MITTEILUNGEN

aus dem

MATHEM. SEMINAR GIESSEN

Herausgegeben von den Professoren
des Mathematischen Instituts der Universität Giessen

Geschäftsführung: D. Gaier, G. Pickert

Sonderdruck aus Heft 149

GIESSEN 1981

SELBSTVERLAG DES MATHEMATISCHEN INSTITUTS
ISSN 0373 - 8221 CODEN: MMUGAU

THE GOLDIE RANK OF PRIME SUPERSOLVABLE GROUP ALGEBRAS

Martin Lorenz

Dedicated to Professor Hermann Boerner
on the occasion of his 75th birthday

The Farkas-Snider-Cliff theorem ([2],[5]) states that if G is a torsion-free polycyclic-by-finite group then, for any field K, the group algebra KG is a domain. It is now tempting to consider, more generally, the class of polycyclic-by-finite groups having no non-trivial finite <u>normal</u> subgroups. These are precisely the polycyclic-by-finite groups G whose group algebras KG are prime rings. In this case, by Goldie's theorem, KG has a classical ring of quotients $Q(KG)$, obtained by inverting the regular elements of KG, and $Q(KG)$ is of the form

$$Q(KG) \cong M_n(D)$$

for some positive integer n and some division ring D. Here, n and D are uniquely determined by KG, and n is called the <u>Goldie rank</u> of KG, written

$$n = \mathrm{rank}(KG) .$$

Thus KG has Goldie rank 1 if and only if KG is a domain. In view of the Farkas-Snider-Cliff theorem, it seems reasonable to try to relate n to the finite subgroups of G. More specifically, D. Farkas has conjectured the following (at least for so-called crystallographic groups; [4,p.593]):

(*) If G is polycyclic-by-finite having no non-identity finite normal subgroups then, for any field K,

$$\mathrm{rank}(KG) = f(G) .= \mathrm{l.c.m.}\{|U| : U \leq G , U \text{ finite}\} .$$

In this note, we offer a few reductions and show that (*) is in fact true for supersolvable groups, that is groups G having a finite series $\langle 1\rangle = G_0 \leq G_1 \leq \dots \leq G_r = G$ with each G_i normal in G and with G_i/G_{i-1} cyclic for all i. This result has been independently obtained by S. Rosset [10]. Our methods depend heavily on a certain special group theoretic property of supersolvable groups (first pointed out by Formanek [6]). Hence there is no hope that they might eventually lead to a solution of the problem for general polycyclic-by-finite groups.

We remark that the above notion of Goldie rank is actually a special case of a more general concept. If R is a ring (always with 1) and M is a right R-module then the <u>Goldie rank of M</u> (also called <u>uniform dimension of M</u>) is the greatest integer n such that M contains a direct sum of n nonzero submodules, or ∞ if no such integer exists. The (right) Goldie rank of R is the Goldie rank of R_R.

Throughout, G will denote a polycyclic-by-finite group and K will be a commutative field. In general, our notation and terminology is as in [9].

§1. Some Reductions

In this section, we collect a few basic facts about (Goldie) ranks of prime polycyclic group algebras. In particular we compare the rank of KG to the ranks of KH for certain subgroups H of G. Much of this is surely well-known, but it doesn't seem to be readily available in the literature.

In the proof of our first lemma, we will repeatedly use the following fact: If $R \subset S$ is an inclusion of rings (with the same 1) with R simple Artinian then rank(R) divides rank(S). ([7, Lemma 3.8])

Lemma 1.1. Assume KG is prime and let H be a subgroup of G.

i. If H is normal in G then KH is prime and $\mathrm{rank}(KH) \mid \mathrm{rank}(KG)$.

ii. If H has finite index then KH is prime and $\mathrm{rank}(KH) \mid \mathrm{rank}(KG) \mid \mathrm{rank}(KH) \cdot [G:H]$.

iii. Assume H is finite and let N denote a torsion-free normal subgroup of finite index in G. Set $U = \langle H, N \rangle$. Then $|H| = \mathrm{rank}(KU) \mid \mathrm{rank}(KG)$.

Proof. (i). Recall that KG is prime if and only if $\Delta^+(G)$, the finite radical of G, is $\langle 1 \rangle$. If H is normal in G, then $\Delta^+(H) \subset \Delta^+(G)$, and so we conclude that KH is prime. Let T denote the set of regular elements of KH. Then, by [9, Lemma 13.3.5(ii)], T is a right divisor set of regular elements in KG, and we have an inclusion of rings

$R = (KH)T^{-1} \subset S = (KG)T^{-1}$. Here, R is simple Artinian, by Goldie's theorem, and S is prime right Noetherian. Thus, as we have remarked above, $\operatorname{rank}(R) \mid \operatorname{rank}(S)$. Finally, $\operatorname{rank}(R) = \operatorname{rank}(KH)$ and $\operatorname{rank}(S) = \operatorname{rank}(KG)$, by [9, Lemma 10.2.13], so that (i) follows.

(ii). Now assume that H has finite index in G. Then, again, $\Delta^+(H) \subset \Delta^+(G)$ and so KH is prime. Let $N = \cap_{g\in G} H^g$ be the normal kernel of H and let T denote the set of regular elements of KN. As above, we can form $R = (KH)T^{-1} \subset S = (KG)T^{-1}$. Since N has finite index in G, it follows that both R and S are simple Artinian. Thus, as in part (i), we conclude that $\operatorname{rank}(KH) = \operatorname{rank}(R) \mid \operatorname{rank}(S) = \operatorname{rank}(KG)$. Moreover, S as a right module over R is free of rank $[G:H]$. Therefore, $S \subset \operatorname{End}(S_R) \cong M_{[G:H]}(R_R) \cong M_{[G:H]r}(D)$, where $r = \operatorname{rank}(R)$ and $R \cong M_r(D)$. Again, we conclude from our above remark that $\operatorname{rank}(S) \mid [G:H]r$, and (ii) is proved.

(iii). In view of part (ii), it suffices to show that $|H| = \operatorname{rank}(KU)$. Set $T = KN \setminus \{0\}$ and consider $R = (KN)T^{-1} \subset S = (KU)T^{-1}$, as in the proof of part (i). Then $S = RH$, a <u>skew</u> group ring of H over R with H acting on R via its conjugation action on KN. This action allows us to make R a right S-module by setting $r\cdot\Sigma_{h\in H}\, r_h h = \Sigma_{h\in H}\, (rr_h)^h$ for $r, r_h \in R$. Using the fact that S is simple Artinian we easily see that $S_S \cong R_S^{|H|}$, whence $\operatorname{rank}(S) = \operatorname{rank}(KU) = |H|\operatorname{rank}(R_S) = |H|$, where the latter holds since R is a domain. This completes the proof of the lemma.

Similar relations also hold for $f(G) = \text{l.c.m.}\{|U| :$ U a finite subgroup of $G\}$. The proof of the following lemma is quite elementary.

Lemma 1.2. For any subgroup H of G, we have $f(H) \mid f(G)$. Moreover, if H has finite index in G, then $f(G) \mid f(H)\cdot[G:H]$.

We close this section with two applications of the above.

Corollary 1.3. Let KG be prime. Then $f(G) \mid \text{rank}(KG)$.

Proof. This follows immediately from Lemma 1.1(iii), since N does always exist [9, Lemma 10.2.5].

Our next corollary reduces the problem of proving (*) to the special case of "tops" of prime power order. Part (ii) is actually a special case of a more general result on crossed products of finite groups [8, Theorem 1.7]. However, in the present situation, the result is a simple consequence of Lemma 1.1.

Corollary 1.4. Fix a torsion-free normal subgroup N of G of finite index. For each prime p, let G_p denote the complete inverse image in G of a Sylow p-subgroup of G/N. Then

i. $f(G) = \Pi_p f(G_p)$.

ii. If KG is prime then $\mathrm{rank}(KG) = \Pi_p \mathrm{rank}(KG_p)$.

In particular, if (*) holds for each KG_p then KG also satisfies (*) .

Proof. We only prove (ii), the proof of (i) being entirely analogous using Lemma 1.2 instead of Lemma 1.1.

Lemma 1.1(ii) implies that, for each p, $\mathrm{rank}(KG_p) \mid \mathrm{rank}(KG) \mid \mathrm{rank}(KG_p)\,[G:G_p]$. Applied to KG_p , Lemma 1.1(ii) also yields $\mathrm{rank}(KG_p) \mid \mathrm{rank}(KN)\,[G_p:N] = [G_p:N]$, where the latter equality follows from the Farkas-Snider-Cliff theorem. We deduce that $\mathrm{rank}(KG_p)$ is the p-part of $\mathrm{rank}(KG)$, and the assertion follows.

Let us quickly note that the above, together with the Farkas-Snider-Cliff theorem, already implies that (*) holds whenever G has a torsion-free normal subgroup N of finite index such that $|G/N|$ is a product of different primes. Indeed, Corollary 1.4 reduces the problem to the case $|G/N| = p$, a prime, and Lemma 1.1(ii), together with the Farkas-Snider-Cliff theorem, further implies that $\mathrm{rank}(KG) = 1$ or $\mathrm{rank}(KG) = p$, with the former being the case if and only if G is torsion-free, that is if and only if $f(G) = 1$.

§2. Torsion-Free Abelian and Infinite Dihedral Tops

Assume G has a normal subgroup N such that G/N is torsion-free. Then, clearly, all finite subgroups of G are contained in N and so we have

$$f(G) = f(N).$$

If G/N is torsion-free abelian then, by [1, Lemma 2.5], we also have

$$\mathrm{rank}(KG) = \mathrm{rank}(KN).$$

Combining these two observations we get the following

Lemma 2.1. Let KG be prime and assume that G has a normal subgroup N such that G/N is torsion-free abelian. If KN satisfies (*) then so does KG .

We now consider the more delicate situation where G has a normal subgroup N such that $G/N = D_\infty$, the infinite dihedral group. Since D_∞ is the free product of two copies of C_2, the cyclic group of order 2, our assumption on G implies that there exist subgroups G_1 and G_2 containing N such that G_i/N has order 2 $(i=1,2)$ and $G \cong G_1 \underset{N}{*} G_2$, the free product of G_1 and G_2 over N. Therefore, the next two lemmas deal with generalized free products of groups and of rings.

We begin with a few brief remarks on free products of rings over a common subring. For details we refer to [3,Chapter 5].

Let D be a ring. Then a <u>D-ring</u> is a ring R together with a fixed ring homomorphism $D \to R$. R is a <u>faithful</u> D-ring if the homomorphism $D \to R$ is an embedding. The D-rings, together with certain obvious morphisms, form a category and free products over D are coproducts in this category which satisfy certain additional technical conditions. The free product of two faithful D-rings R_1 and R_2, if it exists, is usually written as $R_1 \underset{D}{*} R_2$. A sufficient condition for the existence of $R_1 \underset{D}{*} R_2$ is that both R_1/D and R_2/D are free as right D-modules. In this case, $R_1 \underset{D}{*} R_2$ contains R_1 and R_2 as subrings, and if $B_i \cup \{1\}$ is a right D-basis for R_i $(i=1,2)$ then the <u>non-interacting words</u> in $B_1 \cup B_2$, i.e. the formal products in the elements of B_1 and B_2 with neighboring factors in different B_i's and with the empty word representing 1, form a basis of $R_1 \underset{D}{*} R_2$ as a right D-module. ([3,Theorem 5.1.2]). Note that this certainly applies if D is a division ring.

<u>Lemma 2.2.</u> Let D be a division ring and let R_1 and R_2 be faithful D-rings. Then the matrix rings $M_n(R_1)$ and $M_n(R_2)$ are faithful $M_n(D)$-rings, and $M_n(R_1) \underset{M_n(D)}{*} M_n(R_2)$ exists and is isomorphic to $M_n(R_1 \underset{D}{*} R_2)$.

<u>Proof.</u> We consider D as being embedded in R_1 and R_2. For $i = 1, 2$ let $B_i \cup \{1\}$ be a right D-basis of R_i. Identifying elements of R_i with the corresponding diagonal matrices in $M_n(R_i)$, we see that $B_i \cup \{1\}$ is a right $M_n(D)$-basis for $M_n(R_i)$. Thus, as we have remarked above, $S = M_n(R_1) \underset{M_n(D)}{*} M_n(R_2)$ exists and is free as a right $M_n(D)$-module, with basis the non-interacting words in $B_1 \cup B_2$.

Similarly, the non-interacting words in $B_1 \cup B_2$ form a right D-basis for $R_1 \underset{D}{*} R_2$ and, embedded diagonally in $M_n(R_1 \underset{D}{*} R_2)$, these words form a right $M_n(D)$-basis for $M_n(R_1 \underset{D}{*} R_2)$.

Now the embeddings of R_i into $R_1 \underset{D}{*} R_2$ yield embeddings μ_i of $M_n(R_i)$ into $M_n(R_1 \underset{D}{*} R_2)$ which agree on $M_n(D)$. Hence, by the coproduct property, there exists a homomorphism

$$\mu : \; S = M_n(R_1) \underset{M_n(D)}{*} M_n(R_2) \longrightarrow M_n(R_1 \underset{D}{*} R_2)$$

extending μ_1 and μ_2. By the above, μ sends a right $M_n(D)$-basis of S to a right $M_n(D)$-basis of $M_n(R_1 \underset{D}{*} R_2)$. Therefore, μ is bijective and the lemma is proved.

We now return to group algebras.

<u>Lemma 2.3.</u> Assume that G has subgroups N, G_1 and G_2 such that N is normal in G_1 and G_2 and $G = G_1 \underset{N}{*} G_2$. Let T be a right divisor set of regular elements in KN and assume that $g^{-1}Tg \subset T$ for all $g \in G_1 \cup G_2$. Then T is a right divisor set of regular elements in KG_1, KG_2, and KG. Moreover,

$$(KG)T^{-1} \cong (KG_1)T^{-1} \underset{(KN)T^{-1}}{*} (KG_2)T^{-1}.$$

<u>Proof.</u> Note that N is also normal in G and that $g^{-1}Tg \subset T$ holds for all $g \in G$, since G_1 and G_2 generate G. The fact that T is a right divisor set of regular elements in KG_1, KG_2, and KG now follows as in [9, Lemma 13.3.5(ii)]. Set $Q = (KN)T^{-1}$, $R_i = (KG_i)T^{-1}$ $(i=1,2)$, and $R = (KG)T^{-1}$ so that $Q \subset R_i \subset R$ $(i=1,2)$.

For $i = 1, 2$, let $B_i \cup \{1\}$ be a transversal for N in G_i. Then the non-interacting words in $B_1 \cup B_2$ form a transversal for N in G, and hence a basis of R as a right Q-module. Similarly, $B_i \cup \{1\}$ is a right Q-basis of R_i and so the non-interacting words in $B_1 \cup B_2$ form a basis of $R_1 \underset{Q}{*} R_2$ as a right Q-mudule. Finally, the inclusions $Q \subset R_i \subset R$ extend to a map $\varphi\colon R_1 \underset{Q}{*} R_2 \quad R$. Thus φ is the identity on Q and maps a right Q-basis of $R_1 \underset{Q}{*} R_2$ to a right Q-basis of R, whence φ is bijective. The lemma is proved.

We are now ready to prove the main result of this section.

Proposition 2.4. Let KG be prime and assume that G has a normal subgroup N such that G/N is infinite dihedral. If (*) holds for all subgroups of G of infinite index then KG also satisfies (*).

Proof. Our assumption on G implies that G has a normal subgroup M containing N such that G/M has order 2 and M/N is infinite cyclic. By Lemmy 1.1(ii), we have either $\mathrm{rank}(KG) = \mathrm{rank}(KM)$ or $\mathrm{rank}(KG) = 2\cdot\mathrm{rank}(KM)$. Moreover, as we have remarked above, $\mathrm{rank}(KM) = \mathrm{rank}(KN)$ and since KN satisfies (*) , by assumption, we further have $\mathrm{rank}(KN) = f(N)$. Thus:

$$\text{either} \quad \mathrm{rank}(KG) = f(N) \quad \text{or} \quad \mathrm{rank}(KG) = 2\cdot f(N) \; .$$

Now choose subgroups G_1 and G_2 of G so that G_1 and G_2 both contain N as a subgroup of index 2 , and $G \cong G_1 \underset{N}{*} G_2$. By Lemma 1.2, $f(G_i)$ is either equal to $f(N)$ or $2\cdot f(N)$. If $f(G_i) = 2\cdot f(N)$ for some i then $f(G) \geq 2\cdot f(N) \geq \mathrm{rank}(KG)$, by the foregoing, and Corollary 1.3 yields the result in this case. Thus, in the following, we may assume that

$$f(G_1) = f(G_2) = f(N) \; .$$

We remark that this implies that $f(G)$ also equals $f(N)$, but we will not need this information.

Suppose that some KG_i is not prime. Then G_i has a non-trivial finite normal subgroup which has to intersect N trivially, since KN is prime, by Lemma 1.1(i). It follows easily that $f(G_i) = 2\cdot f(N)$, a contradiction. Thus:

KG_1 and KG_2 are both prime, and
$\text{rank}(KG_1) = \text{rank}(KG_2) = \text{rank}(KN)$.

Here, the rank equalities follow since KG_1 and KG_2 satisfy (*), by assumption.

Now let T denote the set of regular elements of KN. Then Lemma 2.3 applies and letting $Q = (KN)T^{-1}$, $R_i = (KG_i)T^{-1}$, and $R = (KG)T^{-1}$ we have $Q \subset R_i \subset R$ $(i=1,2)$ and $R \cong R_1 \underset{Q}{*} R_2$. Moreover, Q, R_1, and R_2 are simple Artinian rings all having the same rank n, say. Therefore, if e is a primitive idempotent of Q, then eQ is a simple right ideal of Q and eR_i is a simple right ideal of R_i for $i = 1, 2$. Thus $D = eQe$ and $D_i = eR_ie$ are division rings with $D \subset D_i$ and $Q \cong M_n(D) \subset M_n(D_i) \cong R_i$. Lemma 2.2, with D_i playing the rôle of R_i, yields $R_1 \underset{Q}{*} R_2 \cong M_n(D_1 \underset{D}{*} D_2)$ so that we have

$$R \cong M_n(D_1 \underset{D}{*} D_2)\ .$$

By [3,Theorem 5.3.2], $D_1 \underset{D}{*} D_2$ is a domain and it follows that $\text{rank}(R) = \text{rank}(KG) = n$. Finally, $n = \text{rank}(Q) = \text{rank}(KN)$

$= f(N) \leq f(G)$ and so the proposition follows from Corollary 1.3.

§3. Supersolvable Groups

It is now a simple matter to obtain (*) for group algebras of supersolvable groups. So let KG be prime with G supersolvable. Arguing by induction on the Hirsch number of G, we may assume that G is infinite and that (*) holds for all subgroups of infinite index in G. By a result of Formanek [6], G has a normal subgroup N with G/N either infinite cyclic or infinite dihedral. In the former case, Lemma 2.1 shows that KG satisfies (*) and in the latter case, the result follows from Proposition 2.4. Thus we have obtained the following

Theorem 3.1. Let KG be prime with G supersolvable. Then $\operatorname{rank}(KG) = \text{l.c.m.}\{|U| : U \text{ a finite subgroup of } G\}$.

REFERENCES

1. K. A. Brown, Artinian quotient rings of group rings, J. Algebra 49(1977), 63 - 80.

2. G. H. Cliff, Zero divisors and idempotents in group rings, Canadian J. Math. 32(1980), 596 - 602.

3. P. M. Cohn, Skew Field Constructions, L.M.S. Lect. Notes 27, Cambridge University Press, Cambridge (1977).

4. D. R. Farkas, Group rings: an annotated questionnaire, Commun. in Algebra 8(1980), 585 - 602.

5. D. R. Farkas and R. L. Snider, K_0 and Noetherian group rings, J. Algebra 42(1976), 192 - 198.

6. E. Formanek, The zero divisor question for supersolvable groups, Bull. Australian Math. Soc. 9(1973), 67 - 71.

7. A. Joseph and L. W. Small, An additivity principle for Goldie rank, Israel J. Math. 31(1978), 105 - 114.

8. M. Lorenz and D. S. Passman, Observations on crossed products and fixed rings, Commun. in Algebra 8(1980), 743 - 779.