Prime Ideals in Group Algebras of Polycyclic-by-Finite Groups :

Vertices and Sources

by

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These notes represent a somewhat expanded version of a talk that I gave in this seminar in November 79. The results presented here are joint work with D.S. Passman.

In Section 1 we describe the machinery that has been developed to study prime ideals in K[G] whith G polycyclic-by-finite. We briefly discuss Roseblade's fundamental work on group algebras of orbitally sound groups and its extension to general polycyclic-by-finite groups by Passman and the author. Although crossed products have played an important role here and some results do in fact hold in this more general setting, we will concentrate on group algebras here. Sections 2 and 3 contain previously unpublished material. The main purpose of these sections is to illustrate the notions of vertex and source for prime ideals in K[G] that were introduced in [4] . Our general point of view in Section 2 is to consider the vertex of a prime P as being given and derive information about P . In particular, we will describe the set $\mathrm{Spec}_H(K[G])$ of all prime ideals in K [G] having a fixed vertex H . Section 3 is devoted to the catenarity problem.

Throughout these notes, G will always be a polycyclic-by-finite group and K will be a commutative field.

§ 1 - Preliminary results

1. A - Induced Ideals ([6], [4]). Let H be a subgroup of G and let L be an ideal of K[H]. Then we let L^G denote the unique largest ideal of K[G] contained in L K[G], that is :

$$L^G = \mathrm{ann}_{K[G]}\ (K[G] / L\,K[G]) = \cap_{g\in G} L^g\,K[G].$$

Any ideal of K[G] of the form $I = L^G$ will be called an induced ideal or, more precisely, induced from H. Let $\pi_H : K[G] \longrightarrow K[H]$ be the projection map sending $\Sigma_{g\in G}\, k_g\, g$ to $\Sigma_{g\in H}\, k_g\, g$. Then L^G can also be characterized as the unique largest ideal of K[G] satisfying $\pi_H(I) \subseteq L$. In particular, the above definition of L^G is left-right symmetric. If H is normal in G, then the above expres- expression for L^G becomes :

$$(1.1) \qquad L^G = (\cap_{g\in G} L^g)\ K[G] = (L^G \cap K[H])\ K[G] .$$

Thus, for an ideal of K[G], being induced from a normal subgroup H of G is the same as being controlled by H, in the usual sense. The basic result that we will need is a follows :

(1.2) Theorem ([4, Theorem 1.7]). Let N be a normal subgroup of G of finite index, let Q be a prime ideal of K[N] and let A be any subgroup of G containing the stabilizer of Q in G, that is $A \supseteq \{g \in G \mid Q^g = g^{-1} Q\, g = Q\}$. Then the induction map $(.)^G$ yields a 1-1 correspondence between the prime ideals T of K[A] with $T \cap K[N] = \cap_{a\in A} Q^a$ and the primes P of K[G] with $P \cap K[N] = \cap_{g\in G} Q^g$. Moreover, if $P = T^G$ as above, then T is the unique minimal covering prime of $P \cap K[A]$ with $\cap_{g\notin A} Q^g \not\subseteq T \cap K[N]$.

We remark that if N is normal in G of finite index and P is a prime ideal of K[G], then $P \cap K[N]$ always has the form $P \cap K[N] = \cap_{g\in G} Q^g$ for some prime ideal Q of K[N] which is unique up to G-conjugacy.

(1.3) Corollary Let N be a normal subgroup of G of finite index and let P be a prime ideal of $K[G]$. Write $P \cap K[N] = \bigcap_{g \in G} Q^g$ for some prime Q of $K[N]$. If A is a normal subgroup of G containing $\mathrm{Stab}_G(Q)$, then $P = (P \cap K[A])\, K[G]$.

Proof By Theorem 1.2, $P = T^G$ for some prime T of $K[A]$ and, by (1.1), $T^G = (T^G \cap K[A])\, K[G]$, since A is normal.

1. B - Orbitally Sound Groups (Roseblade [8]). A subgroup H of G is called orbital if $[G : \mathbb{N}_G(H)] < \infty$. An orbital subgroup H is said to be isolated orbital if and only if H is the only orbital subgroup M with $M \supseteq H$ and $[M : H] < \infty$. In general one defines, for H orbital in G,

$$i_G(H) = \langle M \mid H \subseteq M \subseteq G,\ M \text{ orbital},\ [M : H] < \infty \rangle .$$

One can show (see [8, p. 400/401]) that $[i_G(H) : H] < \infty$ and that $i_G(H)$ is isolated orbital in G. Therefore, $i_G(H)$ is called the isolator of H in G. The definition of $i_G(H)$ makes it clear that we have :

(1.4) $$\mathbb{N}_G(H) \subseteq \mathbb{N}_G(i_G(H)) .$$

The group G is said to be orbitally sound if and only if all isolated orbital subgroups of G are normal. The following important result is due to Roseblade.

(1.5) Theorem ([8, Theorem C2]). Set $\mathrm{nio}(G) = \bigcap_H \mathbb{N}_G(H)$, where the intersection runs over all isolated orbital subgroups of G. Then $\mathrm{nio}(G)$ is an orbitally sound characteristic subgroup of G of finite index. Moreover, $\mathrm{nio}(G)$ contains every orbitally sound normal subgroup of G of finite index and every finite-by-nilpotent normal subgroup of G.

Using the linearity of polycyclic-by-finite groups, Wehrfritz has given an alternate proof for the existence of an orbitally sound normal subgroup of G of finite index. (See [2, § 2]) .

1. C Standard Prime Ideals ([3]). We let $\Delta = \Delta(G)$ denote the f.c.center of G, that is :

$$\Delta = \{ g \in G \mid [G : \mathbb{C}_G(g)] < \infty \} .$$

For any ideal I of $K[G]$ we set $I^{\dagger} = \{ g \in G \mid 1 - g \in I \}$. Thus $I^{\dagger}$ is the kernel of the natural map $G \to K[G]/I$ and hence is normal in G . Following Roseblade [8] , I will be called faithful if $I^{\dagger} = \langle 1 \rangle$ and almost faithful if $I^{\dagger}$ is finite. Note that any ideal I of $K[G]$ is the complete inverse image of a faithful ideal in $K[G/I^{\dagger}]$.

(1.6) Definition ([3]) . A prime ideal P of $K[G]$ is said to be standard if and only if $P = L^G$ for some almost faithful prime ideal L of $K[\Delta]$.

In [3, Proposition 1.4] it is shown that for any almost faithful prime L of $K[\Delta]$ the induced ideal L^G is always prime in $K[G]$. Any standard prime is in particular almost faithful. We call P virtually standard if the image of P in $K[G/P^{\dagger}]$ is standard. Although the defining conditions seem to be very restrictive, virtually standard primes do in fact occur quite often. Indeed, Roseblade's theorem [8, Theorem C 1] can be stated as follows :

If G is orbitally sound then all primes of $K[G]$ are virtually standard.

A converse to this will be proved in Section 2 .

1.D Vertices and Sources of Prime Ideals ([4]) For any subgroup H of G we let :

$$\nabla_G(H)$$

denote the complete inverse image in $\mathbb{N}_G(H)$ of $\Delta(\mathbb{N}_G(H)/H)$. Thus, clearly, $\nabla_G(H) \lhd \mathbb{N}_G(H)$ and $\nabla_G(H)/H$ is finite-by- abelian. Furthermore, if H is orbital then so is $\nabla_G(H)$ and if H is isolated orbital then $\nabla_G(H)/H \cong \mathbb{Z}^n$ for some n. Finally, it is not hard to see that for isolated orbital subgroups H_1 and H_2 we have :

(1.7) $$H_1 \subseteq H_2 \text{ implies } \nabla_G(H_1) \subseteq \nabla_G(H_2)$$

([4, Lemma 3.1]). Let I be an ideal of $K[G]$ and let $N \trianglelefteq G$. Then we say that I is <u>almost faithful sub</u> N if and only if $I^\dagger \subseteq N$ and $[N : I^\dagger] < \infty$.

(1.8) <u>Theorem</u> ([4, Theorem I, II, III]). <u>Let</u> K <u>be a field and</u> let G <u>be a polycyclic-by-finite group</u>.

(i) (Existence) <u>If</u> P <u>is a prime ideal of</u> $K[G]$, <u>then there exists an isolated orbital subgroup</u> H <u>of</u> G <u>and an almost faithful sub</u> H <u>prime ideal</u> L <u>of</u> $K[\nabla_G(H)]$ <u>with</u> $P = L^G$.

(ii) (Uniqueness) <u>In the situation of part</u> (i), H <u>is unique up to conjugation in</u> G <u>and, for a given</u> H, L <u>is unique up to conjugation by</u> $\mathbb{N}_G(H)$.

(iii) (Converse) <u>If</u> H <u>is an isolated orbital subgroup of</u> G <u>and</u> L <u>is an almost faithful sub</u> H <u>prime ideal of</u> $K[\nabla_G(H)]$, <u>then</u> L^G <u>is a prime ideal of</u> $K[G]$.

(1.9) <u>Definition</u> ([4]). Let P be a prime ideal of $K[G]$ and let H and L be as in Theorem 1.8(i),(ii). Then we call H a <u>vertex</u> of P and write :

$$H =_G \mathrm{vx}(P) .$$

Furthermore, we call L a <u>source</u> of P (corresponding to the vertex H).

If P is a given prime of $K[G]$, then $\mathrm{vx}(P)$ can be obtained as follows. Write $P \cap K[\mathrm{nio}(G)] = \bigcap_{g \in G} Q^g$ for some prime ideal Q of $K[\mathrm{nio}(G)]$. (Q is unique up to G-conjugacy). Then we have :

(1.10) $$\mathrm{vx}(P) =_G i_G(Q^\dagger)$$

([4, Theorem 2.4(i)]). If $H = \mathrm{vx}(P)$ is given, then the possible sources of P, for this H , are obtained as follows. Set $A = \mathbb{N}_G(H)$. Then, by *(1.4)* and *(1.10)* $A \supseteq \mathbb{N}_G(Q^\dagger) \supseteq \mathrm{Stab}_G(Q)$ and so Theorem 1.2 implies that there exists a unique prime ideal T of $K[A]$ with $T \cap K[\mathrm{nio}(G)] = \cap_{a\in A} Q^a$ and $T^G = P$. Then we have :

(1.11) The sources of P (for the given H) are precisely the minimal covering primes of $T \cap K[\nabla_G(H)]$.

(See [4 , Theorem 2.4 (ii), proof].) The relations between P and its vertex H and source L are of course quite interesting. For example, if $Q(.)$ denotes the classical ring of quotients, then the centers $\mathfrak{Z}(Q(K[G]/P))$ and $\mathfrak{Z}(Q(K[\nabla_G(H)]/L))$ have the same transcendence degree over K , in short

(1.12) $\qquad \mathrm{c.r.}(P) = \mathrm{c.r.}(L)$

(c.r. = central rank ; see [3].). It follows that, for K nonabsolute, P is primitive if and only if L has finite codimension in $K[\nabla_G(H)]$. Finally, one can give an expression for the height $\mathrm{ht}(P)$ of P involving a certain group theoretic invariant, depending upon $H = \mathrm{vx}(P)$, and the central rank of L . For details we refer to [4].

§ 2. - Vertices of Primes in $K[G]$

The vertex $\mathrm{vx}(P)$ of any prime P in $K[G]$ is, by definition, an isolated orbital subgroup of G . The following lemma shows that all isolated orbital subgroup of G do in fact occur this way.

(2.1) Lemma Let $H \leqslant G$ be an isolated orbital subgroup of G . Then there exists a prime ideal P in $K[G]$ with $\mathrm{vx}(P) =_G H$.

Proof If $H \leqslant G$ is isolated orbital, then $\nabla_G(H)/H \cong \mathbf{Z}^n$ for some n and so the augmentation ideal $L = (\omega H)K[\nabla_G(H)]$ satisfies $K[\nabla_G(H)]/L \cong K[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}]$. Thus L is prime and is clearly almost faithful sub H . By Theorem 1.8(iii) we conclude that $P = L^G$ is a prime ideal of $K[G]$ with $vx(P) =_G H$.

By definition of nio(G), we have for any prime ideal P of $K[G]$:

$$\mathbb{N}_G(vx(P)) \supseteq nio(G) . \qquad (2.2)$$

The extreme case of a normal vertex is certainly of interest.

(2.3) Lemma Let P be a prime ideal of $K[G]$. Then $vx(P)$ is normal in G if and only if P is virtually standard.

Proof First assume that $H = vx(P)$ is normal in G and set $\nabla = \nabla_G(H)$. Then $\nabla \trianglelefteq G$ and, by Theorem 1.8, $P = L^G$ for some prime ideal L of $K[\nabla]$ with $[H : L^\dagger] < \infty$. Since $\nabla \trianglelefteq G$, we have $P = (P \cap K[\nabla])K[G]$ and $P \cap K[\nabla] = \cap_{g\in G} L^g$, by (1.1). In particular, it follows that $P^\dagger = P^\dagger \cap \nabla = \cap_{g\in G} (L^\dagger)^g \subseteq H$. Note that $[H : \cap_{g\in G}(L^\dagger)^g] < \infty$, since each $(L^\dagger)^g$ is a subgroup of H of index $[H : L^\dagger]$ and there are only finitely many such subgroups. Thus $[H : P^\dagger] < \infty$ and, in particular, $[L^\dagger : P^\dagger] < \infty$. Moreover, the definition of ∇ easily implies that $\Delta(G/P^\dagger) = \nabla/P^\dagger$. Thus, if $\bar{\ } : K[G] \to K[G/P^\dagger]$ denotes the natural map then $\bar{\nabla} = \Delta(\bar{G})$, $\bar{L}$ is almost faithful in $K[\bar{\nabla}]$ and $\bar{P} = \bar{L}^{\bar{G}}$. This proves that P is virtually standard.

Conversely, assume that P is virtually standard and let D denote the complete inverse image of $\Delta(G/P^\dagger)$ in G . Then $P = L^G$ for some prime L of $K[D]$ with $[L^\dagger : P^\dagger] < \infty$. Let $H/P^\dagger$ be the torsion subgroup of $D/P^\dagger$. Then $[H : P^\dagger] < \infty$, and H is easily seen to be isolated orbital and normal in G . In particular, L is almost faithful sub H , and since $P = L^G$ and $D = \nabla_G(H)$ we deduce from the Uniqueness Theorem (Theorem 1.8(ii)) that $H = vx(P)$. Thus $vx(P)$ is normal in G , and the lemma is proved.

Recall that, by definition, G is orbitally sound if and only if all isolated orbital subgroups of G are normal. Thus Lemmas 2.1 and 2.3 immediately give the following result :

(2.4) Corollary G is orbitally sound if and only if all primes in $K[G]$ are virtually standard.

Note that this contains Roseblade's Theorem C 1 in [8] as the "only if" -direction.

In the other extrema case, namely $\mathbb{N}_G(vx(P)) = nio(G)$, we have $P = (P \cap K[nio(G)])K[G]$. This follows from the following slightly more general observation, together with (1.1).

(2.5) Lemma Let P be a prime ideal of $K[G]$ and let A be a subgroup of G with $A \supseteq \nabla_G(vx(P))$. Then $P = I^G$ for some prime ideal I of $K[A]$.

Proof By Theroem 1.8(i), P is induced from $\nabla_G(vx(P))$ and, since induction is transitive, P is also induced from A. Thus there exists an ideal I of $K[A]$ with $I^G = P$. Choosing I maximal with this property we can get I to be prime. Indeed, if J_1 and J_2 are ideals of $K[A]$ containing I such that $J_1 . J_2 \subseteq I$, then $J_1^G . J_2^G \subseteq (J_1 . J_2)^G \subseteq I^G = P$, and hence $J_1^G \subseteq P$ or $J_2^G \subseteq P$. The maximality of I now yields $J_1 = I$ or $J_2 = I$, as required.

Now consider prime ideals P_1, and P_2, of $K[G]$ with $P_1 \subseteq P_2$. Then it follows from (1.10) that we have :

$$vx(P_1) \subseteq vx(P_2) \tag{2.6}$$

(up to G-conjugation, of course). For, if we write $P_i \cap K[nio(G)] = \bigcap_{g\in G} Q_i^g$ for suitable primes Q_i of $K[nio(G)]$, we see that $Q_2 \supseteq \bigcap_{g\in G} Q_1^g$ and hence $Q_2 \supseteq Q_1^g$ for some $g \in G$. Replacing Q_1 by a G-conjugate if necessary, we may assume that $Q_2 \supseteq Q_1$. Thus $Q_2^\dagger \supseteq Q_1^\dagger$ and so $i_G(Q_2^\dagger) \supseteq i_G(Q_1^\dagger)$, since $i_G(.)$ is monotonic ([8, § 3.1]). (2.6) now follows from (1.10). Note that (2.6) and (1.7) imply that :

$$\nabla_G(vx(P_1)) \subseteq \nabla_G(vx(P_2)) . \tag{2.7}$$

We now consider the case $vx(P_1) = vx(P_2)$. For any isolated orbital subgroup H of G we set :

$$\mathrm{Spec}_H\,(K[G]) = \{ P \in \mathrm{Spec}(K[G]) \mid vx(P) =_G H \},$$

a nonempty subset of $\mathrm{Spec}(K[G])$, by Lemma 2.1. We have

$$\mathrm{Spec}(K[G]) = \dot{\cup}_H \ \mathrm{Spec}_H(K[G]),$$

a disjoint union with H ranging over a complete set of non-conjugate isolated orbital subgroups of G . Our goal is to describe $\mathrm{Spec}_H(K[G])$ for H a fixed isolated orbital subgroup of G . Set $A = \mathbb{N}_G(H)$ and $\nabla = \nabla_G(H)$ so that $H \trianglelefteq \nabla \trianglelefteq A$ and $\nabla/H = \Delta(\mathbf{A}/H)$. Note also that H is normal and isolated orbital in ∇ , the latter since ∇/H is torsion-free abelian. Thus $\mathrm{Spec}_H(K[\nabla])$ is defined and is in fact easily seen to be identical with the set of all primes of $K[\nabla]$ which are almost faithful sub H . Now A acts on $\mathrm{Spec}_H(K[\nabla])$ by conjugation, and we let

$$\mathcal{S}_H = \mathrm{Spec}_H\,(K[\nabla]) / A$$

denote the set orbits under this action. The A-orbit of $L \in \mathrm{Spec}_H(K[\nabla])$ will be written as $[L]$. We remark that <u>each such orbit is finite</u>. To see this, choose a normal subgroup N of A with $N \subseteq L^\dagger$ and $[H : N] < \infty$. Then we have $\Delta(A/N) = \nabla/N$ and thus there exists a subgroup X of A of finite index which centralizes ∇/N . Clearly, $X \subseteq \mathrm{Stab}_A(L)$ and so the latter has finite index in A. For $L_1, L_2 \in \mathrm{Spec}_H(K[\nabla])$ define :

$$[L_1] \leqslant [L_2] \quad \text{if and only if} \quad L_1 \subseteq L_2{}^a \quad \text{for some } a \in A .$$

If $[L_1] \leqslant [L_2] \leqslant [L_1]$, then $L_1 \subseteq L_2{}^a \subseteq L_1{}^b$ for suitable $a,b \in A$, and the fact that $[L_1]$ is finite implies that we have equality throughout so that $[L_1] = [L_2]$. Thus $\leqslant$ defines a partial order on $\mathcal{S}_H$. Note that, surely, $L^G = (L^a)^G$ for any $L \in \mathrm{Spec}_H(K[\nabla])$ and $a \in A$. Hence the induction map $(.)^G$ can be defined on $\mathcal{S}_H$, and Theorem 1.8 says that $(.)^G$ is a one-to-one map of $\mathcal{S}_H$ onto $\mathrm{Spec}_H(K[G])$. The preimage of $P \in \mathrm{Spec}_H(K[G])$ is the set of all sources of P corresponding to H .

(2.8) Proposition. The map :

$$(.)^G : \mathcal{S}_H \longrightarrow \mathrm{Spec}_H([G])$$

induces a 1-1 correspondence between these two sets such that for any $[L_1]$, $[L_2] \in \mathcal{S}_H$ we have :

$$[L_1]^G \subseteq [L_2]^G \text{ if and only if } [L_1] \leqslant [L_2] .$$

Moreover, in this case :

$$\mathrm{ht}\,([L_2]^G / [L_1]^G) = \mathrm{c.r.}\,(L_1) - \mathrm{c.r.}\,(L_2) .$$

Here, of course, $\mathrm{ht}([L_2]^G / [L_1]^G)$ denotes the height of the prime ideal $[L_2]^G / [L_1]^G$ of $K[G]/[L_1]^G$. Note also that $\mathrm{c.r.}(L_i)$ is surely on invariant of $[L_i]$, since the factor rings corresponding to the elements of $[L_i]$ are pairwise isomorphic. In addition, we know by (1.12) that :

$$\mathrm{c.r.}\,(L_i) = \mathrm{c.r.}\,([L_i]^G) .$$

Proof of (2.8) The fact that $(.)^G$ is one-to-one and onto has been noted above, and $(.)^G$ is clearly order preserving, i.e. $[L_1] \leqslant [L_2]$ implies $[L_1]^G \subseteq [L_2]^G$. It remains to show that, conversely, $[L_1]^G \subseteq [L_2]^G$ implies $[L_1] \leqslant [L_2]$ and to verify the height formula.

Write $P_i = [L_i]^G$ $(i = 1,2)$ and assume that $P_1 \subseteq P_2$. We reconstruct $[L_1]$ and $[L_2]$ by using (1.10) , (1.11) . Thus write $P_i \cap K[\mathrm{nio}(G)] = \cap_{g\in G} Q_i^{\,g}$ for suitables primes Q_i of $K[\mathrm{nio}(G)]$ and, as we have remarked earlier, we may assume that $Q_1 \subseteq Q_2$. By (1.10), we have $H = i_G(Q_1^{\dagger}) = i_G(Q_2^{\dagger})$. As above, let $A = \mathbb{N}_G(H)$. Then (1.4) yields $A \supseteq \mathbb{N}_G(Q_i^{\dagger}) \supseteq \mathrm{Stab}_G(Q_i) \supseteq \mathrm{nio}(G)$ for $i = 1,2$, and hence it follows from Theorem 1.2 that $P_i = T_i^{\,G}$ for certain uniquely determined prime ideals T_i of $K[A]$ with $T_i \cap K[\mathrm{nio}(G)] = \cap_{a\in A} Q_i^{\,a} (i = 1,2)$. In fact, we know that T_i is the unique minimal covering prime of $P_i \cap K[A]$ which does not contain $\cap_{g\in G\setminus A} Q_i^{\,G}$. We claim that $T_2 \supseteq T_1$. If not, then T_2 contains

contains $\bigcap_{g\in G\setminus A} Q_1^g$ and we obtain that $Q_2 \supseteq \bigcap_{a\in A} Q_2^a =$ $T_2 \cap K[nio(G)] \supseteq \bigcap_{g\in G\setminus A} Q_1^g$. Therefore, $Q_2 \supseteq Q_1^g$ for some $g \in G\setminus A$ so that $H = i_G(Q_2\dagger) \supseteq i_G((Q_1^g)\dagger) = (i_G(Q_1\dagger))^g = H^g$. Since H is orbital, it follows that $H = H^g$, contradicting the fact that $g \notin A = \mathbb{N}_G(H)$. Thus we must have $T_2 \supseteq T_1$, and hence $T_2 \cap K[\nabla] \supseteq T_1 \cap K[\nabla]$. (Here, $\nabla = \nabla_G(H)$, as above). By (1.11), the sources of P_i , for the given H , are precisely the minimal corering primes of $T_i \cap K[\nabla]$ $(i = 1,2)$. Since any minimal covering prime of $T_2 \cap K[\nabla]$ contains a minimal covering prime of $T_1 \cap K[\nabla]$, we conclude that $[L_2] \geqslant [L_1]$. This proves the first assertion.

As to the height formula, first note that the members of any chain of primes in $K[G]$ leading from $[L_1]^G$ to $[L_2]^G$ belong to $\mathrm{Spec}_H(K[G])$, by (2.6) . Thus, by the foregoing, we conclude that :

$$\mathrm{ht}\ ([L_2]^G / [L_1]^G) = \mathrm{ht}\ ([L_2] / [L_1]) ,$$

where the latter of course denotes the maximal length, n , of a saturated chain $[L_1] = S_0 \lneq S_1 \lneq \dots \lneq S_n = [L_2]$ with $S_i \in \mathcal{S}_H$. Now each such chain yields a chain $L_1 = I_0 \subsetneq I_1 \subsetneq \dots \subsetneq I_n$ with $I_j \in \mathrm{Spec}_H(K[\nabla])$ and $I_n \in [L_2]$, and conversely. Thus in order to complete the proof of the proposition, it suffices to establish the following sublemma :

<u>Sublemma</u> <u>Let</u> H <u>and</u> ∇ <u>be as above and let</u> $I \subseteq J$ <u>be prime ideals of</u> $K[\nabla]$ <u>which are almost faithful</u> sub H . <u>Then</u> $\mathrm{ht}(J/I) = \mathrm{c.r.}(I) - \mathrm{c.r.}(J)$.

<u>Proof</u> We have $I\dagger \lhd \nabla$ and $\nabla/I^\dagger$ is finite-by-abelian, since $[H : I^\dagger] < \infty$ and ∇/H is abelian. Let $\bar{\ } : K[\nabla] \to K[\nabla/I^\dagger]$ denote the natural map. Then $\bar{I}$ and $\bar{J}$ are primes of $K[\nabla]$ with $\bar{I} \subseteq \bar{J}$ and $\mathrm{c.r.}(\bar{I}) = \mathrm{c.r.}(I)$, $\mathrm{c.r.}(\bar{J}) = \mathrm{c.r.}(J)$. Moreover, $\mathrm{ht}(\bar{J}/\bar{I}) = \mathrm{ht}(J/I)$, and so we may assume that ∇ is finite-by-abelian. In particular, ∇ contains a torsion-free central subgroup Z of finite index. Now $X = I \cap K[Z] \subseteq Y = J \cap K[Z]$ are prime in $K[Z]$, and $\mathrm{c.r.}(X) = \mathrm{c.r.}(I)$, $\mathrm{c.r.}(Y) = \mathrm{c.r.}(J)$ (see [3, Lemma 4.3]). It is not hard to show that $\mathrm{ht}(J/I) =$ $= \mathrm{ht}(Y/X)$. Indeed, $\leqslant$ follows from Incomparability ([4, Lemma 1.3(ii)] , for example), and $\geqslant$ is a consequence of the more general Proposition 3.3 . Thus we

may assume that $\nabla = Z$, and hence $K[\nabla] \cong K[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}]$ for some m . Since the assertion is classical in this case (cf. [7 , p. 84/85]), the sublemma, and hence the proposition, are proved.

The above proposition shows that $\operatorname{Spec}_H(K[G])$ and $\mathcal{S}_H$ may be identified for our purposes. Moreover, as we have seen in the above sublemma, the situation in $\mathcal{S}_H$ is almost classical. So, for example, if n denotes the rank of the free abelian group ∇/H , then we know that any prime ideal L of $K[\nabla]$ which is almost faithful sub H has central rank at most n . Thus, by (1.12), we have for any $P \in \operatorname{Spec}_H(K[G])$.

$$\text{(2.9)} \qquad \text{c.r.}(P) \leqslant \operatorname{rank}(\nabla/H) = n .$$

Proposition 2.8 further implies that any chain of primes in $\operatorname{Spec}_H(K[G])$ has length at most n . On the other hand, if K is nonabsolute then there always exists a chain $L_o \subsetneqq L_1 \subsetneqq \ldots \subsetneqq L_n$ of primes L_i in $K[\nabla]$ with $L_i^\dagger = H$ for all i . Indeed, if we let $\bar{\ }: K[\nabla] \to K[\nabla/H]$ denote the canonical map, then $\bar{\nabla} = \prod_{i=i}^{n} \langle x_i \rangle \cong \mathbb{Z}^n$ can be embedded in K° . For $i = 1,2,\ldots,n$ let $\zeta_i \in K^\circ$ denote the image of x_i under this embedding, and for each $\ell = 0,1,\ldots,n$ let $\varphi_\ell\, K[\bar{\nabla}] \to K[\bar{\nabla}]$ be the K-algebra map given by $\varphi_\ell(x_i) = \zeta_i = \zeta$ for $i \leqslant \ell$ and $\varphi_\ell(x_i) = x_i$ for $i > \ell$. Then each $\bar{L}_\ell = \operatorname{Ker}\varphi_\ell$ is a faithful prime in $K[\bar{\nabla}]$, and $\bar{L}_o \subsetneqq \bar{L}_1 \subsetneqq \ldots \subsetneqq \bar{L}_n$. Thus for nonabsolute K we have :

$$\text{(2.10)} \qquad \dim \operatorname{Spec}_H(K[G]) = \operatorname{rank}(\nabla/H) .$$

This is however no longer true if K is absolute. For, in this case the image of ∇ in every simple homomorphic image of $K[\nabla]$ is finite. Another consequence of Proposition (2.8) is that for any two given primes $P_1 \subseteq P_2$ in $\operatorname{Spec}_H(K[G])$ all saturated chains $P_1 = Q_o \subsetneqq Q_1 \subsetneqq \ldots \subsetneqq Q_r = P_2$ of primes Q_i in $K[G]$ have length $r = \text{c.r.}(P_1) - \text{c.r.}(P_2)$.

§ 3 - Catenarity

It is an interesting question, whether or not the fact described in the last paragraph of Section 2 holds quite generally in $\operatorname{Spec}(K[G])$. That is, given any two primes $P_1 \subseteq P_2$ in $K[G]$, do all saturated chains of primes

$P_1 = Q_0 \subsetneqq Q_1 \subsetneqq \cdots \subsetneqq Q_r = P_2$ have the same length $r = ht(P_2) - ht(P_1)$? In short, are group algebras of polycyclic-by-finite groups G catenary ? Roseblade has proved a positive answer to this for G orbitally sound or, slightly more generally, for G a $\mathfrak{Z}$-group (see [8], [9]). In this section we will show that our methods do at least quite easily yield an extension of Roseblade's result from orbitally sound to orbitally sound-by-finite nilpotent groups, that is polycyclic-by-finite groups G with a normal subgroup N such that N is orbitally sound and G/N is finite nilpotent.

Recall that two primes $P_1 \subseteq P_2$ of $K[G]$ are called <u>adjacent</u> (or <u>neighbors</u>) if there exists no prime in $K[G]$ lying strictly between P_1 and P_2. $K[G]$ catenary if and only if for any two adjacent primes $P_1 \subseteq P_2$ of $K[G]$ we have $ht(P_2) = ht(P_1) + 1$.

(3.1) <u>Lemma</u> <u>Let</u> H <u>be a normal subgroup of</u> G <u>of finite index and assume that</u> $K[H]$ <u>is catenary. Let</u> $P_1 \subseteq P_2$ <u>be adjacent primes of</u> $K[G]$ <u>such that</u> P_1 is <u>induced from</u> H (i.e. $P_1 = (P_1 \cap K[H]).K[G])$. <u>Then</u> $ht(P_2) = ht(P_1) + 1$.

<u>Proof</u> For $i = 1,2$ write $P_i \cap K[H] = \cap_{g\in G} Q_i^g$ for suitable primes Q_i of $K[H]$ such that $Q_1 \subseteq Q_2$. It is well-known (see for example [8, § 8.1]) that $ht(Q_i) = ht(P_i)$. Thus it suffices to show that $ht(Q_2) = ht(Q_1) + 1$ or, since $K[H]$ is catenary, that Q_1 and Q_2 are adjacent. Assume otherwise so that $Q_1 \subsetneqq Q \subsetneqq Q_2$ for some prime Q of $K[H]$. Then we have

$$P_1 \cap K[H] = \cap_{g\in G} Q_1^g \subsetneqq I = \cap_{g\in G} Q^g \subsetneqq P_2 \cap K[H] = \cap_{g\in G} Q_2^g ,$$

where the inclusions are strict since all occuring intersections are finite. It follows that $P_1 = (P_1 \cap K[H])K[G] \subsetneqq IK[G] \subsetneqq (P_2 \cap K[H])K[G] \subsetneqq P_2$. Note that $IK[G]$ is an ideal of $K[G]$ and, moreover, every minimal covering prime of $IK[G]$ intersects $K[H]$ in I (see [5, Lemma 4.1] or [8,Lemma 8]). Since P_2 contains such a minimal covering prime, P , we obtain that $P_1 \subsetneqq P \subsetneqq P_2$, contradicting the fact that P_1 and P_2 are adjacent. Thus Q_1 and Q_2 have to be adjacent, and the lemma is proved.

(3.2) <u>Proposition</u> <u>Assume</u> G <u>is orbitally sound-by-finite nilpotent and let</u> K <u>be any field. Then</u> $K[G]$ <u>is catenary</u>.

<u>Proof</u> Let N be an orbitally sound normal subgroup of G such that G/N is finite nilpotent. We argue by induction on $|G/N|$. The case $G = N$ is due to Roseblade so we assume that $G \supsetneqq N$. Let $P_1 \subseteq P_2$ be adjacent primes in $K[G]$. We have to

show that $ht(P_2) = ht(P_1) + 1$. For this, we may assume P_1 to be faithful. Indeed, writing $P_i \cap K[N] = \bigcap_{g\in G} Q_i^g$ for suitable primes Q_i of $K[N]$ with $Q_1 \subsetneq Q_2$, we have $ht(Q_i) = ht(P_i)$, and $ht(Q_2) = ht(Q_1) + 1$ holds if and only if Q_1 and Q_2 are adjacent, by Roseblade's result. Thus $ht(P_2) = ht(P_1) + 1$ if and only if Q_1 and Q_2 are adjacent, and the latter surely holds if and only if the images of Q_1 and Q_2 under $K[G] \twoheadrightarrow K[G/P_1^{\dagger}]$ are adjacent. Thus P_1 will be faithful in the following.

First assume that the vertex of P_1 , $vx(P_1)$, is normal in G . Then, by Lemma 2.3, P_1 is virtually standard. In particular, since P_1 is faithful, we have $P_1 = (P_1 \cap K[\Delta])K[G]$, where $\Delta = \Delta(G)$ is contained in nio(G), by (1.5). Thus P_1 is induced from nio(G), and since $K[\text{nio}(G)]$ is catenary, by Roseblade's result, we may apply Lemma 3.1 to conclude that $ht(P_2) = ht(P_1) + 1$.

Now assume that $\mathbb{N}_G(vx(P_1))$ is a proper subgroup of G . By (2.2), $\mathbb{N}_G(vx(P_1))$ contains nio(G) and hence N . Since G/N is nilpotent, there exists a proper normal subgroup H of G with $H \supseteq \mathbb{N}_G(vx(P_1))$. By Lemma 2.5, P_1 is induced from H and, by induction, $K[H]$ is catenary. Thus Lemma 3.1 again yields $ht(P_2) = ht(P_1) + 1$, and we are done.

We remark that group algebras of finitely generated abelian-by-finite groups are catenary. This follows from work of Schelter on affine PI-rings ([10]) . We close with a related result on certain crossed products. For the definition and basic facts concerning crossed products we refer to [5] . Here we just note that if $S = R \star G$ is a crossed product of the finite group G over the <u>commutative</u> ring R , then G acts on R , and if P is a prime ideal of S , then $P \cap R = \bigcap_{g\in G} Q^g$ for some prime ideal Q_i of R which is unique up to G-conjugacy and satisfies $ht(Q) = ht(P)$ (See [5, § 4]).

(3.3) <u>Proposition</u> <u>Let</u> R <u>be a finitely generated commutative K-algebra and and let</u> $S = R \star G$ <u>be a crossed product with</u> G <u>a finite group</u>. <u>Let</u> $P_1 \subsetneq P_2$ <u>be primes in</u> S <u>and write</u> $P_i \cap R = \bigcap_{g\in G} Q_i^g$ <u>for suitable primes</u> Q_i <u>of</u> R <u>with</u> $Q_1 \subsetneq Q_2$. <u>If</u> P_1 <u>and</u> P_2 <u>are adjacent then so are</u> Q_1 <u>and</u> Q_2 .

<u>Proof</u> Upon dividing out by $(P_1 \cap R)S$ we may assume that $P_1 \cap R = 0$. Set $T = S/P_1$ and let P denote the image of P_2 in T . Then $R \subseteq T$ and P has height 1 in T . Since G is finite, the fixed subring $R^G = \{r \in R \mid r^g = r$ for all $g \in G\}$ is a finitely generated K-algebra, as R is, and R is a finitely generated module over R^G . Moreover, R^G is central in T . By the Noether normalization theorem ([7, p.91]), R^G contains a subring V such that

$V \cong K[X_1, X_2, \ldots, X_\ell]$ for some ℓ and R^G is a finitely generated module over V. Therefore, T is a finitely generated module over the central subring $V \cong K[X_1, X_2, \ldots, X_\ell]$.

Now assume, by way of contradiction, that $Q_1 \subsetneq Q \subsetneq Q_2$ for some prime Q of R. Then the Incomparability theorem ([1, p.61]) implies that $0 \neq X = Q \cap V \subsetneq X_2 = Q_2 \cap V$. Note that $X_2 = (\cap_{g \in G} Q_2^g) \cap V = P \cap V$, since $V \subset R^G$. Note further that V is integrally closed and T is a prime PI-algebra, being a finitely generated module over a commutative K-algebra. Hence, by Schelter [10, Theorem 3], the Going Down theorem holds for the extension $V \subseteq T$. Thus there exists a prime ideal P' in T with $P' \subseteq P$ and $P' \cap V = X$. In particular, $0 \neq P' \cap V \subsetneq P \cap V$ and so $0 \neq P' \subsetneq P$, contradicting the fact that P has height 1. Therefore, we conclude that Q_1 and Q_2 are adjacent, and the proposition is proved.

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