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On the Gelfand–Kirillov Dimension of Skew Polynomial Rings

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Let R be an algebra over a field k, let δ be a k-derivation of R and let $S = R[t; \delta]$ be the skew polynomial ring obtained from R by adjoining the indeterminate t subject to the relations

$$tr - rt = \delta(r) \qquad (r \in R).$$

In this note, we briefly comment on the relation between the Gelfand–Kirillov dimensions of R and S. More precisely, with $d(\cdot)$ denoting Gelfand–Kirillov dimension over k, we show that

(i) If R is affine over k then $d(S) = d(R) + 1$.

(ii) For any positive integer r, there exist R and δ such that $S = R[t; \delta]$ is affine over k and $d(S) = r$, whereas $d(R) = 0$.

Assertion (i), of course, comes as no surprise and was in fact believed to be true in general, without any further assumption on R. We include the easy proof for the sake of completeness. The examples constructed for (ii) can be used to give an example of a skew polynomial ring $S = R[t; \delta]$ with $d(R) = 0$ but $d(S) = \infty$. This was pointed out to us by John McConnell whom we also wish to thank for bringing the problem discussed in this note to our attention. We also thank G. Michler for helpful discussions.

Our notation is as in [1].

LEMMA (cf. [1, Lemma 3.1c]). *Let R be an affine k-algebra, let δ be a k-derivation of R, and let $S = R[t; \delta]$ be the corresponding skew polynomial ring over R. Then $d(S) = d(R) + 1$.*

Proof. The inequality $d(S) \geqslant d(R) + 1$ is straightforward and so we concentrate on the other inequality. For this, let $V \subseteq R$ be a finite-dimensional k-subspace of R such that $1 \in V$ and $R = k[V]$. Fix a positive integer l such that $\delta(V) \subseteq V^l$ and set $W = k \cdot t + V$, a finite-dimensional

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subspace of S with $S = k[W]$. We claim that for all $n = 0, 1, 2, \ldots$ we have $W^n \subseteq \sum_{i=0}^{n} V^{ln} t^i$. This is of course obvious for $n = 0$, since $W^0 = V^0 = k$. Assume that the assertion holds for $n \geqslant 0$. Then

$$V \cdot W^n \subseteq \sum_{i=0}^{n} V^{ln+1} t^i \subseteq \sum_{i=0}^{n+1} V^{l(n+1)} t^i,$$

and

$$t \cdot W^n \subseteq \sum_{i=0}^{n} t \cdot V^{ln} t^i \subseteq \sum_{i=0}^{n} V^{ln} t^{i+1} + \sum_{i=0}^{n} \delta(V^{ln})\, t^i$$
$$\subseteq \sum_{i=0}^{n+1} V^{l(n+1)} t^i.$$

Here, the latter inclusion holds because $\delta(V^{ln}) \subseteq \sum_{j=0}^{ln-1} V^j \delta(V)\, V^{ln-j-1} \subseteq \sum_{j=0}^{ln-1} V^j V^l V^{ln-j-1} \subseteq V^{l(n+1)}$. Since $W^{n+1} = V \cdot W^n + t \cdot W^n$, our claim is established. We deduce that $\dim_k W^n \leqslant (n+1) \dim_k V^{ln}$, and hence

$$d(S) = \overline{\lim_{n \to \infty}} \frac{\log \dim_k W^n}{\log n} \leqslant 1 + \overline{\lim_{n \to \infty}} \frac{\log \dim_k V^{ln}}{\log n}$$
$$= 1 + d(R),$$

since V^l is a finite-dimensional subspace of R generating R as a k-algebra. This proves the lemma.

Of course, finite generation of R over k is by no means necessary for the equality $d(S) = d(R) + 1$ to hold. Indeed, it clearly suffices to assume that every finite-dimensional k-subspace of R is contained in an affine δ-stable k-subalgebra of R. This is of course always true if $\delta = 0$, and also if R is of the form $R = A_T$, where A is affine over k and T is an Ore subset of regular elements of A. We now construct a family of examples showing that the conclusion of the lemma does not hold in general.

EXAMPLE. Fix a positive integer r. Let $T = k[X_1, X_2, \ldots]$ be the commutative polynomial ring in countably many variables over k and let $I = \langle X_1, X_2, \ldots \rangle$ be the augmentation ideal of T. Define a k-derivation δ on T by setting $\delta(X_i) = X_{i+1}$ $(i = 1, 2, \ldots)$. Then $\delta(I) \subseteq I$, and hence $\delta(I^r) \subseteq I^r$. Thus δ defines a derivation on $R = T/I^r$ which we will also denote by δ. Note that R is locally finite-dimensional so that $d(R) = 0$. Set $S = R[t; \delta]$. We show that $d(S) = r$.

Let x_i denote the image of X_i in R and set $V = k \cdot 1 + k \cdot x_1 + k \cdot t \subseteq S$. Note that V generates S as a k-algebra, since $tx_i - x_i t = x_{i+1}$ holds for all i. The same relation also shows that $x_n \in V^n$ for all $n \geqslant 1$. The set $\mathscr{M}_n$ of all nonzero monomials of the form $x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ in R, with $l_i \geqslant 0$, is linearly independent over k and is contained in $V^{n(r-1)}$, since necessarily $\sum_{i=1}^{n} l_i < r$. Moreover, the number of these monomials is

$$\varphi_r(n) = \sum_{j=0}^{r-1} \binom{n+j-1}{j} = \frac{1}{(r-1)!} n^{r-1} + (\text{terms of lower degree in } n).$$

The elements μt^i with $\mu \in \mathscr{M}_n$ and $0 \leqslant i \leqslant n$ are all linearly independent over k and belong to V^{rn}. On the other hand, V^n is contained in the subspace W_n of S generated by these elements. Indeed, this is clear for $n = 0$. Moreover, $x_1 \mathscr{M}_n \subseteq \mathscr{M}_n$ implies $x_1 W_n \subseteq W_n$, and if $\mu \in \mathscr{M}_n$ and $0 \leqslant i \leqslant n$ then

$$t\mu t^i = \mu t^{i+1} + \delta(\mu)\, t^i \in W_{n+1},$$

since $\delta(\mathscr{M}_n)$ is contained in the k-subspace of R generated by $\mathscr{M}_{n+1}$. By induction, we conclude that $V^n \subseteq W_n$. Summarizing, we have shown that

$$\dim_k V^{rn} \geqslant \dim_k W_n \geqslant \dim_k V^n.$$

Finally, $\dim_k W_n = (n+1)\, \varphi_r(n) = \lambda n^r + p$ for some constant $\lambda > 0$ and some polynomial p in n of degree less than r. Therefore, the first inequality implies that $d(S) \geqslant r$, since V^r is a finite-dimensional k-subspace of S, and the second inequality shows that $d(S) \leqslant r$, since V generates S. Thus we must have $d(S) = r$, as we have claimed.

To obtain an example with $d(S) = \infty$, let R_r, S_r, and δ_r denote the rings and the derivation constructed above so that $S_r = R_r[t; \delta_r]$, $d(S_r) = r$. Now let R denote the subalgebra of the direct product $\prod_{r \geqslant 1} R_r$ (with componentwise operations) generated by the elements of finite support, together with 1 (to make sure that all rings in sight have a 1), and let $\delta = \prod_{r \geqslant 1} \delta_r$ be the derivation of R obtained by letting δ_r act on the rth component in the given manner. Then $S = R[t; \delta]$ contains each S_r, which forces $d(S) = \infty$. On the other hand, R is clearly locally finite-dimensional so that $d(R) = 0$.

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Reference

1. W. Borho and H. Kraft, Über die Gelfand–Kirillov dimension, *Math. Ann.* **220** (1976), 1–24.