CHAINS OF PRIME IDEALS IN ENVELOPING ALGEBRAS OF SOLVABLE LIE ALGEBRAS

MARTIN LORENZ

Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over an algebraically closed field k of characteristic zero and let $U = U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$. The following problem is still open: given any two primes $P \subsetneqq Q$ of U, do all saturated chains of prime ideals $P = P_0 \subsetneqq P_1 \subsetneqq \ldots \subsetneqq P_s = Q$ have the same length s? In short, is U catenary? For abelian Lie algebras, this is of course classical, for in this case U is just an ordinary polynomial ring in finitely many commuting indeterminates over k. More generally, the answer is known to be positive if $\mathfrak{g}$ is nilpotent. In fact, homological techniques yield the result in this case (Malliavin [7], Levasseur [6]) or, alternatively, a proof can be based on the fact that U is catenary whenever the so-called Dixmier map is bicontinuous for $\mathfrak{g}$ (Lorenz and Rentschler: see [7; Proposition 3.2]). Because of the latter, the catenarity problem, in addition to being fascinating in its own right, can also be viewed as a test for the long standing conjecture that the Dixmier map is indeed always bicontinuous (for solvable Lie algebras). We remark that, in general, enveloping algebras of non-solvable Lie algebras are not catenary, as can be seen from the diagrams in [**1**; p. 39], for example.

It is an easy consequence of Corollary 6 below, originally due to Tauvel [**8**], that U is catenary if and only if the following technical condition is satisfied:

(*) for any two primes $P \subset Q$ of U one has

$$\operatorname{ht}(Q/P) = d(U/P) - d(U/Q).$$

Here, as usual, $\operatorname{ht}(Q/P)$ denotes the height of the prime ideal Q/P of U/P, and $d(.)$ denotes Gelfand–Kirillov-dimension over k. (See [**2**] for the definition and basic facts concerning Gelfand–Kirillov-dimension.) The case of (*) when $P = 0$ has been established by Tauvel [**6**]. In this note, we consider a special class of prime ideals in U that we shall call extended primes. In Proposition 5, we show that the formula in (*) holds whenever P is extended from an ideal $\mathfrak{h}$ of $\mathfrak{g}$ such that $U(\mathfrak{h})$ is catenary. This does in particular apply to the zero ideal and thus includes Tauvel's result. Proposition 5 can also be used to establish catenarity for certain almost algebraic Lie algebras (Corollary 8).

The above notation will be kept throughout. In particular, $\mathfrak{g}$ will always denote a finite-dimensional solvable Lie algebra over an algebraically closed field k of characteristic zero. The assumption that k be algebraically closed is not crucial but should help to clarify the exposition. For a detailed study of field extensions in the present context we refer to [**9**].

We recall a few general facts that will be used freely in the sequel. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Then $U(\mathfrak{h})$ is a subalgebra of $U = U(\mathfrak{g})$, and $[\mathfrak{g}, U(\mathfrak{h})] \subset U(\mathfrak{h})$. An ideal I of $U(h)$ is called $\mathfrak{g}$-*stable* if $[\mathfrak{g}, I] \subset I$. In this case, $I_1 = IU$ is an ideal of U with $I_1 \cap U(\mathfrak{h}) = I$. Moreover, if I is prime then so is I_1. Conversely, if J is an ideal of U,

Received 4 August, 1980.

[J. London Math. Soc. (2), 24 (1981), 205–210]

then $J \cap U(\mathfrak{h})$ is a $\mathfrak{g}$-stable ideal of $U(\mathfrak{h})$ which is prime if J is prime. All this holds even for non-solvable Lie algebras, and we refer the reader to [4; Chapter 3] for the details.

Definition. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. An ideal J of $U = U(\mathfrak{g})$ is said to be *extended from* $\mathfrak{h}$ if and only if $J = (J \cap U(\mathfrak{h}))U$. Also, J is called *extended* if J is extended from a suitable proper ideal of $\mathfrak{g}$.

Without proof, we remark that if J is extended from $\mathfrak{h}_1$ as well as from $\mathfrak{h}_2$, then J is extended from $\mathfrak{h}_1 \cap \mathfrak{h}_2$. In particular, there exists a unique smallest ideal of $\mathfrak{g}$ from which J is extended. These remarks will not be used in the following, nor will part (ii) of the following lemma, which is included here because it seems interesting in its own right.

LEMMA 1. *Let P be a prime ideal of U.*

(i) *If the semicentre of $A = U/P$ is strictly larger than the centre of A, then P is extended. More precisely, let a be a nonzero element of A and λ a (nonzero) linear functional on $\mathfrak{g}$ such that $[x, a] = \lambda(x)a$ for all $x \in \mathfrak{g}$. Then P is extended from the ideal* $\ker(\lambda)$ *of* $\mathfrak{g}$.

(ii) *If P is primitive, then P is either maximal or extended.*

Proof. Assertion (i) is [4; 3.3.8]. For (ii), let P be a primitive ideal of U, and assume that P is not extended. By part (i), the latter implies that the semicentre of $A = U/P$ is equal to the centre, Z, of A. The primitivity of P implies that $Z = k$. On the other hand, by Lie's theorem, any nonzero ideal of A intersects the semicentre of A nontrivially and hence contains a unit, by the foregoing. Therefore, A is simple, and (ii) follows.

The next lemma essentially rests on the following noncommutative version of Krull's principal ideal theorem, due to Jategaonkar [5]. Let R be a right Noetherian ring with a nonunit r satisfying $rR = Rr$. Then any prime of R minimal over Rr has height at most 1. For a generalization of this result see [3; Theorem 4.6].

LEMMA 2. *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ and let $P \subset Q$ be $\mathfrak{g}$-stable primes in $U(\mathfrak{h})$. If there exists a prime ideal T with $P \subsetneqq T \subsetneqq Q$, then T can be chosen to be $\mathfrak{g}$-stable.*

Proof. Since P and Q are $\mathfrak{g}$-stable, $\mathfrak{g}$ acts on Q/P via the adjoint action on $U(\mathfrak{h})$. In fact, Q/P is the union of the finite-dimensional $\mathfrak{g}$-subspaces $Q \cap U_n/P \cap U_n$, where $\{U_n \mid n \geqslant 0\}$ denotes the canonical filtration of $U(\mathfrak{h})$. Hence, by Lie's theorem, there exists $0 \neq r \in Q/P$ and $\lambda \in \mathfrak{g}^*$ such that $[x, r] = \lambda(x)r$ for all $x \in \mathfrak{g}$, or $(x+P)r = r((x+P)+\lambda(x))$. Since the elements $x+P$, with $x \in \mathfrak{g}$, generate $R = U(\mathfrak{h})/P$ as a k-algebra, we see that r satisfies $rR = Rr$. Set $I = rR$. Then I is a $\mathfrak{g}$-stable ideal of R, and $I \subset Q/P$. Therefore, Q/P contains a minimal covering prime, X, of I in R. By [4; 3.3.2], any prime of R minimal over I is $\mathfrak{g}$-stable, as I is. In particular, X is $\mathfrak{g}$-stable and, moreover, Jategaonkar's theorem implies that X has height 1 in R. Thus, if $\operatorname{ht}(Q/P) > 1$ then X is strictly smaller than Q/P, and we can take T to be the inverse image of X in $U(\mathfrak{h})$. This proves the lemma.

The following observation will be used in the proof of our next lemma. Suppose that $\mathfrak{k}$ is an ideal of $\mathfrak{g}$ of codimension 1, say $\mathfrak{g} = \mathfrak{k} \oplus kx$. Then $U = \bigoplus_{i \geq 0} U(\mathfrak{k})x^i$, and $U(\mathfrak{k})$ is stable under the derivation $\delta = [x,.]$ of U. Thus U is isomorphic to the Ore extension $U(\mathfrak{k})[x;\delta]$. Now let J be an ideal of U which is extended from $\mathfrak{k}$. Then J can be written as $J = \bigoplus_{i \geq 0} (J \cap U(\mathfrak{k}))x^i$, and U/J is isomorphic to the Ore extension $(U(\mathfrak{k})/J \cap U(\mathfrak{k}))[x;\delta']$, where δ' denotes the derivation of $U(\mathfrak{k})/J \cap U(\mathfrak{k})$ induced by δ.

Recall that $d(.)$ denotes Gelfand–Kirillov-dimension over k.

LEMMA 3. *Let $\mathfrak{h} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \ldots \subset \mathfrak{h}_t = \mathfrak{g}$ be a chain of ideals of $\mathfrak{g}$ such that $\dim_k(\mathfrak{h}_{i+1}) = 1 + \dim_k(\mathfrak{h}_i)$. Let P be a prime ideal of $U = U(\mathfrak{g})$ and set $P_i = (P \cap U(\mathfrak{h}_i))U$, $i = 0, 1, \ldots, t$. Then $P_0 \subset P_1 \subset \ldots \subset P_t = P$ is a chain of prime ideals of U of length*

$$l = t + d(U(\mathfrak{h})/P \cap U(\mathfrak{h})) - d(U/P).$$

Proof. We argue by induction on $t = \dim_k(\mathfrak{g}/\mathfrak{h})$, the case when $\mathfrak{g} = \mathfrak{h}$ being clear. So assume that $t \geq 1$ and write $\mathfrak{k} = \mathfrak{h}_{t-1}$, $U' = U(\mathfrak{k})$ and $P' = P \cap U'$, a prime ideal of U'. Furthermore, set $P'_i = (P \cap U(\mathfrak{h}_i))U'$ for $i = 0, 1, \ldots, t-1$ and let l' denote the length of the chain $P'_0 \subset P'_1 \subset \ldots \subset P'_{t-1} = P'$ in $\operatorname{Spec}(U')$. By induction, we conclude that

$$l' = t - 1 + d(U(\mathfrak{h})/P' \cap U(\mathfrak{h})) - d(U'/P'),$$

where, of course, $P' \cap U(\mathfrak{h}) = P \cap U(\mathfrak{h})$. Now $P_i = P'_i U$ for $i = 0, 1, \ldots, t-1$, and $P_i = P_{i+1}$ holds if and only if $P'_i = P'_{i+1}$. Therefore, $l = l'$ in case $P = P'U$ and $l = l'+1$ otherwise. In the former case, P is extended from $\mathfrak{k}$ and, by our above remarks, we conclude that $U/P \simeq (U'/P')[x;\delta]$, some Ore extension over U'/P'. Now [**2**; Lemma 3.1c] implies that $d(U/P) = d(U'/P') + 1$, and so our above formula for l' becomes

$$l = l' = t - 1 + d(U(\mathfrak{h})/P \cap U(\mathfrak{h})) - d(U/P) + 1.$$

Thus we have finished when P is extended from $\mathfrak{k}$. If $P \supsetneqq P'U$, then we deduce from [**2**; Lemma 3.1c, d and Satz 3.4] that

$$d(U'/P') \leq d(U/P) < d(U/P'U) = d(U'/P') + 1.$$

Since $d(U/P)$ is an integer, by [**2**; Korollar 5.4], we have $d(U'/P') = d(U/P)$ in this case, and the assertion again follows from our above formula for l'. Thus the assertion holds in either case and the lemma is proved.

COROLLARY 4. *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ and let P be a prime ideal of U. Then P is extended from $\mathfrak{h}$ if and only if*

$$d(U/P) = d(U(\mathfrak{h})/P \cap U(\mathfrak{h})) + \dim_k(\mathfrak{g}/\mathfrak{h}).$$

Proof. Since $\mathfrak{g}$ is solvable, we can choose a chain of ideals $\mathfrak{h} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \ldots \subset \mathfrak{h}_t = \mathfrak{g}$ as in Lemma 3. Clearly, P is extended from $\mathfrak{h}$ if and only if $P = P_0$, in the notation of Lemma 3, that is if and only if $l = 0$. The assertion follows from this.

We are now ready to prove our extension of Tauvel's height formula.

PROPOSITION 5. *Let* $\mathfrak{h}$ *be an ideal of* $\mathfrak{g}$ *and assume that* (*) *holds in* $U(\mathfrak{h})$. *Let* $P \subset Q$ *be prime ideals in* $U = U(\mathfrak{g})$ *such that* P *is extended from* $\mathfrak{h}$. *Then*

$$\operatorname{ht}(Q/P) = d(U/P) - d(U/Q).$$

Proof. Write $U' = U(\mathfrak{h})$, $P' = P \cap U'$, $Q' = Q \cap U'$, and set $s = d(U'/P') - d(U'/Q')$. Then $P' \subset Q'$ are $\mathfrak{g}$-stable primes in U', and we claim that there exists a chain

$$P' = P'_0 \subset P'_1 \subset \ldots \subset P'_s = Q'$$

of length s consisting of $\mathfrak{g}$-stable prime ideals of U'. Indeed, choose any maximal chain of $\mathfrak{g}$-stable primes in U' connecting P' to Q'. Then Lemma 2 implies that this chain is in fact saturated in $\operatorname{Spec}(U')$, and hence it must have length s, since (*) holds in $U(\mathfrak{h})$.

For each $i = 0, 1, \ldots, s$ set $P_i = P'_i U$. Furthermore, choose a chain of ideals $\mathfrak{h} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \ldots \subset \mathfrak{h}_t = \mathfrak{g}$ as in Lemma 3 and set $Q_i = (Q \cap U(\mathfrak{h}_i))U$. Then

$$P = P_0 \subset P_1 \subset \ldots \subset P_s = Q'U = Q_0 \subset Q_1 \subset \ldots \subset Q_t = Q$$

is a chain of prime ideals in U of length

$$l = s + t + d(U'/Q') - d(U/Q).$$

Using the definition of s, we can rewrite this as $l = t + d(U'/P') - d(U/Q)$. Since P is extended from $\mathfrak{h}$, Corollary 4 yields that $d(U/P) = d(U'/P') + t$, and so we get

$$l = d(U/P) - d(U/Q).$$

Finally, we clearly have $l \leqslant \operatorname{ht}(Q/P)$, and $\operatorname{ht}(Q/P) \leqslant d(U/P) - d(U/Q)$ holds quite generally, as a consequence of [**2**; Satz 3.4]. Therefore, we obtain the desired equality, $\operatorname{ht}(Q/P) = d(U/P) - d(U/Q)$, and the proof is complete.

We remark that the crucial assumption above is of course that P should be extended from $\mathfrak{h}$, whereas, proceeding by induction on $\dim_k(\mathfrak{g})$, one can usually assume that (*) holds in $U(\mathfrak{h})$.

Note that if $\mathfrak{h}$ is the zero ideal of $\mathfrak{g}$, then $U(\mathfrak{h}) = k$ and this surely satisfies (*). Moreover, the zero ideal of U is clearly extended from the zero ideal of $\mathfrak{g}$. Thus if we take $P = 0$ in Proposition 5 and observe that $d(U) = \dim_k(\mathfrak{g})$, by Corollary 4, for example, then we get Tauvel's height formula.

COROLLARY 6 (Tauvel [**8**]). *Let* Q *be a prime ideal of* $U = U(\mathfrak{g})$. *Then* $\operatorname{ht}(Q) = \dim_k(\mathfrak{g}) - d(U/Q)$.

Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ and let Q be a $\mathfrak{g}$-stable prime ideal of $U(\mathfrak{h})$. Define $\operatorname{ht}_{\mathfrak{g}}(Q)$ to be the maximum possible length of all chains $0 = Q_0 \subsetneqq Q_1 \subsetneqq \ldots \subsetneqq Q_t = Q$ consisting of $\mathfrak{g}$-stable primes in $U(\mathfrak{h})$. Then, clearly, $\operatorname{ht}_{\mathfrak{g}}(Q) \leqslant \operatorname{ht}(Q)$. Our next corollary shows that we have in fact equality here.

COROLLARY 7. *Let* $\mathfrak{h}$ *be an ideal of* $\mathfrak{g}$ *and let* Q *be a* $\mathfrak{g}$*-stable prime ideal of* $U(\mathfrak{h})$. *Then* $\mathrm{ht}_{\mathfrak{g}}(Q) = \mathrm{ht}(Q)$.

Proof. The result is clear when $\mathfrak{h} = 0$. So assume that $\mathfrak{h}$ is nonzero and choose an ideal $\mathfrak{h}'$ of $\mathfrak{g}$ of codimension 1 in $\mathfrak{h}$. Set $U' = U(\mathfrak{h}')$ and $Q' = Q \cap U'$, a $\mathfrak{g}$-stable prime ideal of U'. By induction on $\dim_k(\mathfrak{h})$, we may assume that $\mathrm{ht}_{\mathfrak{g}}(Q') = \mathrm{ht}(Q')$. Since for any $\mathfrak{g}$-stable prime X of U' the extended ideal $XU(\mathfrak{h})$ is a $\mathfrak{g}$-stable prime of $U(\mathfrak{h})$, we conclude immediately that

$$\mathrm{ht}(Q') = \mathrm{ht}_{\mathfrak{g}}(Q') \leqslant \mathrm{ht}_{\mathfrak{g}}(Q'U(\mathfrak{h})) \leqslant \mathrm{ht}_{\mathfrak{g}}(Q).$$

If Q is extended from $\mathfrak{h}'$, then we deduce from Corollaries 4 and 6 that $\mathrm{ht}(Q) = \mathrm{ht}(Q')$, whence $\mathrm{ht}_{\mathfrak{g}}(Q) \geqslant \mathrm{ht}(Q)$, as claimed. If Q is not extended from $\mathfrak{h}'$, then Lemma 3 and Corollary 6 similarly yield that $\mathrm{ht}(Q) = \mathrm{ht}(Q')+1$. Again, the desired inequality $\mathrm{ht}_{\mathfrak{g}}(Q) \geqslant \mathrm{ht}(Q)$ follows, since now $\mathrm{ht}_{\mathfrak{g}}(Q) \geqslant \mathrm{ht}_{\mathfrak{g}}(Q'U(\mathfrak{h}))+1$. This proves the corollary.

A solvable Lie algebra $\mathfrak{g}$ is said to be *almost algebraic* if $\mathfrak{g}$ has the form $\mathfrak{g} = \mathfrak{n} + \mathfrak{s}$, where $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{g}$ and $\mathfrak{s}$ is an abelian subalgebra of $\mathfrak{g}$ acting semisimply on $\mathfrak{n}$. By enlarging $\mathfrak{n}$ and shrinking $\mathfrak{s}$, if necessary, one can always assume that $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{g}$ and that the sum $\mathfrak{n} + \mathfrak{s}$ is direct. As our last application of Proposition 5 we show that at least in the special case where $\mathfrak{n}$ is abelian $U(\mathfrak{g})$ is in fact catenary.

COROLLARY 8. *Assume that* $\mathfrak{g} = \mathfrak{a} + \mathfrak{s}$, *where* $\mathfrak{a}$ *is an abelian ideal of* $\mathfrak{g}$ *and* $\mathfrak{s}$ *is a commutative subalgebra of* $\mathfrak{g}$ *acting semisimply on* $\mathfrak{a}$. *Then* (*) *holds in* $U(\mathfrak{g})$.

Proof. Let $P \subset Q$ be primes in $U = U(\mathfrak{g})$. Note that homomorphic images of $\mathfrak{g}$ clearly inherit the structure of $\mathfrak{g}$ as described in the hypotheses. Thus, upon factoring out the ideal $\mathfrak{g} \cap P$ from $\mathfrak{g}$ if necessary and arguing by induction on $\dim_k(\mathfrak{g})$, we may assume that P is *faithful*, that is $\mathfrak{g} \cap P = 0$. By the assumption on $\mathfrak{g}$, we can write $\mathfrak{a} = \sum_i \mathfrak{a}^{\lambda_i}$, where λ_i are suitable linear functionals on $\mathfrak{g}$ vanishing on $\mathfrak{a}$ such that $\mathfrak{a}^{\lambda_i} = \{a \in \mathfrak{a} \mid [x, a] = \lambda_i(x)a \text{ for all } x \in \mathfrak{g}\}$ is nonzero. Note that the image of $\mathfrak{a}^{\lambda_i}$ in U/P belongs to the semicentre of U/P and is nonzero, as P is faithful. Therefore, if $\lambda_i \neq 0$ for some i, then we conclude from Lemma 1(i) that P is extended from $\mathfrak{h} = \ker(\lambda_i)$, and $\mathfrak{h} = \mathfrak{a} + (\mathfrak{s} \cap \mathfrak{h})$ has the same structure as $\mathfrak{g}$ but smaller dimension. Thus, by induction, we may assume that (*) holds in $U(\mathfrak{h})$. It now suffices to quote Proposition 5 to finish the proof in this case.

Finally, if $\lambda_i = 0$ for all i, then $\mathfrak{g}$ is abelian and the result is classical.

References

1. W. Borho and J. C. Jantzen, "Über primitive Ideale in der Einhüllenden einer halbeinfachen Lie-Algebra", *Invent. Math.*, 39 (1977), 1–53.
2. W. Borho and H. Kraft, "Über die Gelfand–Kirillov-Dimension", *Math. Ann.*, 220 (1976), 1–24.
3. A. W. Chatters, A. W. Goldie, C. R. Hajarnavis and T. H. Lenagan, "Reduced rank in Noetherian rings", *J. Algebra*, 61 (1979), 582–589.
4. J. Dixmier, *Algèbres enveloppantes* (Gauthier–Villars, Paris, 1974).
5. A. V. Jategaonkar, "Relative Krull dimension and prime ideals in right Noetherian rings", *Comm. Algebra*, 4 (1974), 429–468.

6. T. Levasseur, "Sur le spectre premier d'une algèbre enveloppante d'une algèbre de Lie nilpotente", *C. R. Acad. Sci. Paris Sér. A*, 289 (1979), 55–58.
7. M. P. Malliavin, "Caténarité et théorème d'intersection en algèbre non commutative", *Séminaire d'algèbre Paul Dubreil, Proceedings, Paris* 1977–78, Lecture Notes in Mathematics 740 (Springer, Berlin, 1979), pp. 408–431.
8. P. Tauvel, "Sur les quotients premiers de l'algèbre enveloppante d'une algèbre de Lie résoluble", *Bull. Soc. Math. France*, 106 (1978), 177–205.
9. S. Yammine, "Les théoremes de Cohen-Seidenberg en algèbre non commutative", *Séminaire d'algèbre Paul Dubreil, Proceedings, Paris* 1977–78, Lecture Notes in Mathematics 740 (Springer, Berlin, 1979), pp. 120–169.

Universität Essen,
Fachbereich 6, Mathematik,
Postfach 103764,
4300 Essen 1,
West Germany.