weaker assumption on char K), the elements of $S(P)$ which are algebraic over F are in fact exactly the $S(P)^*$-conjugates of elements of R (Proposition 3.2). Most of the work in Sections 2 and 3 covers more general classes of Hilbert–Neumann rings and uses very little of the specific properties of $S(P)$. However, the generalizations being quite obvious, we have for the most part restricted ourselves to the case of $S(P)$.

Finally, in Section 4 we derive the statements about division algebras generated by finitely generated nilpotent groups G made at the beginning from the following result about factors modulo completely prime ideals P in $K[G]$: The classical (division) ring of quotients $Q(K[G]/P)$ determines $(K[G]/P)_{Z^*}$, the localization of $K[G]/P$ at the nonzero elements of the center $Z=Z(K[G]/P)$. In other words, if P_1 and P_2 are completely prime ideals in $K[G_1]$, respectively, $K[G_2]$, with G_1 and G_2 finitely generated nilpotent, and if $Q(K[G_1]/P_1)\cong Q(K[G_2]/P_2)$, then we must have $(K[G_1]/P_1)_{Z_1^*}\cong (K[G_2]/P_2)_{Z_2^*}$ (Theorem 4.1). The latter result had been conjectured by Zalesskii [13] for maximal P. It possibly does extend to arbitrary prime ideals of $K[G]$. The difficulty here is that the lowest term valuation of $S(P)$ (see Section 3) which plays a crucial role in this paper will not be multiplicative for general primes.

This article is related to the work in [3], where the case $P=0$ and G torsion-free is considered in detail. Under these circumstances, D does in fact determine G [3, Main Theorem]. Indeed, the present article grew out of an extended correspondence with Dan Farkas concerning these matters during the early stages of [3]. It is a pleasure to thank him. Thanks are also due to Kenny Brown and Toby Stafford for helpful discussions.

Notations and Conventions

In general our notation follows [11]. All rings are understood to have a 1 which is inherited by subrings. The symbol $(\cdot)^*$ always denotes the subset of nonzero elements (not to be confused with the usual crossed product notation $R*H$) and $Z(\cdot)$ denotes the center. Throughout, K will be a commutative field.

1. The Hilbert–Neumann Construction

This section contains some general facts concerning the so-called Hilbert–Neumann construction. We begin by recalling the basic definitions.

Let H be a group, let R be a ring, and let $R * H$ be a crossed product of H over R. Thus $R * H$ is a free left R-module with basis $\{\bar{x} \mid x \in H\}$ and with a multiplication which is governed by the formula

$$(r_x \bar{x})(r_y \bar{y}) = r_x r_y^{\bar{x}^{-1}} t(x, y)\, \overline{xy}$$

for all $x, y \in H$ and $r_x, r_y \in R$. Here, $t: H \times H \to U(R)$ is a map from $H \times H$ to the group $U(R)$ of units of R and, for fixed $x \in H$, the map ${}^{\bar{x}}: r \mapsto r^{\bar{x}}$ is an automorphism of R. Of course, t and the automorphisms ${}^{\bar{x}}$ have to satisfy certain relations in order to guarantee that $R * H$ will be an associative ring, but the precise form of these relations is quite irrelevant for us (see [7, 1.1], e.g.). By modifying $\bar{1}$ and t if necessary, we can assume that $\bar{1}$ is the identity of $R * H$ and we will identify the elements $r \in R$ with $r\bar{1} \in R * H$, thereby viewing R as a subring of $R * H$. We note that the elements $\bar{x}$ $(x \in H)$ are units in $R * H$, and for all $r \in R$ we have $\bar{x}^{-1} r \bar{x} = r^{\bar{x}}$.

Now assume that, in addition, H is a fully ordered group and let $R * ((H))$ denote the set of all formal sums

$$\alpha = \sum_{x \in H} a_x \bar{x}$$

with $a_x \in R$ and with $\operatorname{Supp} \alpha = \{x \in H \mid a_x \neq 0\}$ a well-ordered subset of H. Thus $R * ((H))$ contains $R * H$ as the subset of elements α with finite support $\operatorname{Supp} \alpha$, and it can be shown that the ring operations of $R * H$ do in fact extend to $R * ((H))$ and make $R * ((H))$ an associative ring containing $R * H$ as a subring (see [10]). $R * ((H))$ will be called a *(crossed) Hilbert–Neumann ring of H over R*. In the following, we shall repeatedly use the *lowest term map*

$$\lambda: R * ((H)) \to \mathscr{H} \cup \{0\}.$$

Here we have set $\mathscr{H} = \{r\bar{x} \mid r \in R^*, x \in H\}$, and λ is defined by $\lambda(0) = 0$ and $\lambda(\alpha) = a_x \bar{x}$ if $\alpha \neq 0$ and x is the least element of $\operatorname{Supp} \alpha$. Note that for $\alpha, \beta \in R * ((H))$ we have

$$\lambda(\alpha\beta) = \lambda(\alpha)\,\lambda(\beta) \qquad \text{provided} \quad \lambda(\alpha)\,\lambda(\beta) \neq 0.$$

The first lemma is a slight extension of the well-known Malcev–Neumann theorem [10, Theorem 5.7; 9]. Only a few modifications are needed to adapt the usual proof of this result to the present situation.

LEMMA 1.1. *Let $S = R * ((H))$ be a Hilbert–Neumann ring of the ordered group H over the ring R. If $0 \neq e = e^2$ is an idempotent of R such that Re is a simple left ideal of R, then eSe is a division ring.*

Proof. We first show that for any nonzero element of the form $ea\bar{x}e$, with $a \in R$ and $x \in H$, there exists $b \in R$ with $eb\bar{x}^{-1}e \cdot ea\bar{x}e = e$. Indeed, $ea\bar{x}e \neq 0$ implies $eae^{\bar{x}^{-1}} \neq 0$ and hence $Reae^{\bar{x}^{-1}} = Re^{\bar{x}^{-1}}$, by the simplicity of $Re^{\bar{x}^{-1}} = (Re)^{\bar{x}^{-1}}$. Fix $r \in R$ with $reae^{\bar{x}^{-1}} = e^{\bar{x}^{-1}}$ and set $b = r^{\bar{x}} \in R$. Then $eb\bar{x}^{-1}e \cdot ea\bar{x}e = e\bar{x}^{-1}reae^{\bar{x}^{-1}}\bar{x} = e\bar{x}^{-1}e^{\bar{x}^{-1}}\bar{x} = e$, as required.

Now let $0 \neq \alpha \in eSe$ be given. We show that $\alpha\beta = e$ for some $\beta \in eSe$. Indeed, by the foregoing, we can adjust the lowest term $\lambda(\alpha)$ of α so that α has the form $\alpha = e - \alpha_0$, where $\alpha_0 \in eSe$ has its support in $\{x \in H \mid x > 1\}$. As in [11, Proof of 13.2.11], one shows that the "geometric series" $\beta = e + \alpha_0 + \alpha_0^2 + \cdots + \alpha_0^n + \cdots$ belongs to eSe and satisfies $\alpha\beta = e$. ∎

Recall that the automorphisms $^{\bar{x}}$ $(x \in H)$ of R induce an action of H on the set of ideals of R and so we can talk about H-stable ideals of R (cf. [8, Sect. 1]). R is called *H-prime* if the product of any two nonzero H-stable ideals of R is nonzero or, equivalently, if for any two nonzero elements $a, b \in R$ we can find $r \in R$ and $x \in H$ with $a^{\bar{x}}rb \neq 0$.

PROPOSITION 1.2. *Let $S = R * ((H))$ be a Hilbert–Neumann ring of the ordered group H over the ring R.*

(i) *S is prime if and only if R is H-prime.*

(ii) *S is Artinian if and only if R is Artinian.*

(iii) *If R is H-prime Artinian, then S is simple Artinian and* $\operatorname{rk} S = \operatorname{rk} R$, *where* $\operatorname{rk}(\cdot)$ *denotes composition length.*

Proof. For any right ideal I of R let $I * ((H))$ denote the set of elements of $S = R * ((H))$ whose R-coefficients belong to I. Then $I * ((H))$ is a right ideal of S with $I * ((H)) \cap R = I$. Moreover, if I is an H-stable ideal, then $I * ((H))$ is an ideal of S and for any right ideal J of R we have $J * ((H)) \cdot I * ((H)) \subset (JI) * ((H))$. These observations immediately imply the necessity of the conditions on R in (i) and (ii).

Now assume that R is H-prime and let $\alpha, \beta \in S^*$ be given. If $\lambda(\alpha) = a\bar{x}$, $\lambda(\beta) = b\bar{y}$ then, by the H-primeness of R, we can find $z \in H$ and $r \in R$ with $arb^{\bar{z}^{-1}} \neq 0$. Therefore, in S we have $0 \neq (a\bar{x})(\bar{x}^{-1}r\bar{z})(b\bar{y}) = \lambda(\alpha(\bar{x}^{-1}r\bar{z})\beta)$ and so $\alpha S\beta$ is nonzero and S is prime. This proves (i).

For (ii), assume that R is Artinian and let J denote the Jacobson radical of R. Then $\mathscr{J} = J * ((H))$ is a nilpotent ideal of S and each $\mathscr{J}^i/\mathscr{J}^{i+1}$ is a finitely generated module over $S/\mathscr{J} \cong (R/J) * ((H))$. Thus we may assume that $J = 0$ and hence $R = \bigoplus_{i=1}^{t} R_i$, a direct sum of simple Artinian rings R_i. Since the automorphisms $^{\bar{x}}$ $(x \in H)$ permute the R_i's, some subgroup U of finite index in H stabilizes them all. Therefore, we have an obvious isomorphism $R * ((U)) \cong \bigoplus_{i=1}^{t} R_i * ((U))$, a direct sum of suitable crossed Hilbert–Neumann rings of U over the R_i's. Since S is a finitely generated module over the subring $R * ((U))$, it suffices to consider the case

$S = R * ((H))$ with $R = M_n(D)$ simple Artinian. Let $e_{ij} \in R$ denote the usual matrix idempotents. Then, by [11, 6.1.5], we have $S \cong M_n(e_{11} S e_{11})$ and, by Lemma 1.1, $e_{11} S e_{11}$ is a division ring. Thus S is Artinian and (ii) is proved.

Finally, except for the assertion concerning the ranks, (iii) follows from (i) and (ii). As to the equality $\operatorname{rk} R = \operatorname{rk} S$, first note that if R is H-prime, then we have $J = 0$ in the above and H permutes the simple components R_i of $R = \bigoplus_{i=1}^{t} R_i$ transitively. Therefore, $\operatorname{rk} R = t \operatorname{rk} R_1$. Moreover, if $e_i \in R$ $(i = 1, 2, \ldots, t)$ are the idempotents corresponding to the above decomposition of R, then, as right S-modules, $S = \bigoplus_{i=1}^{t} e_i S$ and $e_i S = e_1^{\bar{x}_i} S = \bar{x}_i^{-1} e_1 S \cong e_1 S$ for suitable elements $x_i \in H$. Therefore, $\operatorname{rk} S = t \operatorname{length}(e_1 S) = t \operatorname{rk}(e_1 S e_1)$. Finally, letting U denote the stabilizer subgroup of R_1 in H, we obtain quite easily that $e_1 S e_1 \cong R_1 * ((U))$ and, as in the last part of the preceding paragraph, it follows that $\operatorname{rk} R_1 * ((U)) = \operatorname{rk} R_1$. This establishes (iii). ∎

We close this section with a few observations concerning the behavior of regular elements under the embedding $R * H \subset R * ((H))$. In general, regular elements of $R * H$ need not stay regular in $R * ((H))$. The following example is due to Gilmer (see [4, Example 3]). Let R be a ring containing nonzero elements a, b_i $(i = 0, 1, 2, \ldots)$ with $ab_0 = 0$ and $ab_i = b_{i-1}$ for $i \geqslant 1$. Let $C = \langle x \rangle$ be the infinite cyclic group, ordered so that $x > 1$, and set $\alpha = a - x \in R[C]$ and $\beta = \sum_{i \geqslant 0} b_i x^i \in R((C))$ (non-crossed). Then α is regular in $R[C]$, but $\alpha\beta = 0$. To obtain an explicit example of such a ring R, take R to be the endomorphism ring of a countably dimensional vector space with basis $\{v_0, v_1, \ldots\}$, say, and let $a, b_i \in R$ be defined by $a(v_0) = 0$, $a(v_i) = v_{i-1}$ $(i \geqslant 1)$ and $b_i(v_0) = v_i$, $b_i(v_j) = 0$ $(j \neq 0)$. Note that, for all i, we have $a^{i+1} b_i = 0$ but $a^i b_i \neq 0$ so R certainly is not Noetherian. This explains the Noetherian assumption in

LEMMA 1.3. *Let $R * H$ be a crossed product of the ordered group H over the ring R and assume that $R * H$ is semiprime Noetherian. Then regular elements of $R * H$ stay regular in the Hilbert–Neumann ring $R * ((H))$ which extends $R * H$.*

Proof. Arguing as in the first paragraph of the proof of Proposition 1.2, we see that R too is semiprime Noetherian. Now let $\alpha \in R * H$ be regular and set $I = \alpha \cdot R * H$. Then I is an essential right ideal of $R * H$ [11, 10.4.9(ii)] and it follows quite easily that $c(I) = \{r \in R \mid \exists \beta \in I : \lambda(\beta) = r\}$ is an essential right ideal of R. Thus, by Goldie's theorem [11, 10.4.10, Step 1], $c(I)$ contains a regular element. In other words, some right multiple $\alpha_1 = \alpha\beta$ of α has its lowest term of the form $\lambda(\alpha_1) = a$ with a a regular element of R. Therefore, α_1 is clearly regular in $R * ((H))$ and so α has zero left annihilator in $R * ((H))$. By symmetry, α also has zero right annihilator in $R * ((H))$ and the lemma is proved. ∎

2. The Hilbert–Neumann Ring Associated with a Prime Ideal of $K[G]$

In this section, G will always denote a finitely generated nilpotent group and P will be a prime ideal of the group algebra $K[G]$. We will embed $K[G]/P$ into a suitable Hilbert–Neumann ring $R * ((H))$. After replacing G by its image in $K[G]/P$ if necessary, we may clearly assume that P is faithful, i.e., that G embeds into $K[G]/P$. The relevant facts about faithful primes of $K[G]$, all due to Zalesskii [13], are collected in

Lemma 2.1 (Zalesskii). *Let P be a faithful prime ideal of $K[G]$. Then $P = (P \cap K[\Delta])\, K[G]$, where $\Delta = \Delta(G)$ denotes the FC-center of G. Moreover, $Z(K[G]/P) \subset K[\Delta]/P \cap K[\Delta]$ and $K[\Delta]/P \cap K[\Delta]$ is a finitely generated module over $Z(K[G]/P)$.*

Proof. The first assertion is contained in [11, 11.4.5], and the second assertion then follows from [11, 8.4.8] and the fact that $Z(G)$ has finite index in Δ [11, 11.4.3(iii)]. ∎

We fix some *notation* that will be used throughout the remainder of this article. $\Delta = \Delta(G)$ will always denote the FC-center of G and we let

$$R = (K[\Delta]/P \cap K[\Delta])_{Z^*}$$

denote the localization of $K[\Delta]/P \cap K[\Delta]$ at the nonzero elements of $Z = Z(K[G]/P)$. Also, set

$$F = Q(Z(K[G]/P)),$$

the field of fractions of Z. Then, by the lemma, R is a finite-dimensional algebra over F. Moreover, R is G-simple, i.e., R has no nontrivial G-invariant ideals [11, 11.4.7], and the lemma further implies that the localization of $K[G]/P$ at Z^* has the form

$$(K[G]/P)_{Z^*} \cong R * H,$$

a suitable crossed product of $H = G/\Delta$ over R. Here, the elements $\bar{x}$ $(x \in H)$ can be taken to be any fixed set of coset representatives of Δ in G. Now $H = G/\Delta$ is torsion-free nilpotent and hence orderable [11, 11.4.3(iii) and 13.1.6]. We fix an ordering of H and let

$$S(P) = R * ((H))$$

denote the Hilbert–Neumann ring which extends $R * H$. $S(P)$ will be called the *Hilbert–Neumann ring associated with P*. The following lemma describes some properties of $S(P)$.

LEMMA 2.2. *Let $S(P)$ be the Hilbert–Neumann ring associated with a prime ideal P of $K[G]$. Then $S(P)$ is a simple Artinian ring containing the classical ring of quotients $Q(K[G]/P)$ of $K[G]/P$ as a subring. Moreover,* $\operatorname{rk} S(P) = \operatorname{rk} Q(K[G]/P)$ *and* $Z(S(P)) = Z(Q(K[G]/P)) = F$.

Proof. As we have pointed out above, R is finite F-dimensional and G-simple, hence H-prime Artinian. Thus Proposition 1.2(iii) implies that $S(P)$ is simple Artinian with $\operatorname{rk} S(P) = \operatorname{rk} R$. Moreover, by Lemma 1.3, regular elements of $(K[G]/P)_{Z^*} = R * H$ become invertible in $S(P)$. Therefore, the embedding of $K[G]/P$ into $S(P)$ extends to an embedding of $Q(K[G]/P)$ into $S(P)$. Thus we have $R \subset Q(K[G]/P) \subset S(P)$ and these rings are all semisimple Artinian. By considering decompositions of 1 into primitive orthogonal idempotents, it is trivial to see that $\operatorname{rk} R \leqslant \operatorname{rk} Q(K[G]/P) \leqslant \operatorname{rk} S(P)$ so all ranks coincide.

As to the assertion concerning the centers, first note that, with $\mathbb{C}.\ (G)$ denoting centralizers of G, we have $Z(S(P)) = \mathbb{C}_{S(P)}(G) \supset \mathbb{C}_{Q(K[G]/P)}(G) = Z(Q(K[G]/P)) \supset F$. Here all equalities and inclusions are clear, except for the inclusion $\mathbb{C}_{S(P)}(G) \subset Z(S(P))$. To see this, note that if $\alpha = \sum_x a_x \bar{x} \in \mathbb{C}_{S(P)}(G)$ then its lowest term $\lambda(\alpha)$ belongs to $\mathbb{C}_{\mathscr{H}}(G)$, where $\mathscr{H} = \{r\bar{x} \mid r \in R^*, x \in H\}$, and, using the fact that $\operatorname{Supp} \alpha$ is well ordered, we conclude that $a_x \bar{x} \in \mathbb{C}_{\mathscr{H}}(G)$ for all $x \in \operatorname{Supp} \alpha$. But $\mathbb{C}_{\mathscr{H}}(G) \subset Z((K[G]/P)_{Z^*}) = F$, whence $\mathbb{C}_{S(P)}(G) = F$. This proves the lemma. ∎

We now give a quick application of the above depending on a suitable trace map on $S(P)$. With our previous notation, define $\operatorname{tr}: S(P) = R * ((H)) \to R$ by sending each $\alpha = \sum_x a_x \bar{x} \in R * ((H))$ to its identity coefficient $a_1 \in R$. Clearly, tr is left and right R-linear. For the latter, recall our convention that $\bar{1} = 1$, corresponding to the canonical choice of the identity element of G as the representative of the coset $\Delta \in H = G/\Delta$. For convenience, we further normalize the set $\{\bar{x} \mid x \in H\}$ by first fixing a coset representative $\bar{x} \in G$ for each $x \in H = G/\Delta$ with $x > 1$ and then, for $x \in H$ with $x < 1$, taking $(\overline{x^{-1}})^{-1}$ as the representative of x. Note that, with this choice of basis, we have $t(x^{-1}, x) = t(x, x^{-1}) = 1$ for all $x \in H$. Consequently, for $\alpha = \sum_x a_x \bar{x}$, $\beta = \sum_x b_x \bar{x} \in R * ((H)) = S(P)$ we obtain

$$\operatorname{tr}(\alpha\beta) = \sum_x a_x b_{x^{-1}}^{\bar{x}^{-1}}$$

and

$$\operatorname{tr}(\beta\alpha) = \sum_x b_{x^{-1}} a_x^{\bar{x}}.$$

Now R is finite dimensional over F and hence we can define $T(r)$ for $r \in R$ to be the trace of the F-linear endomorphism of R given by right multiplication with r. Then $T: R \to F$ is F-linear with $T(r_1 r_2) = T(r_2 r_1)$ and $T(r^{\bar{x}}) = T(r)$ for all $r, r_1, r_2 \in R$ and $x \in H$. Therefore, the map

$$\tau = T \circ \mathrm{tr}: S(P) = R * ((H)) \to F$$

is F-linear and satisfies $\tau(\alpha\beta) = \tau(\beta\alpha)$ for all $\alpha, \beta \in S(P)$. In particular, τ is identically zero on $[S(P), S(P)]$, the F-linear span of all Lie commutators $\alpha\beta - \beta\alpha$ with $\alpha, \beta \in S(P)$, and since $\tau(1) = \dim_F R$, we obtain

LEMMA 2.3. *Let $S(P) = R * ((H))$ be the Hilbert–Neumann ring associated with the prime ideal P of $K[G]$. If $\dim_F R$ is nonzero in K, then $[S(P), S(P)] \cap F = 0$.* ∎

In particular, if $A_1(K)$ denotes the Weyl algebra over a field K of characteristic 0, then $A_1(K) \not\subset Q(K[G]/P)$.

We remark that the above construction of the trace map $\tau: R * ((H)) \to F$ works under more general circumstances. All that was really needed was the fact that R is finite dimensional over some central subfield F of $R * ((H))$. Lemma 2.3 carries over to this more general class of Hilbert–Neumann algebras.

3. COMPLETELY PRIME IDEALS: ALGEBRAIC ELEMENTS OF $S(P)$

In this section, we will concentrate on the case of a completely prime ideal P of $K[G]$ so that $S(P)$ is a division ring. We will study the elements of $S(P)$ that are algebraic over the center F of $S(P)$. Throughout, we will retain the notation of the previous section. In particular, assuming G to be faithfully embedded into $K[G]/P$, we set $R = (K[\Delta]/P \cap K[\Delta])_{Z^*}$. Note that R now is a finite-dimensional division algebra over F and R^* is a normal subgroup of $\mathscr{H} = \{r\bar{x} \mid r \in R^*, x \in H\}$ with $\mathscr{H}/R^* \cong H$. Also, using the ordering of H, it is easy to see that *$\mathscr{H}$ is the full group of units of $K[G]/P = R * H$* (cf. [11, 13.1.9(ii)]). Note further that, in the present situation, the lowest term map $\lambda: S(P) = R * ((H)) \to \mathscr{H} \cup \{0\}$ is multiplicative. If we follow λ by the map $\mathscr{H} \cup \{0\} \to H \cup \{\infty\}$ which sends 0 to ∞ and $\mathscr{H}$ to H in the canonical fashion, then we obtain the *lowest term valuation* $v: S(P) = R * ((H)) \to H \cup \{\infty\}$. Thus v sends 0 to ∞ and any nonzero element of $R * ((H))$ to the lowest element in its support. With the usual rules for ∞, we have $v(\alpha\beta) = v(\alpha)\, v(\beta)$, $v(\alpha + \beta) \geqslant \min\{v(\alpha), v(\beta)\}$, and $v(\alpha) = \infty$ iff $\alpha = 0$ $(\alpha, \beta \in S(P))$ so that v is indeed a (non-commutative) valuation in the sense of [12].

LEMMA 3.1. *Let $S(P)$ be the Hilbert–Neumann ring associated with a completely prime ideal P of $K[G]$ and let $A \subset S(P)$ be a K-subalgebra which* *is algebraic over $F = Z(S(P))$. Then the lowest term map $\lambda: S(P) \to \mathscr{H} \cup \{0\}$ yields a K-algebra embedding of A into R.*

Proof. We first show that for any $\alpha \in A$ we have $\lambda(\alpha) \in R$. This is of course clear for $\alpha = 0$. So let α be nonzero and let $f_n \alpha^n + f_{n-1}\alpha^{n-1} + \cdots + f_0 = 0$ with $f_i \in F$ be an algebraic relation of α over F. Since $F \subset R$, we have $v(f_i \alpha^i) = v(\alpha)^i$ for all i with $f_i \neq 0$. Hence, if $v(\alpha) \neq 1$, then $v(f_n \alpha^n + f_{n-1}\alpha^{n-1} + \cdots + f_0) = \min\{v(\alpha)^i \mid i \text{ with } f_i \neq 0\} \neq \infty$, contradicting the fact that $v(0) = \infty$. Therefore, we must have $v(\alpha) = 1$ and so $\lambda(\alpha) \in R$.

We conclude that $\lambda|_A = \mathrm{tr}|_A$, where $\mathrm{tr}: S(P) = R * ((H)) \to R$ sends each element of $R * ((H))$ to its identity coefficient (cf. end of Section 2), and hence λ is K-linear on A. Since λ is also multiplicative and maps nonzero elements to nonzero elements, the lemma is proved. ∎

Again, the above arguments cover more general Hilbert–Neumann rings $R * ((H))$. Namely, it suffices to assume that R is a domain and that F is a central subfield of $R * ((H))$ which is contained in R. The lemma becomes false if P is not completely prime, for in this case there exist elements $0 \neq r \in R$ and $1 \neq x \in H$ with $rr^{\bar{x}^{-1}} = 0$ and hence $(r\bar{x})^2 = 0$. However, the argument used at the beginning of the proof shows that, for any prime ideal P of $K[G]$ and any element $\alpha \in S(P)$ that is algebraic over F, we either have $\lambda(\alpha) \in R$ or $\lambda(\alpha)$ is nilpotent. It is not true that algebraic elements of $S(P)$ (or $Q(K[G]/P)$) necessarily have to be contained in R, simply because the set of elements of $S(P)$ (or $Q(K[G]/P)$) that are algebraic over F is invariant under automorphisms but R need not be. For example, if $x \in H$, $x > 1$, then for any $\alpha \in R$ we have in $Q(K[G]/P) \subset S(P)$: $(1-\bar{x})\,\alpha(1-\bar{x})^{-1} = (1-\bar{x})\,\alpha(1+\bar{x}+\bar{x}^2+\cdots) = \alpha + \sum_{i \geqslant 1}(\alpha - \alpha^{\bar{x}^{-1}})\,\bar{x}^i$ which does not belong to R, unless $\alpha = \alpha^{\bar{x}}$. However, we do at least have the following result which is a consequence of the work of Bergman in [1]. The result will not be needed here and mainly serves as an illustration of Bergman's powerful techniques in [1]. We have therefore stated it in a somewhat wider context.

PROPOSITION 3.2. *Let $S = R * ((H))$ be a crossed Hilbert–Neumann ring with R a finite dimensional division algebra over some subfield $F \subset R$ which is central in S and with H an ordered group. Assume that* $\dim_F R$ *is nonzero in F. If $\alpha \in S$ is algebraic over F then there exists an element $\beta \in S$ with* $\lambda(\beta) = 1$ *such that $\beta^{-1}\alpha\beta \in R$.*

Proof. We may of course assume that α is nonzero. By Lemma 3.1, we know that $a = \lambda(\alpha) \in R$. Elaborating on the outline in [1, Sect. 7], we will show that there exists $\beta \in S$ with $\lambda(\beta) = 1$ such that $\gamma = \beta^{-1}\alpha\beta$ commutes with $a = \lambda(\alpha)$. It then follows that $\gamma - a$ is algebraic over F. But $\lambda(\gamma) = \lambda(\alpha) = a \in R$ and so $v(\gamma - a) > 1$. Thus we conclude that $\gamma = a$, as required.

To establish the existence of β, we need one preliminary observation.

Namely, if $x \in H$ is fixed, then the restriction of $\mathrm{ad}\, a = [a, \cdot]$ to $R\bar{x} \subset S$ yields an F-linear endomorphism φ of $R\bar{x}$ such that $\mathrm{Ker}\, \varphi \cap \mathrm{Im}\, \varphi = 0$. Indeed, suppose there exists $0 \neq \delta \in \mathrm{Ker}\, \varphi \cap \mathrm{Im}\, \varphi$. Then $\delta = [a, r\bar{x}]$ for some $r \in R$ and δ commutes with a. Thus, in S, we obtain $1 = \delta^{-1}[a, r\bar{x}] = [a, \delta^{-1} r\bar{x}]$ which contradicts Lemma 2.3. Therefore, we must have $\mathrm{Ker}\, \varphi \cap \mathrm{Im}\, \varphi = 0$, and since $R\bar{x}$ is finite dimensional over F, we conclude that $R\bar{x} = \mathrm{Ker}\, \varphi \oplus \mathrm{Im}\, \varphi = \mathbb{C}_{R\bar{x}}(a) \oplus [a, R\bar{x}]$.

Now, as in the proof of [1, Theorems 3.1 and 7.1], consider the set $\mathscr{X} = \mathscr{X}(\alpha)$ of all triples (x, γ, β) with $1 < x \in H \cup \{\infty\}$, $\gamma \in S$ such that $\lambda(\gamma) = a$, $\gamma a = a\gamma$ and $\mathrm{Supp}\, \gamma < x$ (i.e., all support elements of γ are less than x), and $\beta \in S$ such that $\lambda(\beta) = 1$, $\mathrm{Supp}\, \beta < x$ and $v(\beta\gamma\beta^{-1} - \alpha) = x$. Then $\mathscr{X}$ can be partially ordered by setting $(x, \gamma, \beta) \leqslant (x', \gamma', \beta')$ iff $x \leqslant x'$ and γ' and β' extend γ, respectively, β (i.e., γ' and γ agree on $\mathrm{Supp}\, \gamma$ and similarly for β' and β). By [1], $\mathscr{X}$ contains a maximal element, say (x, γ, β), and the result will follow if we can show that $x = \infty$, i.e., $\beta\gamma\beta^{-1} = \alpha$. So suppose to the contrary that $x = v(\beta\gamma\beta^{-1} - \alpha) < \infty$ and write $\lambda(\beta\gamma\beta^{-1} - \alpha) = a_x \bar{x}$. By the result of the preceding paragraph, we can write $a_x \bar{x} = r\bar{x} + [a, s\bar{x}]$ with $r, s \in R$ and $r\bar{x}a = ar\bar{x}$. Set $\gamma_1 = \gamma - r\bar{x}$ and $\beta_1 = \beta + s\bar{x}$ so that γ_1 and β_1 surely extend γ, respectively, β. Then $\beta_1^{-1} = \beta^{-1} - s\bar{x} +$ (higher terms) and so

$$\beta_1 \gamma_1 \beta_1^{-1} - \alpha = \beta\gamma\beta^{-1} - \alpha + (s\bar{x}a - as\bar{x} - r\bar{x}) + \text{(higher terms)}$$
$$= a_x \bar{x} - a_x \bar{x} + \text{(higher terms)}.$$

This shows that $x_1 = v(\beta_1 \gamma_1 \beta_1^{-1} - \alpha) > x$ so that $(x_1, \gamma_1, \beta_1) > (x, \gamma, \beta)$ which is the desired contradiction. ∎

4. Invariants

THEOREM 4.1. *Let G be a finitely generated nilpotent group and let P be a completely prime ideal of $K[G]$. Then the classical division ring of quotients $Q(K[G]/P)$ determines the localization $(K[G]/P)_{Z^*}$ of $K[G]/P$ at the nonzero elements of $Z = Z(K[G]/P)$.*

Proof. We have to show that if P_1 and P_2 are completely prime ideals of $K[G_1]$, respectively, $K[G_2]$, with G_1 and G_2 finitely generated nilpotent, such that $D_1 = Q(K[G_1]/P_1)$ and $D_2 = Q(K[G_2]/P_2)$ are isomorphic K-division algebras, then $(K[G_1]/P_1)_{Z_1^*}$ and $(K[G_2]/P_2)_{Z_2^*}$ are isomorphic. For this, we may clearly assume that P_1 and P_2 are faithful and we will do so in the following. The notation established in the previous sections will be used here with indices (1 or 2) attached.

Fix a K-algebra isomorphism $\varphi: D_1 \simeq D_2$. We view D_2 as being embedded in $S(P_2)$ and set $\Phi = \lambda_2 \circ \varphi$, with $\lambda_2: S(P_2) \to \mathscr{H}_2 \cup \{0\}$ denoting the lowest term map. Thus Φ is a multiplicative map from D_1 to $\mathscr{H}_2 \cup \{0\}$. We proceed in a number of steps.

Step 1. The restriction of Φ to $R_1 = (K[\Delta_1]/P_1 \cap K[\Delta_1])_{Z_1^*}$ induces a K-algebra isomorphism of R_1 onto R_2.

Since φ maps $F_1 = Z(D_1)$ onto $F_2 = Z(D_2)$ and R_1 is finite dimensional over F_1, we see that $\varphi(R_1)$ is algebraic over F_2. Thus, by Lemma 3.1, the lowest term map λ_2 yields a K-algebra embedding of $\varphi(R_1)$ into R_2, and hence $\Phi = \lambda_2 \circ \varphi$ is a K-algebra embedding of R_1 into R_2 which maps F_1 onto F_2. In particular, $\dim_{F_1} R_1 = \dim_{\Phi(F_1)} \Phi(R_1) \leqslant \dim_{F_2} R_2$. By symmetry, equality must hold here which forces Φ to be onto.

Step 2. The restriction of Φ to $\mathscr{H}_1 = \{r\bar{x} \mid r \in R_1^*, x \in H_1 = G_1/\Delta_1\}$ induces a group isomorphism of $\mathscr{H}_1$ onto $\mathscr{H}_2$.

Recall that R_1^* is a normal subgroup of $\mathscr{H}_1$ containing the center F_1^* of $\mathscr{H}_1$ and such that $\mathscr{H}_1/R_1^* \simeq H_1$ is nilpotent. It follows easily that any non-identity normal subgroup of $\mathscr{H}_1$ intersects R_1^* non-trivially. Inasmuch as Φ is injective on R_1, by Step 1, we conclude that Φ is injective on $\mathscr{H}_1$. To prove surjectivity, let $\alpha \in \mathscr{H}_2 \subset D_2$ be given and write $\alpha = \varphi(\beta\gamma^{-1}) = \varphi(\beta)\, \varphi(\gamma)^{-1}$ with $\beta, \gamma \in (K[G_1]/P_1)_{Z_1^*} = R_1 * H_1$. Then $\alpha = \lambda_2(\alpha) = \Phi(\beta)\, \Phi(\gamma)^{-1}$ and so it suffices to show that $\Phi(\beta), \Phi(\gamma) \in \Phi(\mathscr{H}_1)$. Write $\beta = \sum_x b_x \bar{x}$ with $b_x \in R_1$. Then the elements $\Phi(b_x\bar{x}) \in \mathscr{H}_2$, for $x \in \operatorname{Supp} \beta$, belong to distinct cosets of R_2^* in $\mathscr{H}_2$. Indeed, $R_2^*\Phi(b_x\bar{x}) = R_2^*\Phi(b_y\bar{y})$ implies that $\Phi(\bar{y}\bar{x}^{-1}) \in R_2^* = \Phi(R_1^*)$ and, by the injectivity of Φ, we get $\bar{y}\bar{x}^{-1} \in R_1^*$ so that $y = x$. Thus, with $\nu_2 \colon S(P_2) \to H_2 \cup \{\infty\}$ denoting the lowest term valuation, we see that the elements $\nu_2(\varphi(b_x\bar{x})) \in H_2$ are pairwise distinct, say $\nu_2(\varphi(b_{x_0}\overline{x_0}))$ is the smallest. Then $\Phi(\beta) = \lambda_2(\varphi(\beta)) = \lambda_2(\varphi(b_{x_0}\overline{x_0})) = \Phi(b_{x_0}\overline{x_0}) \in \Phi(\mathscr{H}_1)$, as required. Similarly, $\Phi(\gamma) \in \Phi(\mathscr{H}_1)$ and the assertion of Step 2 follows.

Step 3. The isomorphism $\theta \colon (K[G_1]/P_1)_{Z_1^*} \simeq (K[G_2]/P_2)_{Z_2^*}$.

As before, write $(K[G_i]/P_i)_{Z_i^*} = R_i * H_i$ for $i = 1, 2$, and for $\alpha = \sum_x a_x\bar{x} \in R_1 * H_1$ define $\theta(\alpha) = \sum_x \Phi(a_x\bar{x})$. Then θ is well defined, by uniqueness of expression in $R_1 * H_1$. Also, θ is K-linear, by Step 1, and multiplicative, since Φ is multiplicative. As we have seen above, the elements $\Phi(a_x\bar{x})$ with $x \in \operatorname{Supp} \alpha$ belong to distinct cosets of R_2^* in $\mathscr{H}_2$. This shows that $\theta(\alpha) \neq 0$ for $\alpha \neq 0$, and since each element of $R_2 * H_2$ is a sum of elements of $\mathscr{H}_2$, θ is also surjective, by Step 2. This completes the proof of the theorem. ∎

If, in the above theorem, P is also maximal then $Z = Z(K[G]/P)$ is a field and so $Q(K[G]/P)$ does in fact determine $K[G]/P$. This is no longer true if P is not maximal. For example, take a simple transcendental extension $K(X)$ of K and let $G_1 = \langle a \rangle \cong \mathbb{Z}$ and $G_2 = \langle b, c \rangle \cong \mathbb{Z} \times \mathbb{Z}$. Furthermore, let $\varphi_i \colon K[G_i] \to K(X)$ $(i = 1, 2)$ be given by $\varphi_1(a) = X$ and $\varphi_2(b) = X^2 + 1$, $\varphi_2(c) = X^3 + 1$. If P_i denotes the kernel of φ_i in $K[G_i]$, then $Q(K[G_i]/P_i) \cong K(X)$ in both cases. Furthermore, $P_1 = 0$ so $K[G_1]/P_1 \cong K[X]_X$ and

$K[G_2]/P_2 \cong K[X^2, X^3]_{X^2+1, X^3+1}$. Since the former ring is integrally closed but the latter ring is not, we see that $K[G_1]/P_1$ and $K[G_2]/P_2$ are non-isomorphic.

Finally, we rephrase the theorem for divison algebras generated by finitely generated nilpotent groups.

COROLLARY 4.2. *Let $D = K(G)$ be a division K-algebra generated by a finitely generated nilpotent group G and let F denote the center of D. Then the F-subalgebra of D generated by G is determined up to isomorphism by D. In particular, D determines the following data (up to isomorphism):*

(a) *the division subalgebra $K(\Delta)$ of D generated by the FC-center $\Delta = \Delta(G)$ of G,*

(b) *the subgroup $K(\Delta)^*G$ of D, and*

(c) *the group G/Δ.*

Proof. With the appropriate translations, all assertions have been established in the course of the proof of Theorem 4.1. We briefly indicate how to derive the corollary from the statement of the theorem. Write $D \cong Q(K[G]/P)$ with P completely prime in $K[G]$. Then the F-subalgebra, A say, of D generated by G is the image of $(K[G]/P)_{Z^*}$ in D and hence, by the theorem, is determined up to isomorphism by D. Therefore, D also determines the group of units $U(A)$ of A. Since $U((K[G]/P)_{Z^*}) = \mathscr{H} = R^*\bar{H}$, as we have observed in Section 3, we see that $U(A) = K(\Delta)^*G$. The invariance of $K(\Delta)$ follows from its description as the set of elements of A which are algebraic over the center of D. Finally, $G/\Delta \cong K(\Delta)^*G/K(\Delta)$. ∎

We remark that if, in the above, G is torsion-free, then we have $\Delta = Z(G)$ and $K(\Delta) = F = Z(D)$. Thus, in this case, D determines F^*G. In general, this need not be true. For example, let D denote the quaternions over $K = \mathbb{R}$, the real numbers, and take $G_1 = Q_8$, the quaternion group of order 8 in D^*, and $G_2 = \langle Q_8, \varepsilon \rangle$, where $\varepsilon \in \mathbb{C}$ is a primitive 8th root of unity. Then G_1 and G_2 both generate D over K, and G_1 is nilpotent of class 2, whereas G_2 has class 3. Therefore, $\mathbb{R}^*G_1$ and $\mathbb{R}^*G_2$ are not isomorphic.

REFERENCES

1. G. M. BERGMAN, Conjugates and *n*th roots in Hahn–Laurent group rings, *Bull. Malaysian Math. Soc.* **1** (2) (1978), 29–41; Historical addendum **2** (2) (1979), 41–42.
2. W. BORHO, P. GABRIEL, AND R. RENTSCHLER, "Primideale in Einhüllenden auflösbarer Lie-Algebren," Springer, Berlin/Heidelberg/New York, 1973.
3. D. R. FARKAS, A. H. SCHOFIELD, R. L. SNIDER, AND J. T. STAFFORD, The isomorphism question for division rings of group rings, *Proc. Amer. Math. Soc.* **85** (1982), 327–330.

4. D. E. FIELDS, Zero divisors and nilpotent elements in power series rings, *Proc. Amer. Math. Soc.* **27** (1971), 427–433.
5. D. HILBERT, "Grundlagen der Geometrie," Teubner, Leipzig, 1899.
6. T. LEVI-CIVITA, Sugli infiniti ed infinitesimi attulai quali elementi analitici, *Atti Ist. Veneto Sci. Lett. Arti* **IV** (VII) (1892–1893), 1765–1815.
7. M. LORENZ, Primitive ideals in crossed products and rings with finite group actions, *Math. Z.* **158** (1978), 285–294.
8. M. LORENZ AND D. S. PASSMAN, Prime ideals in crossed products of finite groups, *Israel J. Math.* **33** (1979), 89–132.
9. A. I. MALCEV, On embedding of group algebras in a division algebra, *Dokl. Akad. Nauk SSSR* **60** (1948), 1499–1501. [Russian]
10. B. H. NEUMANN, On ordered division rings, *Trans. Amer. Math. Soc.* **66** (1949), 202–252.
11. D. S. PASSMAN, "The Algebraic Structure of Group Rings," Wiley–Interscience, New York, 1977.
12. O. F. G. SCHILLING, Noncommutative valuations, *Bull. Amer. Math. Soc.* **51** (1945), 297–304.
13. A. E. ZALESSKII, Irreducible representations of finitely generated nilpotent torsion-free groups, *Math. Notes* **9** (1971), 117–123.