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ADDENDUM — PRIME IDEALS IN CROSSED PRODUCTS OF FINITE GROUPS

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ABSTRACT
In this note, we offer a simpler, alternate approach to the work of Section 3 of "Prime ideals in crossed products of finite groups." Indeed, by using the induced ideal map ^{G} instead of the $^\vee$ map, we have eliminated many of the unpleasant computations of the original argument.

Paper [2] is concerned for the most part with prime ideals in crossed products $R * G$ of finite groups and the proof of its main result [2, theorem 1.3] is essentially divided into three parts. Part 1 yields a one-to-one correspondence between suitable prime ideals when the coefficient ring R is prime and culminates in [2, theorem 2.5]. Part 2 introduces certain maps $^\vee$ and $^\delta$ which yield another one-to-one correspondence between suitable prime ideals when R is G-prime and culminates in [2, theorem 3.6]. Part 3 is concerned with proving the nilpotence of J, the intersection of the minimal prime ideals of $R * G$. The latter requires work because the multiplication formula [2, lemma 3.3 (ii)] given for the map $^\vee$ contains an additional factor M which causes difficulties. To overcome this, it was necessary in [2] to introduce the concept of an R-cancelable ideal and to do a number of rather unpleasant computations.

Recently, an alternate extremely useful characterization of the $^\vee$ map was discovered in [3]. This characterization, namely the induced ideal map denoted by ^{G}, is certainly more natural than $^\vee$ and simpler and better understood. Indeed, it clearly yields the correct approach to the necessary correspondences in [2, section 3]. Furthermore, it also satisfies an honest multiplication formula, which thereby trivializes the work of Part 3. This latter formula, given in Lemma 3.4 (i), follows essentially from a short argument due to Deskins [1]. We would like to thank Dr. R. N. Gupta for pointing this reference out to us in the context of an entirely different problem.

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The goal of this addendum is therefore to redo and substantially shorten and simplify [2, section 3] using the induced ideal map ^{G}. Sections 1, 2 and 3 of this note precisely parallel those of [2].

§1. Introduction

This is unchanged.

§2. Prime coefficient rings

This section is essentially unchanged. The unpleasant [2, lemma 2.8] is no longer needed and should be deleted.

The definition of R-cancelable ideals given before [2, lemma 2.3] could be simplified slightly by dropping the requirement that A be a G-invariant ideal. This is clear since, in the prime ring R, every nonzero ideal contains a nonzero G-invariant ideal. In fact the whole concept can be eliminated here by appropriately rephrasing [2, lemma 2.3 (i)]. Note however that either change would require a slight amplification in the proof of [2, theorem 5.6].

We include the following result for the sake of completeness.

COROLLARY 2.8. *Let $R * G$ be a crossed product of the finite group G over the prime ring R. Then we have $0^d = 0$ and $0^u = 0$. In particular $R * G$ is prime if and only if $E = C^t[G_{\text{inn}}]$ is G-prime.*

PROOF. The equation $0^u = 0$ is obvious and the equation $0^d = 0$ follows immediately from [2, lemma 2.1 (i)]. [2, theorem 2.5] now yields the result.

§3. G-prime coefficient rings

This must be totally changed and should read as follows.

This section contains the proofs of theorems 1.2 and 1.3 of [2]. Throughout, G will denote a finite group and $R * G$ will be a crossed product of G over R. Then there is a well defined action of G on the set of ideals of R and we will assume that R is a G-prime ring. Recall that this means, by definition, that the product of any two nonzero G-invariant ideals of R is nonzero. Certainly, this condition is satisfied if there exists a prime ideal Q of R such that $\bigcap_{x\in G} Q^x = 0$. For, if A_1 and A_2 are G-invariant ideals with $A_1A_2 = 0$, then $A_1A_2 \subset Q$ so $A_i \subset Q$ for some i. Using the G-invariance of A_i, we deduce that $A_i \subset \bigcap_{x\in G} Q^x$ and hence $A_i = 0$. Part (i) of the following lemma shows that, conversely, in any G-prime ring R one can find such a prime Q.

We remark on a simple property of semiprime rings. Suppose R is semiprime and let A and B be ideals of R with $AB = 0$. Then $(BA)^2 = 0$ so $BA = 0$. In view of this, left and right annihilators of ideals are equal and we will just use the notation "ann."

LEMMA 3.1. *Let $R * G$ be given and assume that R is G-prime. Then*

(i) *R contains a prime ideal Q with $\bigcap_{x \in G} Q^x = 0$. In particular, R is semiprime.*

(ii) *Any prime ideal of R contains a conjugate Q^x of Q and so $\{Q^x \mid x \in G\}$ are precisely the minimal primes of R.*

(iii) *Let H denote the stabilizer of Q in G and let $N = \operatorname{ann}_R Q$. Then H is a subgroup of G,*

$$N = \bigcap_{x \notin H} Q^x \neq 0$$

and

$$0 = N\bar{x}N = N \cap N^x = N \cap Q$$

for all $x \in G \setminus H$.

(iv) *If A is any nonzero ideal of R with $A \subset N$, then $\operatorname{ann}_R A = Q$.*

PROOF. (i) Since G is finite, an easy application of Zorn's lemma shows that there exists an ideal Q of R maximal with respect to the property that $\bigcap_{x \in G} Q^x = 0$. Now suppose A_1 and A_2 are ideals of R containing Q with $A_1 A_2 \subset Q$ and set $B_i = \bigcap_{x \in G} A_i^x$. Then B_1 and B_2 are G-invariant and since $B_1 B_2 \subset A_1 A_2$ we have

$$B_1 B_2 \subset \bigcap_{x \in G} (A_1 A_2)^x \subset \bigcap_{x \in G} Q^x = 0.$$

Since R is G-prime, we conclude that $B_i = 0$ for some i and then the maximality of Q implies that $A_i = Q$. Thus Q is a prime ideal of R.

(ii) Any prime ideal of R certainly contains $\bigcap_{x \in G} Q^x = 0$ and consequently contains some Q^x, since G is finite. Furthermore, there are no inclusion relations between the primes $\{Q^x \mid x \in G\}$. For, if $Q \subsetneqq Q^x$, then $Q \subsetneqq Q^{x^n}$ for all $n \geqq 1$, so by taking $n = |G|$ we obtain the contradiction $Q \subsetneqq Q$.

(iii) If N denotes the annihilator of Q, then $NQ = 0$ yields $NQ \subset Q^x$ for all $x \in G$. Thus if $x \in G \setminus H$, we deduce from (ii) that $N \subset Q^x$ and we have shown that $N \subset \bigcap_{x \notin H} Q^x$. Conversely, since

$$\Big(\bigcap_{x \notin H} Q^x\Big) Q \subset \bigcap_{x \in G} Q^x = 0,$$

we have $N = \operatorname{ann} Q \supset \bigcap_{x \notin H} Q^{\bar{x}}$ and therefore equality occurs. We remark that if $H = G$, then by definition $N = \bigcap_{x \notin H} Q^{\bar{x}} = R$. In any case, by (ii) above we have $Q \not\supset N$ and hence $N \neq 0$. Moreover, $N \cap Q = \bigcap_{x \in G} Q^{\bar{x}} = 0$ and if $x \notin H$ then $N \subset Q^{\overline{x^{-1}}} = Q^{\bar{x}^{-1}}$ so $N^{\bar{x}} \subset Q$. Thus for $x \notin H$ we have $\bar{x}^{-1} N\bar{x}N \subset N \cap N^{\bar{x}} \subset N \cap Q = 0$ and hence $0 = N\bar{x}N = N \cap N^{\bar{x}}$.

(iv) If $A \subset N$ then clearly $AQ = 0$ so $Q \subset \operatorname{ann} A$. Conversely, suppose $AB = 0$. If $A \neq 0$ then $A \not\subset Q$, since $N \cap Q = 0$ by part (iii). Thus $AB = 0 \subset Q$ implies that $B \subset Q$. This shows that $Q = \operatorname{ann} A$.

Notation. The notation of the preceding lemma will be kept throughout this section. Thus Q will denote a minimal prime of the G-prime ring R, N will be its annihilator in R and H will denote the stabilizer of Q in G. Moreover, we set $M = \Sigma_{x \in G} N^{\bar{x}}$ so that M is a nonzero G-invariant ideal of R.

Part (ii) of the next lemma is crucial for the work of this section. Part (i) is needed for its proof.

Lemma 3.2. *Let H and N be as above.*

(i) *Let V be a nonzero right R-submodule of $N\bar{G}$ and let $T = \{x_1, x_2, \cdots, x_n\}$ be a subset of G with $x_1 = 1$. Suppose that $V \cap R * T \neq 0$ but $V \cap R * T' = 0$ for all $T' \subsetneq T$. Then $T \subset H$.*

(ii) *Let I be an ideal of $R * G$. Then there exists a nonzero G-invariant ideal E of R (depending upon I) with*

$$EI \subset \bar{G}N(I \cap R * H)\bar{G}.$$

Proof. (i) By assumption there exists $0 \neq \alpha = \Sigma_{i=1}^n r_i \bar{x}_i \in V \cap R * T$. Since $V \subset N\bar{G}$ we have $r_i \in N$ for $i = 1, 2, \cdots, n$ and the minimality condition on T implies that $r_i \neq 0$ for all i. Now for any $q \in Q$ we have $\alpha q = \Sigma_{i=1}^n r_i q^{\bar{x}_i^{-1}} \bar{x}_i \in V$ and the first summand on the right, namely $r_1 q$, is contained in $NQ = 0$ and hence is zero. Thus the minimality condition on T yields $r_i q^{\bar{x}_i^{-1}} = 0$ for all i and we have $r_i Q^{\bar{x}_i^{-1}} = 0$. Thus $r_i^{\bar{x}_i} Q = 0$ so $r_i^{\bar{x}_i} \in N = \operatorname{ann} Q$ and therefore, since $r_i \in N$, we have $0 \neq r_i^{\bar{x}_i} \in N \cap N^{\bar{x}_i}$. Lemma 3.1 (iii) now implies that $x_i \in H$.

(ii) Suppose first that $NI = 0$. Then since I is G-invariant we have

$$MI = \Big(\sum_{x \in G} N^{\bar{x}}\Big) I = 0 = \bar{G}N(I \cap R * H)\bar{G}$$

and so we may take $E = M$ in this case. Thus we may assume that $V = NI \neq 0$. Note that V satisfies the hypotheses of part (i) and, in addition, V is a right ideal of $R * G$. Let $\mathcal{T}$ denote the set of all subsets T of G such that $V \cap R * T \neq 0$, $V \cap R * T' = 0$ for all $T' \subsetneq T$ and $1 \in T$. Then since V is a right ideal of $R * G$ it follows easily that $\mathcal{T}$ is a finite nonempty collection of subsets of G. Furthermore each $T \in \mathcal{T}$ satisfies $T \subset H$ by part (i).

For each $T = \{x_1 = 1, x_2, \cdots, x_n\} \in \mathcal{T}$, let A_T denote the set

$$A_T = \left\{ r \in R \,\middle|\, \text{there exists } \beta = \sum_{i=1}^{n} r_i \bar{x}_i \in V \text{ with } r_1 = r \right\}.$$

Note that A_T is a nonzero ideal of R which is contained in N, since V is an R-subbimodule of $N\bar{G}$. Set $D = \bigcap_{T \in \mathcal{T}} A_T$. Since $\mathcal{T}$ is finite and $A_T \subset N$ for all $T \in \mathcal{T}$, it follows from Lemma 3.1 (iii) that $D \neq 0$. We show now, by induction on $m = |\operatorname{Supp} \alpha|$, that if $\alpha \in V$ then $D^{m+1}\alpha \subset \bar{G}N(I \cap R * H)\bar{G}$. The case $m = 0$ is of course trivial.

Now let $\alpha \in V$ be given with $|\operatorname{Supp} \alpha| = m > 0$ and suppose the assertion holds for all elements $\gamma \in V$ of smaller support size. Choose $T \subset \operatorname{Supp} \alpha$ minimal with respect to the property that $V \cap R * T \neq 0$. If $y \in T$, then $\operatorname{Supp} \alpha\bar{y}^{-1} = (\operatorname{Supp} \alpha)y^{-1} \supset Ty^{-1}$, Ty^{-1} also has this minimal property since $V = NI$ is a right ideal of $R * G$, and $1 \in Ty^{-1}$. Since it clearly suffices to show that $D^{m+1}\alpha\bar{y}^{-1} \subset \bar{G}N(I \cap R * H)\bar{G}$, we can replace α by $\alpha\bar{y}^{-1}$ and T by Ty^{-1} and hence we can assume that $1 \in T$. Thus $1 \in \operatorname{Supp} \alpha$ and $T \in \mathcal{T}$.

Let $c = \operatorname{tr} \alpha$ be the identity coefficient of α and let $d \in D \subset A_T$. Then, by definition of A_T, there exists an element $\beta \in V \cap R * T$ with $\operatorname{tr} \beta = d$. Thus $\gamma = d\alpha - \beta c \in V$ and, since $\operatorname{Supp} \beta \subset \operatorname{Supp} \alpha$ and $\operatorname{tr} \gamma = 0$, we have $|\operatorname{Supp} \gamma| < m$. By induction, we deduce that $D^m\gamma \subset \bar{G}N(I \cap R * H)\bar{G}$ and hence $D^m d\alpha \subset \bar{G}N(I \cap R * H)\bar{G} + D^m\beta c$. Now we have observed above that $T \subset H$ and hence $\beta c \in V \cap R * H \subset I \cap R * H$. Thus since $m \geqq 1$ and $D \subset N$, we have $D^m\beta c \subset N(I \cap R * H) \subset \bar{G}N(I \cap R * H)\bar{G}$. We conclude therefore that $D^m d\alpha \subset \bar{G}N(I \cap R * H)\bar{G}$ and, since this holds for all $d \in D$, we have $D^{m+1}\alpha \subset \bar{G}N(I \cap R * H)\bar{G}$. The induction step is proved.

In particular, if $k = |G|$, we deduce from the above that $D^{k+1}V = D^{k+1}NI \subset \bar{G}N(I \cap R * H)\bar{G}$. But observe the $D^{k+1}N \neq 0$ since $D \subset N$, $D \neq 0$ and $Q \cap N = 0$, by Lemma 3.1 (iii). Thus if E is defined by

$$E = \{r \in R \mid rI \subset \bar{G}N(I \cap R * H)\bar{G}\}$$

then E is not zero because $E \supset D^{k+1}N \neq 0$. On the other hand, E is certainly a G-invariant ideal of R since I and $\bar{G}N(I \cap R * H)\bar{G}$ are ideals of $R * G$. Thus we have an appropriate $E \neq 0$ with $EI \subset \bar{G}N(I \cap R * H)\bar{G}$ and the proof is complete.

Note that $Q * H$ is an ideal of $R * H$. Roughly speaking, our method in this section is to pass from $R * G$ to $(R * H)/(Q * H) \simeq (R/Q) * H$ and thus reduce the general problem to the case of prime coefficient rings where the results of [2, section 2] can be applied. The following definition introduces the necessary machinery.

DEFINITION. (i) For any ideal L of $R * H$ we set

$$L^G = \bigcap_{x \in G} (L\bar{G})^{\bar{x}} = \bigcap_{x \in G} L^{\bar{x}}\bar{G}.$$

We will see below that L^G is an ideal of $R * G$ and we will characterize it in several different ways.

(ii) If I is an ideal of $R * G$, then we set

$$I_H = \{\alpha \in R * H \mid N\alpha \subset I\}.$$

Since N is H-invariant, it follows easily that I_H is an ideal of $R * H$. Moreover $N(Q * H) = 0$ shows that $I_H \supset Q * H$, and clearly $I_H \supset I \cap R * H$.

We remark that if V is an $R * H$-module with $L = \mathrm{ann}_{R*H} V$, then it is fairly easy to see that L^G, as defined above, is given by $L^G = \mathrm{ann}_{R*G} V^G$ where V^G denotes the induced right $R * G$-module $V^G = V \otimes_{R*H} R * G$. Thus the ^{G} notation here is natural and classical and we trust that it will not be confused with ^{G} as used in [2, section 4] in the study of rings with group actions.

Note that there is a well defined trace map $\tau : R * G \to R * H$ given by $\tau(\Sigma_{x \in G} r_x \bar{x}) = \Sigma_{x \in H} r_x \bar{x}$. In other words, τ truncates each element of $R * G$ to the partial sum of those terms in its support corresponding to group elements contained in H. It is clear that τ is both a right and a left $R * H$-module homomorphism. Hence if I is a right ideal of $R * G$, then $\tau(I)$ is a right ideal of $R * H$ and it is easy to see that $I \subset \tau(I)\bar{G}$. Similarly, if I is a left ideal of $R * G$, then $\tau(I)$ is a left ideal of $R * H$ and $I \subset \bar{G}\tau(I)$.

LEMMA 3.3. *Let L be an ideal of $R * H$. Then*

(i) *L^G is the unique largest two sided ideal of $R * G$ contained in $L\bar{G}$.*

(ii) *L^G is the unique largest two sided ideal of $R * G$ satisfying $\tau(I) \subset L$.*

PROOF. (i) If I is an ideal of $R * G$ contained in $L\bar{G}$, then since I is G-invariant we have $I \subset \bigcap_{x \in G} (L\bar{G})^{\bar{x}} = L^G$. On the other hand, since $L^G = \bigcap_{x \in G} (L\bar{G})^{\bar{x}} = \bigcap_{x \in G} L^{\bar{x}}\bar{G}$, we see that L^G is clearly a left R-module, a right $R * G$-module and it is G-invariant. Thus it is a two sided ideal and hence the largest such contained in $L\bar{G}$.

(ii) Let I be an ideal of $R * G$. If $I \subset L\bar{G}$ then clearly $\tau(I) \subset \tau(L\bar{G}) = L$. On the other hand, if $\tau(I) \subset L$, then $I \subset \tau(I)\bar{G} \subset L\bar{G}$. Thus the result follows immediately from (i).

Note that the condition in (ii) above is right-left symmetric and hence we can see that the definition of L^G is also right-left symmetric. Now obviously, the maps ^{G} and $_H$ are monotone, as are the maps ^{u} and ^{d} of [2, section 2]. In fact, as we will see, the maps ^{G} and $_H$ behave similarly to ^{u} and ^{d} in many other respects. Indeed the following two lemmas are the analogs of [2, lemmas 2.3 and 2.4].

LEMMA 3.4. *If L_1 and L_2 are ideals of $R * H$, then*
(i) $L_1^G L_2^G \subset (L_1 L_2)^G$,
(ii) $L_1^G \cap L_2^G = (L_1 \cap L_2)^G$.

PROOF. (i) Since L_2^G is an ideal of $R * G$, we have $\bar{G} L_2^G \subset L_2^G$ and hence

$$L_1^G L_2^G \subset L_1 \bar{G} L_2^G \subset L_1 L_2^G \subset L_1 L_2 \bar{G}.$$

Thus, since $L_1^G L_2^G$ is an ideal of $R * G$, Lemma 3.3 (i) yields $L_1^G L_2^G \subset (L_1 L_2)^G$.
(ii) Since ^{G} is monotone we have $(L_1 \cap L_2)^G \subset L_1^G \cap L_2^G$. Conversely, since $\tau(L_1^G \cap L_2^G) \subset L_1 \cap L_2$ we have $L_1^G \cap L_2^G \subset (L_1 \cap L_2)^G$ by Lemma 3.3 (ii).

LEMMA 3.5. *Given the above notation.*
(i) *Let L be an ideal of $R * H$ with $L \cap R \supset Q$. Then $\bar{G} N L \bar{G} \subset L^G \subset L\bar{G}$ and $L \subset (L^G)_H$.*
(ii) *If I is an ideal of $R * G$, then $M(I_H)^G \subset I$. Moreover, there exists a nonzero G-invariant ideal E of R with $EI \subset (I_H)^G$.*

PROOF. (i) If $x \in H$, then $\bar{x} N L \bar{G} \subset L\bar{G}$ since L is an ideal of $R * H$. If $x \notin H$, then

$$\bar{x} N L \bar{G} = N^{\bar{x}^{-1}} L^{\bar{x}^{-1}} \bar{G} \subset Q(R * G) \subset L\bar{G}$$

since $N^{\bar{x}^{-1}} \subset Q \subset L$ for $x \notin H$. Thus $\bar{G} N L \bar{G} \subset L\bar{G}$, and since $\bar{G} N L \bar{G}$ is an ideal of $R * G$, we have $\bar{G} N L \bar{G} \subset L^G \subset L\bar{G}$. In particular $NL \subset L^G$ and hence, by definition, we have $L \subset (L^G)_H$.
(ii) We have $N(I_H)^G \subset N I_H \bar{G} \subset I\bar{G} = I$, where the second inclusion holds by definition of I_H. Since I and $(I_H)^G$ are G-stable, it follows that

$$M(I_H)^G = \sum_{x \in G} N^{\bar{x}} (I_H)^G = \sum_{x \in G} (N(I_H)^G)^{\bar{x}} \subset I.$$

This proves the first assertion in (ii). As to the second, we know by Lemma 3.2 (ii) that $EI \subset \bar{G} N (I \cap R * H) \bar{G}$ for a suitable nonzero G-invariant ideal E of R.

Furthermore $I \cap R * H \subset I_H$ and the latter is an ideal of $R * H$ containing Q. Thus by (i) above, we conclude that

$$EI \subset \bar{G}N(I \cap R * H)\bar{G} \subset \bar{G}NI_H\bar{G} \subset (I_H)^G.$$

LEMMA 3.6. *Given the above notation.*

(i) *If I is an ideal of $R * G$ with $I \cap R = 0$, then $I_H \cap R = Q$. In addition, $0_H = Q * H$.*

(ii) *If L is an ideal of $R * G$ with $L \cap R = Q$, then $L^G \cap R = 0$. In addition, $(Q * H)^G = 0$.*

PROOF. (i) Since $I \cap R = 0$, we have

$$I_H \cap R = \{r \in R \mid Nr \subset I \cap R = 0\} = \operatorname{ann}_R N = Q,$$

by Lemma 3.1 (iv). Similarly, since $\operatorname{ann}_R N = Q$, we have $0_H = Q * H$.

(ii) Since $L\bar{G} \cap R = (L\bar{G} \cap R * H) \cap R = L \cap R = Q$, it follows that

$$L^G \cap R = \bigcap_{x \in G} (L\bar{G})^x \cap R = \bigcap_{x \in G} (L\bar{G} \cap R)^x = \bigcap_{x \in G} Q^x = 0.$$

Similarly

$$(Q * H)^G = \bigcap_{x \in G} Q^x\bar{G} = \left(\bigcap_{x \in G} Q^x\right)\bar{G} = 0$$

and the result follows.

The following is the main result of this section. We prove it simultaneously with Lemma 3.8.

THEOREM 3.7. *Let $R * G$ be a crossed product of the finite group G over the ring R. Assume that R is G-prime and let Q be a minimal prime of R with H the stabilizer of Q in G. Then the maps ^{G} and ${}_H$ yield a one-to-one correspondence between the prime ideals P of $R * G$ with $P \cap R = 0$ and the prime ideals L of $R * H$ with $L \cap R = Q$. More precisely:*

(i) *If P is a prime ideal of $R * G$ with $P \cap R = 0$, then P_H is a prime ideal of $R * H$ with $P_H \cap R = Q$ and $P = (P_H)^G$.*

(ii) *Let L be a prime ideal of $R * H$ with $L \cap R = Q$. Then L^G is a prime ideal of $R * G$ with $L^G \cap R = 0$ and $L = (L^G)_H$.*

LEMMA 3.8. *Let $R * G$ be given with R a G-prime ring. If P is a prime ideal of $R * G$ with $P \cap R = 0$ and if I is an ideal of $R * G$ properly containing P, then $I \cap R \neq 0$.*

PROOF. We start with an observation on a form of cancelation. Let L be a prime ideal of $R * H$ with $L \cap R = Q$ and suppose we have $EI \subset L^G$, where I is an ideal of $R * G$ and E is a nonzero G-invariant ideal of R. Then $(E * H)I \subset L^G \subset L\bar{G}$ and, by applying the trace map $\tau : R * G \to R * H$, we have $(E * H)\tau(I) \subset L$. But certainly $E * H \not\subset L$ since E is G-invariant and $L \cap R = Q$ satisfies $\bigcap_{x \in G} Q^x = 0$. Thus since L is prime we deduce that $\tau(I) \subset L$ and hence that $I \subset L^G$, by Lemma 3.3 (ii).

(i) Let P be a prime ideal of $R * G$ with $P \cap R = 0$ and set $L = P_H$. By Lemma 3.6 (i), we have $L \cap R = P_H \cap R = Q$. Let us first observe, by Lemma 3.5 (ii), that $P \supset M(P_H)^G = (M * G)(P_H)^G$. Thus since P is prime and $M * G \not\subset P$, we see that $P \supset (P_H)^G = L^G$. Next we show that L is prime. Indeed if L_1 and L_2 are ideals of $R * H$ containing L with $L \supset L_1L_2$, then Lemma 3.4 (i) yields

$$P \supset L^G \supset (L_1L_2)^G \supset L_1^G L_2^G.$$

Hence, since P is prime, $P \supset L_i^G$ for some i, and then, by Lemma 3.5 (i), since $L_i \cap R \supset L \cap R = Q$ we have

$$L = P_H \supset (L_i^G)_H \supset L_i.$$

Hence L is a prime ideal of $R * H$ with $L \cap R = Q$. Finally, by Lemma 3.5 (ii), there exists a nonzero G-invariant ideal E of R with $EP \subset (P_H)^G = L^G$. We therefore conclude from the above mentioned cancelation property that $P \subset L^G$, so we have equality and part (i) is proved.

(ii) Now let L be a prime ideal of $R * H$ with $L \cap R = Q$ and set $P = L^G$. Then Lemma 3.6 (ii) asserts that $P \cap R = 0$. Suppose I is any ideal, including P itself, satisfying $I \supset P$ and $I \cap R = 0$. Then, by Lemmas 3.5 (i) and 3.6 (i), we have

$$I_H \supset P_H = (L^G)_H \supset L$$

and $I_H \cap R = Q$. Let $\bar{\ } : R * H \to (R * H)/(Q * H) = (R/Q) * H$ denote the natural homomorphism and observe that both I_H and L contain the kernel $Q * H$ of this map. Thus $\bar{I}_H \supset \bar{L}$, $\bar{L}$ is a prime ideal of $\bar{R} * H$ and clearly $\bar{I}_H \cap \bar{R} = 0$. We therefore conclude from [2, lemma 2.6] that $\bar{I}_H = \bar{L}$ and hence that $I_H = L$ since $L \supset Q * H$. In particular, we have $I_H = P_H = (L^G)_H = L$.

Furthermore, by Lemma 3.5 (ii), there exists a nonzero G-invariant ideal E of R with

$$EI \subset (I_H)^G = L^G = P.$$

Therefore, by the cancelation property discussed in the beginning of this proof, we deduce that $I \subset L^G = P$ and hence that $I = P$. It is now a simple matter to see that P is prime. Indeed if I_1 and I_2 are ideals of $R * G$ properly containing P, then by the above we must have $I_1 \cap R \neq 0$ and $I_2 \cap R \neq 0$. Hence, since R is G-prime,

$$0 \neq (I_1 \cap R)(I_2 \cap R) \subset I_1 I_2 \cap R$$

so $I_1 I_2 \not\subset P$ and P is prime. This completes the proof of Theorem 3.7.

Finally, for Lemma 3.8, let P be a prime ideal of $R * G$ with $P \cap R = 0$ and let I be an ideal of $R * G$ properly containing P. By the above, $P = L^G$ for some prime ideal L of $R * H$ with $L \cap R = Q$. Thus the argument of the preceding paragraph now applies to yield $I \cap R \neq 0$ and the lemma is proved.

We remark that Theorem 3.7 has important applications in [3] to the study of prime ideals in group algebras of polycyclic-by-finite groups. Because of this, it is interesting to observe that the above work is entirely independent of [2, section 2] except for one application of Incomparability, [2, lemma 2.6]. However, Incomparability is easy to prove in the case of Noetherian rings, see for example [3, lemma 1.3 (ii)]. Thus the preceding arguments can clearly yield an alternate approach, independent of [2], to the work of [3, section 1].

Returning to the general G-prime situation, it is clear that the canonical map $\tilde{}: R * H \to (R * H)/(Q * H) = (R/Q) * H$ yields a one-to-one correspondence between the primes L of $R * H$ with $L \cap R = Q$ and the primes $\tilde{L}$ of $\tilde{R} * H = (R/Q) * H$ with $\tilde{L} \cap \tilde{R} = 0$. Thus the maps ^{G} and ${}_H$ combine with $\tilde{}$ to yield a one-to-one correspondence between the set $\mathcal{P}$ of prime ideals of $R * G$ having trivial intersection with R and the primes of $\tilde{R} * H$ having trivial intersection with $\tilde{R}$. For example, the following is an immediate consequence of Theorem 3.7 and Lemma 3.6.

Corollary 3.9. *With the above notation, $R * G$ is a prime ring if and only if $(R * H)/(Q * H) = (R/Q) * H$ is prime.*

Moreover, since $\tilde{R}$ is prime, we can then apply the maps ^{u} and ^{d} of [2, theorem 2.5] to $\tilde{R} * H$ to obtain a one-to-one correspondence between $\mathcal{P}$ and the H-prime ideals of a certain twisted group algebra $C^t[H_{\text{inn}}]$, where C is the extended centroid of $\tilde{R}$. However, instead of formalizing this further, we will content ourselves with proving the main results of [2], namely [2, theorems 1.2 and 1.3], which now follow quite easily.

Proof of [2, theorem 1.2]. In view of the comments made in [2, section 1], this is an immediate consequence of Lemma 3.8.

PROOF OF [2, THEOREM 1.3]. We are given the crossed product $R * G$ with G finite and with R a G-prime ring. As usual, let Q be a minimal prime of R, as in Lemma 3.1 (i), and let H be the stabilizer of Q in G. Furthermore, let $\tilde{}: R * H \to (R * H)/(Q * H) = (R/Q) * H$ denote the natural map. By [2, lemma 2.7], applied to the crossed product $\tilde{R} * H$, we see that there are finitely many primes $\tilde{L}$ of $\tilde{R} * H$ with $\tilde{L} \cap \tilde{R} = 0$. Indeed, if these are $\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_n$, then $n \leqq |H| \leqq |G|$. Moreover $\tilde{T} = \tilde{L}_1 \cap \tilde{L}_2 \cap \cdots \cap \tilde{L}_n$ is the unique largest nilpotent ideal of $\tilde{R} * H$ and $\tilde{T}^{|H|} = 0$.

For each i, let L_i denote the complete inverse image of $\tilde{L}_i$ in $R * H$. Then it follows immediately from Theorem 3.7 that $P_i = L_i^G$ for $i = 1, 2, \cdots, n$ are precisely the prime ideals of $R * G$ having trivial intersection with R. Moreover, set $T = L_1 \cap L_2 \cap \cdots \cap L_n$ so that T is the complete inverse image of $\tilde{T}$ in $R * H$. Since $\tilde{T}^{|H|} = 0$, we then have $T^{|H|} \subset Q * H$. Setting $J = T^G$ we conclude from Lemmas 3.4 (i) (ii) and 3.6 (ii) that

$$J = T^G = L_1^G \cap L_2^G \cap \cdots \cap L_n^G = P_1 \cap P_2 \cap \cdots \cap P_n$$

and

$$J^{|H|} = (T^G)^{|H|} \subset (T^{|H|})^G \subset (Q * H)^G = 0.$$

Thus

$$(P_1 \cap P_2 \cap \cdots \cap P_n)^{|H|} = J^{|H|} = 0$$

and this of course implies that J is the unique largest nilpotent ideal of $R * G$ since each P_i, being prime, contains all nilpotent ideals of the crossed product.

Finally, let P be any prime ideal of $R * G$. Then P contains the nilpotent ideal $P_1 \cap P_2 \cap \cdots \cap P_n$ and hence $P \supset P_i$ for some i. This shows that the minimal primes of $R * G$ are the minimal members of the set $\{P_1, P_2, \cdots, P_n\}$. But $P_i \supset P_j$ implies, by Lemma 3.8, that $P_i = P_j$, since both primes have trivial intersection with R, and hence that $i = j$. This shows that $P_1, P_2, \cdots, P_n$ are precisely the minimal primes of $R * G$ and the theorem is proved.

Corollaries 3.9 and 3.10 of [2] should now be included here. However, since these results require no change in either their statements or proofs, we will not bother to include them. Furthermore, we remark that a quite different proof of [2, corollary 3.9] now appears as [4, theorem 7 (ii)].

We close this paper with an alternate characterization of the maps ^{G} and ${}_H$. Part (i) below of course relates ^{G} to the ${}^\vee$ map of [2, section 3].

PROPOSITION 3.10. *Let $R * G$ be given with R a G-prime ring.*

(i) *If L is a prime ideal of $R * H$ with $L \cap R = Q$, then*

$$L^G = \{\alpha \in R * G \mid M\alpha \subset \bar{G}NL\bar{G}\}.$$

(ii) *If P is a prime ideal of $R * G$ with $P \cap R = 0$, then P_H is the unique minimal covering prime of $P \cap R * H$ with $N \not\subset P_H \cap R$.*

Proof. (i) Let L be given with $L \cap R = Q$ and set $L^\vee = \{\alpha \in R * G \mid M\alpha \subset \bar{G}NL\bar{G}\}$. Since $L^G \subset L\bar{G}$, we have $NL^G \subset NL\bar{G} \subset \bar{G}NL\bar{G}$ and, since both L^G and $\bar{G}NL\bar{G}$ are G-invariant, we deduce that $ML^G \subset \bar{G}NL\bar{G}$. Thus $L^G \subset L^\vee$. Conversely by definition and Lemma 3.5 (i), we have

$$(M * G)L^\vee = ML^\vee \subset \bar{G}NL\bar{G} \subset L^G.$$

Thus since L^G is prime, by Theorem 3.7 (ii), and $M * G \not\subset L^G$, we conclude that $L^\vee \subset L^G$.

(ii) Let P be given with $P \cap R = 0$. Then by definition and Theorem 3.7 (i), we know that P_H is a prime ideal of $R * H$ containing $P \cap R * H$. Furthermore, $P_H \cap R = Q$ and hence $P_H \cap R \not\supset N$. Now let L be any prime ideal of $R * H$ with $L \supset P \cap R * H$ and $L \cap R \not\supset N$. Then, by definition,

$$L \supset P \cap R * H \supset NP_H = (N * H)P_H$$

and hence, since $L \not\supset N * H$, we have $L \supset P_H$. This shows first, by taking $P_H \supset L \supset P \cap R * H$, that P_H is a minimal covering prime of $P \cap R * H$. Then we see that P_H is the unique such minimal covering prime with the additional property that $P_H \cap R \not\supset N$. This completes the proof.

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