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COMPLETELY PRIME IDEALS IN ORE EXTENSIONS

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An ideal P of the ring R is said to be <u>completely prime</u> if R/P is a domain. Clearly, such an ideal is prime, whereas in general prime ideals need not be completely prime. However, if R is commutative, then all its prime ideals are in fact completely prime. More interestingly, Dixmier [2] has shown that the same occurs for enveloping algebras of finite-dimensional solvable Lie algebras over fields of characteristic zero. In this short note we point out that, under certain conditions, the property that all prime ideals of R are completely prime is inherited by Ore extensions of the form $R[X;\delta]$, where δ is a derivation of R . Our proof is extracted from the proof of Dixmier's theorem as given in Borho-Gabriel-Rentschler [1, p.50].

Throughout this note, R will denote a right Noetherian algebra over a commutative field k of characteristic zero, and δ will be a k-derivation of R .

The Ore extension $S = R[X;\delta]$ is obtained from R by adjoining an indeterminate X subject to the equation

$$Xr - rX = \delta(r) \quad (r \in R).$$

We list a few well-known facts that will be needed later on. Proofs can be found in [1, §4], for example.

a) For any prime ideal P of S, the intersection $P \cap R$ is prime in R.

b) Suppose R is a domain. Then the set C of nonzero elements of R satisfies the (right) Ore condition in R, and $D = RC^{-1}$ is a division algebra over k. Moreover, δ has a unique extension from R to D given by $\delta(rc^{-1}) = -rc^{-1}\delta(c)c^{-1} + \delta(r)c^{-1}$ $(r \in R, c \in C)$. Finally, C also is a right Ore set in $S = R[X;\delta]$, and SC^{-1} has the form

$$SC^{-1} \simeq D[X;\delta],$$

where δ on the right denotes the extended derivation.

c) Suppose R is a division algebra. If δ is inner, that is there exists an element $d \in R$ such that $\delta(r) = dr - rd$ for all $r \in R$, then replacing X by $Y = X - d$ we see that $S = R[Y]$ is an ordinary polynomial ring in the commuting indeterminate Y over R. In this case, every ideal of S is generated by a polynomial in $Z(S) = Z(R)[Y]$. If δ is not inner, then $S = R[X;\delta]$ is a simple ring.

We are now ready to prove the main result of this note.

<u>Theorem.</u> Let R be an algebra over a field k of characteristic zero and suppose that for all extension fields K of k, $R_K = R \otimes_k K$ is right Noetherian and all prime ideals of R_K are completely prime.

Then the same holds for $S = R[X;\delta]$, where δ is any k-derivation of R.

Proof. If K is an extension field of k, then δ can be uniquely extended to a K-derivation δ_K of R_K, and $S \otimes_k K \simeq R_K[X;\delta_K]$. The latter ring is surely Noetherian, as R_K is, and replacing R_K by R it suffices to show that all primes of S are completely prime.

So let P be a prime ideal of S. By (a) above, the intersection $Q = P \cap R$ is prime in R, and hence is completely prime. Moreover, $\delta(Q) \subset Q$ and so QS is an ideal of S satisfying $S/QS \simeq (R/Q)[X;\delta']$, where δ' denotes the derivation of R/Q induced by δ and where the isomorphism is given by $[\Sigma r_i X^i + QS] \to \Sigma[r_i + Q]X^i$. Let $^-: S \to S/QS$ denote the natural map. Then $\bar{S} = \bar{R}[X;\delta']$, $\bar{R}$ is a right Noetherian domain, and $\bar{P}$ is a prime ideal of $\bar{S}$ with $\bar{P} \cap \bar{R} = 0$. Let $\mathcal{C}$ denote the set of nonzero elements of $\bar{R}$. Then, by (b) above, we know that $\bar{S}\mathcal{C}^{-1}$ exists and has the form $\bar{S}\mathcal{C}^{-1} \simeq D[X;\delta']$, where $D = \bar{R}\mathcal{C}^{-1}$ is a division algebra over k and δ' is the extended k-derivation of D. Since $\bar{P}$ is disjoint from $\mathcal{C}$, it follows that $P_1 = \bar{P}\mathcal{C}^{-1}$ is a prime ideal of $\bar{S}\mathcal{C}^{-1}$ such that $P_1 \cap \bar{S} = \bar{P}$. Thus $S/P \simeq \bar{S}/\bar{P} \subset \bar{S}\mathcal{C}^{-1}/P_1$, and it suffices to show that P_1 is completely prime.

Note that for any extension field K of k, $D \otimes_k K$ is right Noetherian and all its primes are completely prime. To see this, just write $R_1 = \bar{R} \otimes_k K \subset D_K = D \otimes_k K$ and identify $\mathcal{C}$ with $\mathcal{C} \otimes 1 \subset D_K$. Then $D_K = R_1\mathcal{C}^{-1}$, and this is right Noetherian, as R_1 is. Moreover, if B is any prime of D_K, then we have for the classical rings of quotients $\mathcal{Q}(D_K/B) \simeq \mathcal{Q}(R_1/B \cap R_1)$ (see [1, Satz 2.10]). Finally, the latter ring is a division algebra, since $R_1/B \cap R_1$ is a prime homomorphic image of $R_K = R \otimes_k K$. Hence, starting again with fresh notation, we may assume that P is a prime ideal of $S = R[X;\delta]$, where R satisfies our original assumptions and, in addition, is a division algebra over k.

Since S is a domain, we may clearly assume that P is nonzero. By (c) above, this forces δ to be inner so that $S = R[Y]$ for a suitable commuting indeterminate Y, and P is generated by an irreducible polynomial $f \in Z(R)[Y]$. Write $F = Z(R)$, a commutative extension field of k, and set $K = F[Y]/(f)$, a finite algebraic extension of F. Then

$$S/P \simeq (R \otimes_F F[Y])/(R \otimes_F (f)) \simeq R \otimes_F K ,$$

and we have to show that $R \otimes_F K$ has no zero divisors. But $R \otimes_F K$ is a homomorphic image of R_K and, being isomorphic to S/P, $R \otimes_F K$ is prime. Thus, by assumption on R, we conclude that $R \otimes_F K$ is indeed a domain, and the theorem is proved.

As an immediate corollary, we note that if R satisfies the hypotheses of the theorem and if T is any ring constructed from R by a finite succession of Ore extensions with k-derivations, then all prime ideals of T are completely prime. Since enveloping algebras of finite-dimensional solvable Lie algebras over fields of characteristic zero are particular examples of this kind, with $R = k$ the ground field, we do in particular recover Dixmier's original result. Of course, there are many more examples other than enveloping algebras of solvable Lie algebras that can be constructed this way.

We close with a few simple examples showing that the main hypotheses of the theorem are indeed necessary.

Example 1. Let S denote the Weyl algebra over a field F of positive characteristic p, i.e. $S = F[X][Y; ']$, where $'$ denotes the usual differentiation of polynomials in $F[X]$. The center C of S equals $F[X^p, Y^p]$, an ordinary polynomial ring in X^p and Y^p, and maximal ideals in C extend to maximal ideals in S (see [3]). In particular, $P = (X^p - 1)S + (Y^p - 1)S$ is maximal in S. Since $(X - 1)^p = X^p - 1 \in P$ but $X - 1 \notin P$, we see that P is not completely prime. Hence the result fails to hold for fields of positive characteristic.

Example 2. Let $\mathbb{H}$ denote the real quaternions. Then $P = (X^2 + 1)\mathbb{H}[X]$ is an ideal of $\mathbb{H}[X]$ such that $\mathbb{H}[X]/P \simeq M_2(\mathbb{C})$. Thus P is prime but not completely prime. Hence the assumption that all prime ideals in R are completely prime is not strong enough to ensure the same holds in S.

Example 3. Let $R = k[X, X^{-1}]$ be the Laurent polynomial ring over k and let α be the k-algebra automorphism of R given by $\alpha(X) = X^{-1}$. Let $T = R[Y;\alpha]$ be the Ore extension obtained from R by adjoining Y with $rY = Y\alpha(r)$ for all $r \in R$. Then $P = (Y^2 - 1)T$ is a prime ideal of T which is not completely prime, since $(Y + 1)(Y - 1) \in P$ but $Y + 1$, $Y - 1 \notin P$. This shows that the corresponding result for Ore extensions with automorphisms instead of derivations is false.

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