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ON THE GELFAND-KIRILLOV DIMENSION OF NOETHERIAN PI-ALGEBRAS

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Let $d(.)$ denote Gelfand-Kirillov dimension over a fixed base field k. In this note we prove the following.

THEOREM. *Let* R *be a Noetherian* PI-*algebra over* k *and let* N *denote its nilpotent radical. Then* $d(R) = d(R/N) = \max_P d(R/P)$, *where* P *ranges over the minimal prime ideals of* R.

Here, the last equality is of course clear from [3, Lemma 3.1e]. The corresponding result for (Gabriel-Rentschler) Krull dimension is well-known and is in fact an easy consequence of the so-called partitivity of Krull dimension: For any two modules $M \subseteq N$ over a ring R one has $\text{Kdim } N = \sup\{\text{Kdim } M, \text{Kdim } N/M\}$ whenever these Krull dimensions do exist. In general, this relation does not hold for Gelfand-Kirillov dimensions of modules. Indeed, G. Bergman has recently constructed finitely generated modules $M \subseteq N$ over an affine PI-ring R such that $d(N) > \sup\{d(M), d(N/M)\}$ [2]. Another, non-finitely generated, example, due to McConnell [6], is as follows. Let $R = k\{X,Y\}/\langle Y\rangle^2$, where $k\{X,Y\}$ denotes the free k-algebra generated by X and Y, and set $N = \langle Y\rangle$, the ideal of R generated by Y. Then, as right R-modules, one has $d(R/N) = d(N) = 1$ but $d(R) = 2$. Note that R is an affine PI-algebra, yet the conclusion of the theorem does not hold for R.

Our theorem has the following consequence.

COROLLARY. *Let* R *be a Noetherian* PI-*algebra. Then* $d(R)$ *is either infinite or an integer. If* R *is also affine then* $d(R) = \dim R$, *the prime length of* R.

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Proof. By the theorem, we immediately reduce to the case where R is prime. Let C denote the center of R. Then, using the fact that $Q(R)$ is obtained by localizing at the non-zero elements of C and is a finitely generated module over $Q(C)$, one easily shows that $d(R) = d(C)$ [5]. But $d(C)$ is either infinite or an integer, by [3, Satz 3.2]. In fact, $d(C)$ equals the transcendence degree of C over k, i.e. the maximal number of algebraically independent elements of C over k. By [1. Theorem 7.1], the latter is equal to dim R, and the corollary is proved.

The reader should contrast this with the examples constructed in [3, (2.10 and (2.11)] which show that for any real number $\gamma \geq 2$ there exists an affine (in fact, 2-generated) PI-algebra R with $d(R) = \gamma$.

Throughout this note, k will denote a commutative field. For the basic facts concerning Gelfand-Kirillov dimension we refer to [3].

We begin with two observations that will be needed later on. At least part (i) of the following lemma is well-known, but we include the short argument for the reader's convenience.

LEMMA 1. *Let* R *be a* k-*algebra*.

i. If $S \subseteq R$ is a subalgebra such that R is finitely generated as a (right or left) S-module then $d(R) = d(S)$.

ii. Assume that R is right Noetherian and let $e_1, e_2, \ldots, e_t \in R$ be an orthogonal set of idempotents with $\sum_i e_k = 1$. Then $d(R) = \max_i d(e_k R e_i)$.

Proof. (i). Only $d(R) \leq d(S)$ has to be proved. To see this, just note that R is a subalgebra of $\mathrm{End}_S(R)$ which in turn is a subquotient of some matrix algebra $M_n(S)$ over S. Therefore, $d(R) \leq d(M_n(S))$ and the assertion follows, since $d(M_n(S)) = d(S \otimes_k M_n(k)) = d(S)$, by [3, Lemma 3.1a].

(ii). If $e = e^2 \in R$ then Re is a Noetherian right eRe-module. Indeed, if $I \subseteq Re$ is an eRe-submodule then $IR \cap Re = IRe = IeRe = I$. In particular, each Re_i is finitely generated as a right $e_i Re_i$-module and so R is finitely generated as a right module over $S = \oplus_i e_i Re_i \subseteq R$. Finally, $d(S) = \max_i d(e_i Re_i)$, by [3, Lemma 3.1b]. Hence the assertion follows from part (i).

The next lemma contains the computations needed for the proof of the theorem.

LEMMA 3. *Let* R *be a* k-*algebra and let* N *be a nilpotent ideal of* R *which is finitely generated as a right ideal of* R. *Assume that* R/N *is a finitely generated module over some commutative subalgebra. Then* $d(R) = d(R/N)$.

Proof: We only have to show that $d(R) \leq d(R/N)$ holds and for this we may clearly assume that $d(R/N) = d < \infty$. Let $^{-} : R \to R/N$ denote the canonical map. Using Lemma 1 (i), we immediately reduce to the case where R is commutative.

Fix a finite generating set Γ for N as a right ideal of R. Since N is nilpotent, we may assume that products of elements of Γ are contained in Γ.

Now let V be a finite dimensional k-subspace of R with $1 \in V$. Then by the Noether normalization theorem, the subalgebra $k[\bar{V}] \subseteq \bar{R}$ is a finitely generated module over some polynomial ring $S = k[x_1, x_2, \ldots, x_r] \subseteq k[\bar{V}]$. Here, $r = d(S) \leq d$. Choose preimages $y_i \in k[V] \subseteq R$ with $\bar{y}_i = x_i$ and set $T = k[y_1, \ldots, y_r] \subseteq k[V]$. Furthermore, let $g_1, g_2, \ldots, g_p \in k[V]$ be preimages for the generators of $k[\bar{V}]$ over S and let $v_1, v_2, \ldots, v_q$ be a k-basis for V. Then there exists elements t_{hi}, $t_{hij} \in T$ and n_i, $n_{ij} \in N$ such that

$$(1) \qquad v_i = w_i + n_i \quad \text{with} \quad w_i = \sum_{h=1}^{p} g_h t_{hi} \quad (i=1,2,\ldots,q)$$

and

$$(2) \qquad g_i g_j = \sum_{h=1}^{p} g_h t_{hij} + n_{ij} \quad (i,j=1,2,\ldots,p).$$

Choose a finite dimensional k-subspace W of R such that

$$(3) \qquad V \subseteq W \quad \text{and} \quad [V,V],\ V\cdot\Gamma,\ \{n_i, n_{ij} \text{ all } i,j\} \subseteq \Gamma\cdot W.$$

Moreover, choose an integer $s \geq 1$ such that $y_i, g_h, t_{hi}, t_{hij}, w_i \in V^s$, the vector space generated by the product of length $\leq s$ with factors $\in V$. From (3) we deduce that

$$(4) \qquad [V^s,V^s] = \sum_{h=0}^{2s-2} V^h[V,V]V^{2s-2-h} \subseteq \sum_{h=0}^{2s-2} V^h\Gamma\cdot W\, W^{2s-2-h} \subseteq \Gamma\cdot W^{2s-1} .$$

Now consider a monimial $v_{i_1}v_{i_2} \cdots v_{i_n} \in V^n$. Successively writing $v_{i_h} = w_{i_h} + n_{i_h}$ $(h=1,2,\ldots n)$ we get

$$v_{i_1}v_{i_2} \cdots v_{i_n} = w_{i_1}w_{i_2} \cdots w_{i_n} + n_{i_1}v_{i_2}v_{i_3} \cdots v_{i_n} + + w_{i_1}n_{i_2}v_{i_3} \cdots v_{i_n} + \cdots + w_{i_1}w_{i_2} \cdots w_{i_{n-1}}n_{i_n} .$$

By (3), all summands but the first one belongs to $\Gamma\cdot W^{sn}$. Thus we have

$$v_{i_1}v_{i_2} \cdots v_{i_n} \in w_{i_1}w_{i_2} \cdots w_{i_n} + \Gamma\cdot W^{sn} .$$

Now consider $w_{i_1}w_{i_2} \cdots w_{i_n} = w$. By (1), w is a sum of monomials of the form $g_{h_1}t_{h_1i_1}g_{h_2}t_{h_2i_2} \cdots g_{h_n}t_{h_ni_n}$. But

$$g_{h_1}t_{h_1i_1}g_{h_2}t_{h_2i_2} \cdots g_{h_n}t_{h_ni_n} = g_{h_1}g_{h_2} \cdots g_{h_n}t_{h_1i_1} \cdots t_{h_ni_n} + c,$$

where c is a sum of monomials of the form $c_1c_2 \cdots c_h[c_{h+1},c_{h+2}]c_{h+3} \cdots c_{2n}$ with each $c_h \in \{g_m, t_{hi} | m,h,i\}$. Using (3) and (4), we see that $c \in \Gamma\cdot W^{2sn-1}$.

Next, successively using (2), we can express $g_{h_1} \cdots g_{h_n}t_{h_1i_1} \cdots t_{h_ni_n}$ as a sum of monomials of the form

$$g_1\beta t_{h_1i_1} \cdots t_{h_ni_n} ,$$

where β denotes a monomial of length $n-1$ in the t_{hij}'s, plus a sum of monomials of the form

$$\gamma n_{ij}\delta ,$$

where γ denotes a monomial in the g_h's and δ denotes a monomial in the t_{hi}'s such that $\gamma n_{ij}\delta$ has length $2n-1$. By (3), the latter type of monomials again belongs to $\Gamma\cdot W^{2ns-1}$. The first type on monomials, $g_1 t_{h_1i_1} \cdots t_{h_ni_n}$, can be rewritten by expressing each t_{hij} and t_{hi} in terms of the y_i's. So let Y denote the subspace of T generated by $1, y_1, y_2, \ldots, y_r$ and choose $f \geq 1$ with $f_{hij}, t_{hi} \in Y^f$. Then, summarizing the above, we have shown that

$$(5) \quad V^n \subset \sum_{h=1}^{p} g_h Y^{f(2n-1)} + \Gamma \cdot W^{sn} + \Gamma \cdot W^{2sn-1}$$
$$\subseteq \sum_{h=1}^{p} g_h Y^{2fn} + \Gamma \cdot W^{2sn} .$$

Finally, each monomial $y_{i_1} y_{i_2} \ldots y_{i_m} \in Y^m$ can be written as

$$y_{i_1} y_{i_2} \ldots y_{i_m} = y_1^{e_1} y_2^{e_2} \ldots y_r^{e_r} + b ,$$

where b belongs to $\sum_{h=0}^{m} Y^h [Y,Y] Y^{m-(1+2)}$ and $\Sigma e_i = m$. Moreover, by (3) and (4), we have

$$Y^h [Y,Y] Y^{m-(h+2)} \subseteq V^{sh} \Gamma \cdot W^{2s-1} \cdot W^{s(m-h-2)} \subseteq \Gamma \cdot W^{sm-1}.$$

Thus, if Y_m denotes the k-subspace of R generated by the "ordered" monomials $y_1^{e_1} y_2^{e_2} \ldots y_r^{e_r}$ with $\Sigma e_i \leq m$, then we have, for all m,

$$(6) \quad Y^m \subset Y_m + \Gamma W^{sm-1} .$$

Using this with $m = 2fn$, (5) yields

$$V^n \subseteq \sum_{h=1}^{p} g_h y_{2fn} + \sum_{h=1}^{p} g_h \Gamma \cdot W^{2fsn-1} + \Gamma \cdot W^{2sn}$$
$$\subseteq \sum_{h=1}^{p} g_h Y_{2fn} + \Gamma \cdot W^{s+2fsn-1} + \Gamma \cdot W^{2sn}$$
$$\subseteq \sum_{h=1}^{p} g_h Y_{2fn} + \Gamma \cdot W^{h \cdot n}$$

for a suitable constant $h \geq 1$. Set $X = \sum_{h=1}^{p} g_h Y_{2fn}$ and $V_1 = W^h$. Then, since $\dim_K Y_{2fn} \leq \eta\, n^r$ for a suitable constant $\eta > 0$, we get $\dim_K X \leq \xi n^r \leq \xi n^d$ for a suitable $\xi > 0$. Thus we have shown that

$$(7) \quad V^n \subseteq X + \Gamma \cdot V_1^{\,n}, \text{ with } V_1 \subseteq R \text{ finite dimensional and}$$
$$\dim_K X \leq \xi\, n^d, \quad \xi > 0.$$

We now iterate (7) by letting V_1 play the role of V above, etc. This gives $V^n \subseteq X + \Gamma\cdot V_1{}^n \subseteq X + \Gamma\cdot(X_1 + \Gamma\cdot V_2{}^n) \subseteq \ldots \subseteq X + \Gamma\cdot(X_1 + \Gamma\cdot(X_2 + \ldots + \Gamma\cdot(X_1 + \Gamma\cdot V_{h+1}{}^n)) \ldots)$, where each V_i is a finite dimensional k-subspace of R and $\dim_k X_i \leq \xi_i n^d$ for suitable constants $\xi_i > 0$. Finally, if t denotes the nilpotence of N, then we have

$$X + \Gamma\cdot(X_1 + \Gamma\cdot(X_2 + \ldots + \Gamma\cdot(X_{t-1} + \Gamma\cdot V_t{}^n)) \ldots) = X + \Gamma\cdot(X_1 + \Gamma\cdot(X_2 + \ldots + \Gamma\cdot X_{t-1})) \ldots) .$$

Thus, since Γ is closed under products, we deduce that

$$V^n \subseteq X + \Gamma\cdot \sum_{i=1}^{t-1} X_i .$$

Therefore, $\dim_K V^n \leq \xi n^d + |\Gamma|\cdot \sum_{i=1}^{t-1} \xi_i n^d = \lambda_V n^d$ for some constant $\lambda_V > 0$. The assertion of the lemma now follows, since V was arbitrary.

We remark that the hypotheses of Lemma 3 do not force R to be a finitely generated module over some commutative subalgebra. (See Sarraille [7].) We are now ready to prove the theorem.

Proof of the theorem. Let R be a Noetherian PI-algebra over k. We have to show that $d(R) = d(R/P)$ for some prime ideal P of R.

Since R is Noetherian, we have $0 = \bigcap_{j=1}^{h} I_j$ for certain meet-irreducible. By Gordon [4], R has an Artinian ring of quotients Q. Let N be the nilpotent radical of R and let $\bar{\ }:R \to R/N$ denote the canonical map. S Since $\bar{\ }$ maps regular elements of R to regular elements of $\bar{R}$, we can extend $\bar{\ }$ to a map $\bar{\ }:Q \to \bar{Q} = Q(\bar{R})$ with kernel NQ, and hence we have an isomorphism $\bar{Q} \cong Q/NQ$.

We claim that $d(\bar{Q}) = d(\bar{R})$. Indeed, $\bar{Q} = \bigoplus_{i=1}^{t} Q(\bar{R}/P_i)$, where $P_i (i=1,2,\ldots,t)$ are the minimal primes of $\bar{R}$. Since each $Q(\bar{R}/P_i)$ is obtained from $\bar{R}/P_i$ by central localization, we have $d(Q(\bar{R}/P_i)) = d(\bar{R}/P_i)$ for all i, by [3, Lemma 31f], and hence $d(\bar{Q}) = \max_i d(Q(\bar{R}/P_i)) = \max_i d(\bar{R}/P_i) = d(\bar{R})$, as we have claimed. Clearly, $d(R) \leq d(Q)$ and so it suffices to show that $d(Q) = d(\bar{Q})$.

For this, let $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_t \in \bar{Q}$ denote the idempotents corresponding to the decomposition $\bar{Q} = \bigoplus_{i=1}^{t} Q(\bar{R}/P_i)$. There are orthogonal idempotents $e_i \in Q (i=1,2,\ldots,t)$ which are preimages for the $\bar{e}_i$'s and satisfy $\sum_{i=1}^{t} e_i = 1$. By Lemma 1(ii), we know that $d(Q) = \max_i d(e_iQe_i)$. Moreover, we have $e_iQe_i/e_iNQe_i \cong \overline{e_iQe_i} = Q(\bar{R}/P_i) \subseteq \bar{Q}$. Since $Q(\bar{R}/P_i)$ is a finitely generated module over its center, we can apply Lemma 3, with e_iQe_i playing the role of R and e_iNQe_i playing the role of N, to conclude that $d(e_iQe_i) = d(Q(R/P_i))$. Therefore, $d(Q) = \max_i d(Q(\bar{R}/P_i)) = d(\bar{Q})$, which completes the proof of the theorem.

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