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PRIME IDEALS IN FIXED RINGS II

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Recently there has been some interest in so-called "Additivity Principles" [2] which, for a ring extension $S \subset R$ and a prime ideal P of R, relate the Goldie rank of R/P to the Goldie ranks of S/Q, for all primes Q of S which are minimal over $P \cap S$.

In this note, we prove such a theorem for the ring extension $R^G \subset R$, where R^G is the fixed subring of a finite group G acting as automorphisms of R, such that $|G|^{-1} \in R$. Our result improves the bound on Goldie ranks obtained in [4].

We also include a few additional remarks on prime ideals in fixed rings.

We first require, form [4], some facts about the relationship between prime ideals in R and R^G. For P a prime ideal of R, let

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$A = \bigcap_{g \in G} P^g$; A is a semiprime ideal of R which is G-stable (A is called a G-prime ideal). Let $\mathcal{a} = \{A \cap R^G | A = \bigcap_g P^g$, P a prime ideal of R$\}$. It is clear in the above that $A \cap R^G = P \cap R^G$.

Theorem A [4, Proposition 4.2] let R be a ring, and G a finite group acting as automorphisms on R such that $|G|^{-1} \varepsilon R$. Then

1) $\mathcal{a}$ is a set of semiprime ideals of R^G, and is in one-to-one correspondence with the set of G-prime ideals of R.
2) Any prime Q of R^G is minimal over some unique $I \varepsilon \mathcal{a}$.
3) There are only finitely many primes of R^G minimal over any $I \varepsilon \mathcal{a}$, say $Q_1, \ldots, Q_m$, where $m \leq |G|$. Moreover, $I = \bigcap_{i=1}^{m} Q_i$.

Two primes Q_1, Q_2 of R^G are said to be equivalent if they are minimal over the same $I \varepsilon \mathcal{a}$. For properties of this equivalence, see [4, Proposition 3.5]; in particular, equivalent primes have the same height.

We also require the additivity principle mentioned at the beginning. For a ring C, rk(C) denotes the Goldie rank of C (also called the Goldie dimension).

Theorem B [2, Lemma 3.8]. Let $A \subset B$ be Artinian rings with the same unit element. Let P be a prime ideal of B, and let $Q_1, \ldots, Q_r$ be the primes of A which are minimal over $P \cap A$. Then there exist positive integers $z_1, \ldots, z_r$ such that

$$\mathrm{rk}(B/P) = \sum_{i=1}^{r} z_i \, \mathrm{rk}(A/Q_i).$$

We are now able to prove our main theorem.

Theorem: Let R be a ring, and G a finite group acting as automorphisms of R with $|G|^{-1} \varepsilon R$. Let P be a prime ideal of R, and say that $P \cap R^G = Q_1 \cap Q_2 \cap \cdots \cap Q_m$, where the $\{Q_i\}$ are prime ideals of R^G. If R/P is a Goldie ring, then:

1) R^G/Q_i is a Goldie ring, all $i = 1,\ldots,n$

2) There exist positive integers $z_1,\ldots,z_m$ such that.

$$\mathrm{rk}(R/P) = \sum_{i=1}^{m} z_i \, \mathrm{rk}(R^G/Q_i)$$

proof: Since R/P is Goldie, R/P^g is also Goldie for each $g \varepsilon G$, since $R/P \simeq R/P^g$. Thus $\bar{R} = R/\bigcap_g P^g$ is Goldie since it is a subdirect product of the $\{R/P^g\}$. $\bar{R}$ has an induced G-action, since $\bigcap_g P^g$ is G-stable, and moreover, $\bar{R}^G = \overline{R^G}$ since $|G|^{-1} \varepsilon R$ (the mapping $\rho(x) = |G|^{-1} \sum_{g \varepsilon G} x^g$ is a projection of R onto R^G). By passing to $\bar{R}$ we may assume that R is Goldie, $P \cap R^G = (0)$, and $(0) = Q_1 \cap \cdots \cap Q_m$, where $\{Q_i\}$ are the minimal primes of R^G.

Now by a theorem of Kharchenko [3], R being Goldie implies that R^G is also Goldie; moreover if T is the set of regular elements of R^G, then the elements of T are regular in R and $Q(R) = RT^{-1}$ [1], where Q(R) is the semi-simple Artinian quotient ring of R.

Since R^G is Goldie and Q_i is a minimal prime, R^G/Q_i is also Goldie, proving 1).

Let $A = Q(R^G)$ and $B = Q(R)$. Since $P \cap T = \emptyset$, PT^{-1} is prime in B and $rk(B/PT^{-1}) = rk(R/P)$. Also, $Q_1T^{-1}, \ldots, Q_mT^{-1}$ are precisely the primes of $A = R^GT^{-1}$, and $rk(A/Q_iT^{-1}) = rk(R^G/Q_i)$. The proof is now finished by applying Theorem B to the Artinian rings $A \subset B$. □

We first give an example to show that some hypothesis about $|G|$ is required.

Example 1: A PI ring R of characteristic $p \neq 0$, and $G \subseteq Aut(R)$ of order p, and a prime ideal P of R so that R/P is Goldie of rank 3 but $P \cap R^G = Q$ is a prime ideal of R^G of rank 2. Thus the additivity principle does not hold.

proof: Let k be a field of characteristic $p \neq 0$ and let k[x,y] be the commutative polynomial ring over k. Define $\phi: k[x,y] \to k[x,y]$ by $\phi(y) = x$, $\phi(x) = 0$. Let $B = M_3(k[x,y])$ and let

$$A = \left\{ \begin{pmatrix} \begin{pmatrix} a & b \\ xc & d \end{pmatrix} & 0 \\ 0 & \phi(d) \end{pmatrix},\ a,b,c,d \in k[x,y] \right\} .$$

Note that $A \subset B$ with the same unit element, and that A is prime with $rk(A) = 2$.

Now let $R = \left\{ \begin{pmatrix} a & b_1 \\ 0 & b_2 \end{pmatrix},\ a \in A, b_i \in B \right\}$. Let $g \in Aut(R)$ be given by conjugation by

$S = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix} \in R$; since $S^P = I$, g has order P. Let $G = \langle g \rangle$;

then $R^G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a \in A, b \in B \right\}$. Now let $P = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, a \in A, b \in B \right\}$,

a prime ideal of R. $R/P \cong B$, which is Goldie of rank 3. However

$P \cap R^G = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, b \in B \right\} = Q$, a prime, and $R^G/Q \cong A$, which is Goldie of rank 2.

Our next example shows that the conclusions of the theorem do not hold if one begins with a prime Q of R^G. Moreover, it provides an example of two equivalent primes of R^G which have different depths (as was noted above, equivalent primes always have the smae height).

Example 2: A prime ring R which is not Goldie, and $G \subseteq Aut(R)$ with $|G|^{-1} \in R$ such that R^G has two minimal primes Q_1, Q_2, with $Q_1 \cap Q_2 = (0)$, such that R^G/Q_1 is Goldie but R^G/Q_2 is not. Moreover, Q_1 is maxim al but Q_2 is not.

proof: Let k be a field containing a primitive nth root of 1, say α, for some $n > 1$. Let V be a countable dimensional vector space over k, and let $R = Hom_k(V,V)$. Choose a basis $\{v_1, v_2, \ldots v_n, \ldots\}$ for V, and define $T \in R$ by $Tv_1 = \alpha v_1$ and $Tv_i = v_i$, $i \geq 2$. Let $g \in Aut(R)$ be defined to be conjugation by T. Then $G = \langle g \rangle$ has order n, and $R^G = k\, v_1 \oplus R'$, where $R' = Hom_k(V', V')$ and V' is the subspace with basis $\{v_i, i \geq 2\}$. Let $Q_1 = (0, R')$ and $Q_2 = (kv_1, 0)$. Since $R^G/Q_1 \cong k$, Q_1 is maximal and R^G/Q_1 is Goldie. However, $R^G/Q_2 \cong R'$, which is not Goldie. Finally, let S denote the socle of R'. Since R'/S is simple, $Q = (kv_1, S)$ is a maximal ideal of R^G. Thus Q_2 is not maximal. □

We remark, however, that if $|G|^{-1} \varepsilon R$ and Q is a prime ideal of R^G such that for every prime Q_i equivalent to Q, R^G/Q_i is Goldie, then R/P is Goldie for any prime P with $P \cap R^G = (\bigcap_g P^g) \cap R^G = \bigcap_{Q_i \sim Q} Q_i$, and the conclusion of the Theorem holds. For in that case, one may pass to $\bar{R} = R/\bigcap_g P^g$ and $\bar{R}^G = R^G/\bigcap Q_i$ as before. Since $\bar{R}^G$ is Goldie, $\bar{R}$ is Goldie by [3], and so $\bar{R}/\bar{P} \cong R/P$ is Goldie since $\bar{P}$ is a minimal prime of $\bar{R}$.

We close with one final example to illustrate the pathology which can occur when a finite group acts on a non-commutative ring and $|G|^{-1} \notin R$.

<u>Example 3</u>: A prime ring R, and a finite group $G \subseteq \mathrm{Aut}(R)$ such that R has an infinite number of primes P, and $P \cap R^G = (0)$ for all P.

<u>proof</u>: We use an example of G. Bergman. Let k be a field of characteristic $p \neq 0$, which contains a primitive nth root of 1, say α, for some $n > 1$. Let $k\{x,y\}$ denote the free algebra in x and y, and let $R = M_2(k\{x,y\})$. Let G be the group generated by conjugation by $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$. Then $|G| = np^2$, and $R^G = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \,\middle|\, \alpha \varepsilon k \right\}$. Certainly R has infinitely many primes, as $k\{x,y\}$ does, and any prime satisfies $P \cap R^G = (0)$.

The situation in the above example cannot occur if either R is commutative or $|G|^{-1} \varepsilon R$, for in those cases G is transitive on the set of primes which have a common intersection with R^G (that is, $P_1 \cap R^G = P_2 \cap R^G$ implies $P_2 = P_1^g$, for some $g \varepsilon G$).

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