ADDENDUM — PRIME IDEALS IN CROSSED PRODUCTS OF FINITE GROUPS

BY

MARTIN LORENZ AND D. S. PASSMAN

ABSTRACT
In this note, we offer a simpler, alternate approach to the work of Section 3 of "Prime ideals in crossed products of finite groups." Indeed, by using the induced ideal map $\mathcal{G}$ instead of the $\sigma$ map, we have eliminated many of the unpleasant computations of the original argument.

Paper [2] is concerned for the most part with prime ideals in crossed products $R*G$ of finite groups and the proof of its main result [2, theorem 1.3] is essentially divided into three parts. Part 1 yields a one-to-one correspondence between suitable prime ideals when the coefficient ring $R$ is prime and culminates in [2, theorem 2.5]. Part 2 introduces certain maps $\nu$ and $\lambda$ which yield another one-to-one correspondence between suitable prime ideals when $R$ is $G$-prime and culminates in [2, theorem 3.6]. Part 3 is concerned with proving the nilpotence of $J$, the intersection of the minimal prime ideals of $R*G$. The latter requires work because the multiplication formula [2, lemma 3.3 (ii)] given for the map $\nu$ contains an additional factor $M$ which causes difficulties. To overcome this, it was necessary in [2] to introduce the concept of an $R$-cancelable ideal and to do a number of rather unpleasant computations.

Recently, an alternate extremely useful characterization of the $\nu$ map was discovered in [3]. This characterization, namely the induced ideal map denoted by $\mathcal{G}$, is certainly more natural than $\nu$ and simpler and better understood. Indeed, it clearly yields the correct approach to the necessary correspondences in [2, section 3]. Furthermore, it also satisfies an honest multiplication formula, which thereby trivializes the work of Part 3. This latter formula, given in Lemma 3.4 (i), follows essentially from a short argument due to Deskins [1]. We would like to thank Dr. R. N. Gupta for pointing this reference out to us in the context of an entirely different problem.

Received October 20, 1979
The goal of this addendum is therefore to redo and substantially shorten and simplify [2, section 3] using the induced ideal map $\sigma$. Sections 1, 2 and 3 of this note precisely parallel those of [2].

§1. Introduction

This is unchanged.

§2. Prime coefficient rings

This section is essentially unchanged. The unpleasant [2, lemma 2.8] is no longer needed and should be deleted.

The definition of $R$-cancelable ideals given before [2, lemma 2.3] could be simplified slightly by dropping the requirement that $A$ be a $G$-invariant ideal. This is clear since, in the prime ring $R$, every nonzero ideal contains a nonzero $G$-invariant ideal. In fact the whole concept can be eliminated here by appropriately rephrasing [2, lemma 2.3 (i)]. Note however that either change would require a slight amplification in the proof of [2, theorem 5.6].

We include the following result for the sake of completeness.

**Corollary 2.8.** Let $R \ast G$ be a crossed product of the finite group $G$ over the prime ring $R$. Then we have $0^d = 0$ and $0^u = 0$. In particular $R \ast G$ is prime if and only if $E = C'[G_m]$ is $G$-prime.

**Proof.** The equation $0^u = 0$ is obvious and the equation $0^d = 0$ follows immediately from [2, lemma 2.1 (i)]. [2, theorem 2.5] now yields the result.

§3. $G$-prime coefficient rings

This must be totally changed and should read as follows.

This section contains the proofs of theorems 1.2 and 1.3 of [2]. Throughout, $G$ will denote a finite group and $R \ast G$ will be a crossed product of $G$ over $R$. Then there is a well defined action of $G$ on the set of ideals of $R$ and we will assume that $R$ is a $G$-prime ring. Recall that this means, by definition, that the product of any two nonzero $G$-invariant ideals of $R$ is nonzero. Certainly, this condition is satisfied if there exists a prime ideal $Q$ of $R$ such that $\bigcap_{x \in G} Q^x = 0$. For, if $A_1$ and $A_2$ are $G$-invariant ideals with $A_1A_2 = 0$, then $A_1A_2 \subset Q$ so $A_i \subset Q$ for some $i$. Using the $G$-invariance of $A_n$ we deduce that $A_i \subset \bigcap_{x \in G} Q^x$ and hence $A_i = 0$. Part (i) of the following lemma shows that, conversely, in any $G$-prime ring $R$ one can find such a prime $Q$. 
We remark on a simple property of semiprime rings. Suppose \( R \) is semiprime and let \( A \) and \( B \) be ideals of \( R \) with \( AB = 0 \). Then \( (BA)^2 = 0 \) so \( BA = 0 \). In view of this, left and right annihilators of ideals are equal and we will just use the notation "ann."

**Lemma 3.1.** Let \( R \ast G \) be given and assume that \( R \) is \( G \)-prime. Then

(i) \( R \) contains a prime ideal \( Q \) with \( \bigcap_{x \in G} Q^x = 0 \). In particular, \( R \) is semiprime.

(ii) Any prime ideal of \( R \) contains a conjugate \( Q^x \) of \( Q \) and so \( \{Q^x \mid x \in G\} \) are precisely the minimal primes of \( R \).

(iii) Let \( H \) denote the stabilizer of \( Q \) in \( G \) and let \( N = \text{ann}_R Q \). Then \( H \) is a subgroup of \( G \),

\[
N = \bigcap_{x \not\in H} Q^x \neq 0
\]

and

\[
0 = N\overline{x}N = N \cap N^x = N \cap Q
\]

for all \( x \in G \setminus H \).

(iv) If \( A \) is any nonzero ideal of \( R \) with \( A \subseteq N \), then \( \text{ann}_R A = Q \).

**Proof.** (i) Since \( G \) is finite, an easy application of Zorn's lemma shows that there exists an ideal \( Q \) of \( R \) maximal with respect to the property that \( \bigcap_{x \in G} Q^x = 0 \). Now suppose \( A_1 \) and \( A_2 \) are ideals of \( R \) containing \( Q \) with \( A_1 A_2 \subseteq Q \) and set \( B_i = \bigcap_{x \in G} A_i^x \). Then \( B_1 \) and \( B_2 \) are \( G \)-invariant and since \( B_1 B_2 \subseteq A_1 A_2 \) we have

\[
B_1 B_2 \subseteq \bigcap_{x \in G} (A_1 A_2)^x \subseteq \bigcap_{x \in G} Q^x = 0.
\]

Since \( R \) is \( G \)-prime, we conclude that \( B_i = 0 \) for some \( i \) and then the maximality of \( Q \) implies that \( A_i = Q \). Thus \( Q \) is a prime ideal of \( R \).

(ii) Any prime ideal of \( R \) certainly contains \( \bigcap_{x \in G} Q^x = 0 \) and consequently contains some \( Q^x \), since \( G \) is finite. Furthermore, there are no inclusion relations between the primes \( \{Q^x \mid x \in G\} \). For, if \( Q \nsubseteq Q^x \), then \( Q \nsubseteq Q^{x_n} \) for all \( n \geq 1 \), so by taking \( n = |G| \) we obtain the contradiction \( Q \nsubseteq Q \).

(iii) If \( N \) denotes the annihilator of \( Q \), then \( NQ = 0 \) yields \( NQ \subseteq Q^x \) for all \( x \in G \). Thus if \( x \in G \setminus H \), we deduce from (ii) that \( N \subseteq Q^x \) and we have shown that \( N \subseteq \bigcap_{x \not\in H} Q^x \). Conversely, since

\[
\left( \bigcap_{x \not\in H} Q^x \right) Q \subseteq \bigcap_{x \in G} Q^x = 0,
\]
we have \( N = \text{ann } Q \supseteq \bigcap_{x \in H} Q^x \) and therefore equality occurs. We remark that if \( H = G \), then by definition \( N = \bigcap_{x \in H} Q^x = R \). In any case, by (ii) above we have \( Q \not\supset N \) and hence \( N \neq 0 \). Moreover, \( N \cap Q = \bigcap_{x \in G} Q^x = 0 \) and if \( x \not\in H \) then \( N \subset Q^{x^{-1}} = Q^{x^{-1}} \) so \( N^x \subset Q \). Thus for \( x \not\in H \) we have \( x^{-1}N \cap N = 0 \) and hence \( N = N \subset N \subset N^x \).

(iv) If \( A \subset N \) then clearly \( AQ = 0 \) so \( Q \subset \text{ann } A \). Conversely, suppose \( AB = 0 \). If \( A \neq 0 \) then \( A \not\subset Q \), since \( N \cap Q = 0 \) by part (iii). Thus \( AB = 0 \subset Q \) implies that \( B \subset Q \). This shows that \( Q = \text{ann } A \).

**Notation.** The notation of the preceding lemma will be kept throughout this section. Thus \( Q \) will denote a minimal prime of the \( G \)-prime ring \( R \), \( N \) will be its annihilator in \( R \) and \( H \) will denote the stabilizer of \( Q \) in \( G \). Moreover, we set \( M = \Sigma_{x \in G} N^x \) so that \( M \) is a nonzero \( G \)-invariant ideal of \( R \).

Part (ii) of the next lemma is crucial for the work of this section. Part (i) is needed for its proof.

**Lemma 3.2.** Let \( H \) and \( N \) be as above.

(i) Let \( V \) be a nonzero right \( R \)-submodule of \( N^G \) and let \( T = \{ x_1, x_2, \ldots, x_n \} \) be a subset of \( G \) with \( x_i = 1 \). Suppose that \( V \cap R \ast T \neq 0 \) but \( V \cap R \ast T' = 0 \) for all \( T' \subsetneq T \). Then \( T \subset H \).

(ii) Let \( I \) be an ideal of \( R \ast G \). Then there exists a nonzero \( G \)-invariant ideal \( E \) of \( R \) (depending upon \( I \)) with 

\[
EI \subset \tilde{G}N(I \cap R \ast H)\tilde{G}.
\]

**Proof.** (i) By assumption there exists \( 0 \neq \alpha = \Sigma_{i=1}^n r_i \tilde{x}_i \in V \cap R \ast T \). Since \( V \subset N^G \) we have \( r_i \in N \) for \( i = 1, 2, \ldots, n \) and the minimality condition on \( T \) implies that \( r_i \neq 0 \) for all \( i \). Now for any \( q \in Q \) we have \( \alpha q = \Sigma_{i=1}^n r_i q \tilde{x}_i \in V \) and the first summand on the right, namely \( r_i q \), is contained in \( NQ = 0 \) and hence is zero. Thus the minimality condition on \( T \) yields \( r_i q \tilde{x}_i = 0 \) for all \( i \) and we have \( r_i Q \tilde{x}_i = 0 \). Thus \( r_i Q = 0 \) so \( r_i \tilde{x}_i \in N = \text{ann } Q \) and therefore, since \( r_i \in N \), we have \( 0 \neq r_i \tilde{x}_i \in N \cap N^x \). Lemma 3.1 (iii) now implies that \( x_i \in H \).

(ii) Suppose first that \( NI = 0 \). Then since \( I \) is \( G \)-invariant we have 

\[
MI = \left( \sum_{x \in G} N^x \right) I = 0 = \tilde{G}N(I \cap R \ast H)\tilde{G}.
\]

and so we may take \( E = M \) in this case. Thus we may assume that \( V = NI \neq 0 \). Note that \( V \) satisfies the hypotheses of part (i) and, in addition, \( V \) is a right ideal of \( R \ast G \). Let \( T \) denote the set of all subsets \( T \) of \( G \) such that \( V \cap R \ast T \neq 0 \), \( V \cap R \ast T' = 0 \) for all \( T' \subsetneq T \) and \( 1 \in T \). Then since \( V \) is a right ideal of \( R \ast G \) it
follows easily that $\mathcal{F}$ is a finite nonempty collection of subsets of $G$. Furthermore each $T \in \mathcal{F}$ satisfies $T \subseteq H$ by part (i).

For each $T = \{x_1 = 1, x_2, \cdots, x_n\} \in \mathcal{F}$, let $A_T$ denote the set

$$A_T = \left\{ r \in R \mid \text{there exists } \beta = \sum_{i=1}^{n} r_i x_i \in V \text{ with } r_i = r \right\}.$$

Note that $A_T$ is a nonzero ideal of $R$ which is contained in $N$, since $V$ is an $R$-subbimodule of $NG$. Set $D = \bigcap_{T \in \mathcal{F}} A_T$. Since $\mathcal{F}$ is finite and $A_T \subseteq N$ for all $T \in \mathcal{F}$, it follows from Lemma 3.1 (iii) that $D \neq 0$. We show now, by induction on $m = |\text{Supp } \alpha|$, that if $\alpha \in V$ then $D^{m-1}\alpha \subseteq GN(I \cap R \ast H)G$. The case $m = 0$ is of course trivial.

Now let $\alpha \in V$ be given with $|\text{Supp } \alpha| = m > 0$ and suppose the assertion holds for all elements $\gamma \in V$ of smaller support size. Choose $T \subseteq \text{Supp } \alpha$ minimal with respect to the property that $V \cap R \ast T \neq 0$. If $\gamma \in T$, then $\text{Supp } \alpha^{-1} = (\text{Supp } \alpha) \gamma^{-1} \supseteq Ty^{-1}$, $Ty^{-1}$ also has this minimal property since $V = NI$ is a right ideal of $R \ast G$, and $1 \in Ty^{-1}$. Since it clearly suffices to show that $D^{m-1}\alpha \gamma^{-1} \subseteq GN(I \cap R \ast H)G$, we can replace $\alpha$ by $\alpha \gamma^{-1}$ and $T$ by $Ty^{-1}$ and hence we can assume that $1 \in T$. Thus $1 \in \text{Supp } \alpha$ and $T \in \mathcal{F}$.

Let $c = \text{tr } \alpha$ be the identity coefficient of $\alpha$ and let $d \in D \subseteq A_T$. Then, by definition of $A_T$, there exists an element $\beta \in V \cap R \ast T$ with $\text{tr } \beta = d$. Thus $\gamma = da - bc \in V$ and, since $\text{Supp } \beta \subseteq \text{Supp } \alpha$ and $\text{tr } \gamma = 0$, we have $|\text{Supp } \gamma| < m$. By induction, we deduce that $D^{m}\gamma \subseteq GN(I \cap R \ast H)G$ and hence $D^{m}da \subseteq GN(I \cap R \ast H)G + D^{m}bc$. Now we have observed above that $T \subseteq H$ and hence $\beta c \in V \cap R \ast H \subseteq I \cap R \ast H$. Thus since $m \geq 1$ and $D \subseteq N$, we have $D^{m}bc \subseteq N(I \cap R \ast H) \subseteq GN(I \cap R \ast H)G$. We conclude therefore that $D^{m}da \subseteq GN(I \cap R \ast H)G$ and, since this holds for all $d \in D$, we have $D^{m-1}\alpha \subseteq GN(I \cap R \ast H)G$. The induction step is proved.

In particular, if $k = |G|$, we deduce from the above that $D^{k+1}V = D^{k+1}NI \subseteq GN(I \cap R \ast H)G$. But observe the $D^{k+1}N \neq 0$ since $D \subseteq N$, $D \neq 0$ and $Q \cap N = 0$, by Lemma 3.1 (iii). Thus if $E$ is defined by

$$E = \{ r \in R \mid rI \subseteq GN(I \cap R \ast H)G \}$$

then $E$ is not zero because $E \supseteq D^{k+1}N \neq 0$. On the other hand, $E$ is certainly a $G$-invariant ideal of $R$ since $I$ and $GN(I \cap R \ast H)G$ are ideals of $R \ast G$. Thus we have an appropriate $E \neq 0$ with $EI \subseteq GN(I \cap R \ast H)G$ and the proof is complete.

Note that $Q \ast H$ is an ideal of $R \ast H$. Roughly speaking, our method in this
section is to pass from $R \ast G$ to $(R \ast H)/(Q \ast H) = (R/Q) \ast H$ and thus reduce the general problem to the case of prime coefficient rings where the results of [2, section 2] can be applied. The following definition introduces the necessary machinery.

**DEFINITION.** (i) For any ideal $L$ of $R \ast H$ we set

$$L^G = \bigcap_{x \in G} (L\tilde{G})^x = \bigcap_{x \in G} L^x\tilde{G}.$$  

We will see below that $L^G$ is an ideal of $R \ast G$ and we will characterize it in several different ways.

(ii) If $I$ is an ideal of $R \ast G$, then we set

$$I_H = \{ \alpha \in R \ast H \mid N\alpha \subseteq I \}.$$ 

Since $N$ is $H$-invariant, it follows easily that $I_H$ is an ideal of $R \ast H$. Moreover $N(Q \ast H) = 0$ shows that $I_H \supseteq Q \ast H$, and clearly $I_H = I \cap R \ast H$.

We remark that if $V$ is an $R \ast H$-module with $L = \text{ann}_{R \ast H} V$, then it is fairly easy to see that $L^G$, as defined above, is given by $L^G = \text{ann}_{R \ast G} V^G$ where $V^G$ denotes the induced right $R \ast G$-module $V^G = V \otimes_{R \ast H} R \ast G$. Thus the $G$ notation here is natural and classical and we trust that it will not be confused with $\bar{G}$ as used in [2, section 4] in the study of rings with group actions.

Note that there is a well defined trace map $\tau : R \ast G \to R \ast H$ given by $\tau(\sum_{x \in G} r_x \bar{x}) = \sum_{x \in H} r_x \bar{x}$. In other words, $\tau$ truncates each element of $R \ast G$ to the partial sum of those terms in its support corresponding to group elements contained in $H$. It is clear that $\tau$ is both a right and a left $R \ast H$-module homomorphism. Hence if $I$ is a right ideal of $R \ast G$, then $\tau(I)$ is a right ideal of $R \ast H$ and it is easy to see that $I \subseteq \tau(I)\tilde{G}$. Similarly, if $I$ is a left ideal of $R \ast G$, then $\tau(I)$ is a left ideal of $R \ast H$ and $I \subseteq \tilde{G}\tau(I)$.

**LEMMA 3.3.** Let $L$ be an ideal of $R \ast H$. Then

(i) $L^G$ is the unique largest two sided ideal of $R \ast G$ contained in $L\tilde{G}$.

(ii) $L^G$ is the unique largest two sided ideal of $R \ast G$ satisfying $\tau(I) \subseteq L$.

**PROOF.** (i) If $I$ is an ideal of $R \ast G$ contained in $L\tilde{G}$, then since $I$ is $G$-invariant we have $I \subseteq \bigcap_{x \in G} (L\tilde{G})^x = L^G$. On the other hand, since $L^G = \bigcap_{x \in G} (L\tilde{G})^x = \bigcap_{x \in G} L^x\tilde{G}$, we see that $L^G$ is clearly a left $R$-module, a right $R \ast G$-module and it is $G$-invariant. Thus it is a two sided ideal and hence the largest such contained in $L\tilde{G}$.

(ii) Let $I$ be an ideal of $R \ast G$. If $I \subseteq L\tilde{G}$ then clearly $\tau(I) \subseteq \tau(L\tilde{G}) = L$. On
the other hand, if \( \tau(I) \subseteq L \), then \( I \subseteq \tau(I) \tilde{G} \subseteq L \tilde{G} \). Thus the result follows immediately from (i).

Note that the condition in (ii) above is right-left symmetric and hence we can see that the definition of \( L^G \) is also right-left symmetric. Now obviously, the maps \( ^G \) and \( ^H \) are monotone, as are the maps \( ^a \) and \( ^d \) of [2, section 2]. In fact, as we will see, the maps \( ^G \) and \( ^H \) behave similarly to \( ^a \) and \( ^d \) in many other respects. Indeed the following two lemmas are the analogs of [2, lemmas 2.3 and 2.4].

**Lemma 3.4.** If \( L_1 \) and \( L_2 \) are ideals of \( R \ast H \), then

(i) \( L_1^G L_2^G \subseteq (L_1 L_2)^G \),

(ii) \( L_1^G \cap L_2^G = (L_1 \cap L_2)^G \).

**Proof.** (i) Since \( L_2^G \) is an ideal of \( R \ast G \), we have \( \tilde{G} L_2^G \subseteq L_2^G \) and hence

\[
L_1^G L_2^G \subseteq L_1 \tilde{G} L_2^G \subseteq L_1 L_2^G \subseteq L_1 L_2 \tilde{G}.
\]

Thus, since \( L_1^G L_2^G \) is an ideal of \( R \ast G \), Lemma 3.3 (i) yields \( L_1^G L_2^G \subseteq (L_1 L_2)^G \).

(ii) Since \( ^G \) is monotone we have \( (L_1 \cap L_2)^G \subseteq L_1^G \cap L_2^G \). Conversely, since \( \tau(L_1^G \cap L_2^G) \subseteq L_1 \cap L_2 \) we have \( L_1^G \cap L_2^G \subseteq (L_1 \cap L_2)^G \) by Lemma 3.3 (ii).

**Lemma 3.5.** Given the above notation.

(i) Let \( L \) be an ideal of \( R \ast H \) with \( L \subseteq R \subseteq Q \). Then \( \tilde{G} N \tilde{L} \tilde{G} \subseteq L^G \subseteq L \tilde{G} \) and \( L \subseteq (L^G)_H \).

(ii) If \( I \) is an ideal of \( R \ast G \), then \( M(I_H)^G \subseteq I \). Moreover, there exists a nonzero \( G \)-invariant ideal \( E \) of \( R \) with \( E I \subseteq (I_H)^G \).

**Proof.** (i) If \( x \in H \), then \( x \tilde{N} \tilde{L} \tilde{G} \subseteq L \tilde{G} \) since \( L \) is an ideal of \( R \ast H \). If \( x \notin H \), then

\[
x \tilde{N} \tilde{L} \tilde{G} = N^x L^x \tilde{G} \subseteq Q(R \ast G) \subseteq L \tilde{G}
\]

since \( N^x \subseteq Q \subseteq L \) for \( x \notin H \). Thus \( \tilde{G} N \tilde{L} \tilde{G} \subseteq L \tilde{G} \), and since \( \tilde{G} N \tilde{L} \tilde{G} \) is an ideal of \( R \ast G \), we have \( \tilde{G} N \tilde{L} \tilde{G} \subseteq L^G \subseteq L \tilde{G} \). In particular \( N \subseteq L^G \) and hence, by definition, we have \( L \subseteq (L^G)_H \).

(ii) We have \( N(I_H)^G \subseteq NI_H \tilde{G} \subseteq I \tilde{G} = I \), where the second inclusion holds by definition of \( I_H \). Since \( I \) and \( (I_H)^G \) are \( G \)-stable, it follows that

\[
M(I_H)^G = \sum_{x \in G} N^x(I_H)^G = \sum_{x \in G} (N(I_H)^G)^x \subseteq I.
\]

This proves the first assertion in (ii). As to the second, we know by Lemma 3.2 (ii) that \( E I \subseteq \tilde{G} N(I \cap R \ast H) \tilde{G} \) for a suitable nonzero \( G \)-invariant ideal \( E \) of \( R \).
Furthermore $I \cap R \ast H \subseteq I_h$ and the latter is an ideal of $R \ast H$ containing $Q$. Thus by (i) above, we conclude that

$$EI \subseteq \tilde{G}N(I \cap R \ast H)\tilde{G} \subseteq \tilde{G}NI_h\tilde{G} \subseteq (I_h)^G.$$ 

**Lemma 3.6.** Given the above notation.

(i) If $I$ is an ideal of $R \ast G$ with $I \cap R = 0$, then $I_h \cap R = Q$. In addition, $0_h = Q \ast H$.

(ii) If $L$ is an ideal of $R \ast G$ with $L \cap R = Q$, then $L^G \cap R = 0$. In addition, $(Q \ast H)^G = 0$.

**Proof.** (i) Since $I \cap R = 0$, we have

$$I_h \cap R = \{r \in R \mid Nr \subseteq I \cap R = 0\} = \text{ann}_R N = Q,$$

by Lemma 3.1 (iv). Similarly, since $\text{ann}_R N = Q$, we have $0_h = Q \ast H$.

(ii) Since $L\tilde{G} \cap R = (L\tilde{G} \cap R \ast H) \cap R = L \cap R = Q$, it follows that

$$L^G \cap R = \bigcap_{x \in G} (L\tilde{G})^x \cap R = \bigcap_{x \in G} (L\tilde{G} \cap R)^x = \bigcap_{x \in G} Q^x = 0.$$ 

Similarly

$$(Q \ast H)^G = \bigcap_{x \in G} Q^x \tilde{G} = \left(\bigcap_{x \in G} Q^x\right) \tilde{G} = 0$$

and the result follows.

The following is the main result of this section. We prove it simultaneously with Lemma 3.8.

**Theorem 3.7.** Let $R \ast G$ be a crossed product of the finite group $G$ over the ring $R$. Assume that $R$ is $G$-prime and let $Q$ be a minimal prime of $R$ with $H$ the stabilizer of $Q$ in $G$. Then the maps $^G$ and $^h$ yield a one-to-one correspondence between the prime ideals $P$ of $R \ast G$ with $P \cap R = 0$ and the prime ideals $L$ of $R \ast H$ with $L \cap R = Q$. More precisely:

(i) If $P$ is a prime ideal of $R \ast G$ with $P \cap R = 0$, then $P_h$ is a prime ideal of $R \ast H$ with $P_h \cap R = Q$ and $P = (P_h)^G$.

(ii) Let $L$ be a prime ideal of $R \ast H$ with $L \cap R = Q$. Then $L^G$ is a prime ideal of $R \ast G$ with $L^G \cap R = 0$ and $L = (L^G)_h$.

**Lemma 3.8.** Let $R \ast G$ be given with $R$ a $G$-prime ring. If $P$ is a prime ideal of $R \ast G$ with $P \cap R = 0$ and if $I$ is an ideal of $R \ast G$ properly containing $P$, then $I \cap R \neq 0$.

**Proof.** We start with an observation on a form of cancelation. Let $L$ be a
prime ideal of $R^*H$ with $L \cap R = Q$ and suppose we have $EI \subset L^G$, where $I$ is an ideal of $R^*G$ and $E$ is a nonzero $G$-invariant ideal of $R$. Then $(E^*H)I \subset L^G \subset L^G_H$ and, by applying the trace map $\tau: R^*G \to R^*H$, we have $(E^*H)\tau(I) \subset L$. But certainly $E^*H \not\subset L$ since $E$ is $G$-invariant and $L \cap R = Q$ satisfies $\bigcap_{x \in G} Q^x = 0$. Thus since $L$ is prime we deduce that $\tau(I) \subset L$ and hence that $I \subset L^G$, by Lemma 3.3 (ii).

(i) Let $P$ be a prime ideal of $R^*G$ with $P \cap R = 0$ and set $L = P_H$. By Lemma 3.6 (i), we have $L \cap R = P_H \cap R = Q$. Let us first observe, by Lemma 3.5 (ii), that $P \supset M(P_H)^G = (M^*G)(P_H)^G$. Thus since $P$ is prime and $M^*G \not\subset P$, we see that $P \supset (P_H)^G = L^G$. Next we show that $L$ is prime. Indeed if $L_1$ and $L_2$ are ideals of $R^*H$ containing $L$ with $L_1 \cap L_2$, then Lemma 3.4 (i) yields

$$P \supset L^G \supset (L_1L_2)^G \supset L_1^G L_2^G.$$ 

Hence, since $P$ is prime, $P \supset L_i^G$ for some $i$, and then, by Lemma 3.5 (i), since $L_i \cap R \supset L \cap R = Q$ we have

$$L = P_H \supset (L_i^G)_H \supset L_i.$$ 

Hence $L$ is a prime ideal of $R^*H$ with $L \cap R = Q$. Finally, by Lemma 3.5 (ii), there exists a nonzero $G$-invariant ideal $E$ of $R$ with $EP \subset (P_H)^G = L^G$. We therefore conclude from the above mentioned cancellation property that $P \subset L^G$, so we have equality and part (i) is proved.

(ii) Now let $L$ be a prime ideal of $R^*H$ with $L \cap R = Q$ and set $P = L^G$. Then Lemma 3.6 (ii) asserts that $P \cap R = 0$. Suppose $I$ is any ideal, including $P$ itself, satisfying $I \supset P$ and $I \cap R = 0$. Then, by Lemmas 3.5 (i) and 3.6 (i), we have

$$I_H \supset P_H = (L^G)_H \supset L$$ 

and $I_H \cap R = Q$. Let $\tilde{\gamma}: R^*H \to (R^*H)/(Q^*H) = (R/Q)^*H$ denote the natural homomorphism and observe that both $I_H$ and $L$ contain the kernel $Q^*H$ of this map. Thus $\tilde{I}_H \supset \tilde{L}$, $\tilde{L}$ is a prime ideal of $\tilde{R}^*H$ and clearly $\tilde{I}_H \cap \tilde{R} = 0$. We therefore conclude from [2, lemma 2.6] that $\tilde{I}_H = \tilde{L}$ and hence that $I_H = L$ since $L \supset Q^*H$. In particular, we have $I_H = P_H = (L^G)_H = L$.

Furthermore, by Lemma 3.5 (ii), there exists a nonzero $G$-invariant ideal $E$ of $R$ with

$$EI \subset (I_H)^G = L^G = P.$$ 

Therefore, by the cancelation property discussed in the beginning of this proof,
we deduce that \( I \subseteq L^G = P \) and hence that \( I = P \). It is now a simple matter to see that \( P \) is prime. Indeed if \( I_1 \) and \( I_2 \) are ideals of \( R \ast G \) properly containing \( P \), then by the above we must have \( I_1 \cap R \neq 0 \) and \( I_2 \cap R \neq 0 \). Hence, since \( R \) is \( G \)-prime,

\[
0 \neq (I_1 \cap R)(I_2 \cap R) \subseteq I_1I_2 \cap R
\]

so \( I_1I_2 \not\subseteq P \) and \( P \) is prime. This completes the proof of Theorem 3.7.

Finally, for Lemma 3.8, let \( P \) be a prime ideal of \( R \ast G \) with \( P \cap R = 0 \) and let \( I \) be an ideal of \( R \ast G \) properly containing \( P \). By the above, \( P = L^G \) for some prime ideal \( L \) of \( R \ast H \) with \( L \cap R = Q \). Thus the argument of the preceding paragraph now applies to yield \( I \cap R \neq 0 \) and the lemma is proved.

We remark that Theorem 3.7 has important applications in [3] to the study of prime ideals in group algebras of polycyclic-by-finite groups. Because of this, it is interesting to observe that the above work is entirely independent of [2, section 2] except for one application of Incomparability, [2, lemma 2.6]. However, Incomparability is easy to prove in the case of Noetherian rings, see for example [3, lemma 1.3 (ii)]. Thus the preceding arguments can clearly yield an alternate approach, independent of [2], to the work of [3, section 1].

Returning to the general \( G \)-prime situation, it is clear that the canonical map \( ^*: R \ast H \twoheadrightarrow (R \ast H)/(Q \ast H) = (R/Q) \ast H \) yields a one-to-one correspondence between the primes \( L \) of \( R \ast H \) with \( L \cap R = Q \) and the primes \( \tilde{L} \) of \( \tilde{R} \ast H = (R/Q) \ast H \) with \( \tilde{L} \cap \tilde{R} = 0 \). Thus the maps \( ^* \) and \( ^H \) combine with \( ^* \) to yield a one-to-one correspondence between the set \( \mathcal{P} \) of prime ideals of \( R \ast G \) having trivial intersection with \( R \) and the primes of \( \tilde{R} \ast H \) having trivial intersection with \( \tilde{R} \). For example, the following is an immediate consequence of Theorem 3.7 and Lemma 3.6.

**Corollary 3.9.** With the above notation, \( R \ast G \) is a prime ring if and only if \( (R \ast H)/(Q \ast H) = (R/Q) \ast H \) is prime.

Moreover, since \( \tilde{R} \) is prime, we can then apply the maps \( ^* \) and \( ^H \) of [2, theorem 2.5] to \( \tilde{R} \ast H \) to obtain a one-to-one correspondence between \( \mathcal{P} \) and the \( H \)-prime ideals of a certain twisted group algebra \( C'[H_{\text{fin}}] \), where \( C \) is the extended centroid of \( \tilde{R} \). However, instead of formalizing this further, we will content ourselves with proving the main results of [2], namely [2, theorems 1.2 and 1.3], which now follow quite easily.

**Proof of [2, Theorem 1.2].** In view of the comments made in [2, section 1], this is an immediate consequence of Lemma 3.8.
Proof of [2, Theorem 1.3]. We are given the crossed product $R * G$ with $G$ finite and with $R$ a $G$-prime ring. As usual, let $Q$ be a minimal prime of $R$, as in Lemma 3.1 (i), and let $H$ be the stabilizer of $Q$ in $G$. Furthermore, let $\sim : R * H \to (R * H)/(Q * H) = (R/Q) * H$ denote the natural map. By [2, lemma 2.7], applied to the crossed product $\tilde{R} * H$, we see that there are finitely many primes $\tilde{L}$ of $\tilde{R} * H$ with $\tilde{L} \cap \tilde{R} = 0$. Indeed, if these are $\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_n$, then $n \leq |H| \leq |G|$. Moreover $\tilde{T} = \tilde{L}_1 \cap \tilde{L}_2 \cap \cdots \cap \tilde{L}_n$ is the unique largest nilpotent ideal of $\tilde{R} * H$ and $\tilde{T}^{[H]} = 0$.

For each $i$, let $L_i$ denote the complete inverse image of $\tilde{L}_i$ in $R * H$. Then it follows immediately from Theorem 3.7 that $P_i = L_i^G$ for $i = 1, 2, \cdots, n$ are precisely the prime ideals of $R * G$ having trivial intersection with $R$. Moreover, set $T = L_1 \cap L_2 \cap \cdots \cap L_n$ so that $T$ is the complete inverse image of $\tilde{T}$ in $R * H$. Since $\tilde{T}^{[H]} = 0$, we then have $T^G \subset Q * H$. Setting $J = T^G$ we conclude from Lemmas 3.4 (i) (ii) and 3.6 (ii) that

$$J = T^G = L_1^G \cap L_2^G \cap \cdots \cap L_n^G = P_1 \cap P_2 \cap \cdots \cap P_n$$

and

$$J^{[H]} = (T^G)^{[H]} \subset (T^{[H]})^G \subset (Q * H)^G = 0.$$  

Thus

$$(P_1 \cap P_2 \cap \cdots \cap P_n)^{[H]} = J^{[H]} = 0$$

and this of course implies that $J$ is the unique largest nilpotent ideal of $R * G$ since each $P_i$, being prime, contains all nilpotent ideals of the crossed product.

Finally, let $P$ be any prime ideal of $R * G$. Then $P$ contains the nilpotent ideal $P_1 \cap P_2 \cap \cdots \cap P_n$ and hence $P \supset P_i$ for some $i$. This shows that the minimal primes of $R * G$ are the minimal members of the set $\{P_1, P_2, \cdots, P_n\}$. But $P_i \supset P_i$ implies, by Lemma 3.8, that $P_i = P_n$ since both primes have trivial intersection with $R$, and hence that $i = j$. This shows that $P_1, P_2, \cdots, P_n$ are precisely the minimal primes of $R * G$ and the theorem is proved.

Corollaries 3.9 and 3.10 of [2] should now be included here. However, since these results require no change in either their statements or proofs, we will not bother to include them. Furthermore, we remark that a quite different proof of [2, corollary 3.9] now appears as [4, theorem 7 (ii)].

We close this paper with an alternate characterization of the maps $^G$ and $^H$. Part (i) below of course relates $^G$ to the $^r$ map of [2, section 3].

**Proposition 3.10.** Let $R * G$ be given with $R$ a $G$-prime ring.

(i) If $L$ is a prime ideal of $R * H$ with $L \cap R = Q$, then

$$L^G = \{a \in R * G \mid Ma \subset \tilde{G}NL\tilde{G}\}.$$
(ii) If $P$ is a prime ideal of $R \ast G$ with $P \cap R = 0$, then $P_H$ is the unique minimal covering prime of $P \cap R \ast H$ with $N \subseteq P_H \cap R$.

**Proof.** (i) Let $L$ be given with $L \cap R = Q$ and set $L^* = \{a \in R \ast G \mid Ma \subseteq \mathcal{G}NL_G \}$. Since $L^G \subseteq L_G$, we have $NL^G \subseteq NL_G \subseteq \mathcal{G}NL_G$ and, since both $L^G$ and $\mathcal{G}NL_G$ are $G$-invariant, we deduce that $ML^G \subseteq \mathcal{G}NL_G$. Thus $L^G \subseteq L^*$. Conversely by definition and Lemma 3.5 (i), we have

$$(M \ast G)L^* = ML^* \subseteq \mathcal{G}NL_G \subseteq L^G.$$ 

Thus since $L^G$ is prime, by Theorem 3.7 (ii), and $M \ast G \nsubseteq L^G$, we conclude that $L^* \subseteq L^G$.

(ii) Let $P$ be given with $P \cap R = 0$. Then by definition and Theorem 3.7 (i), we know that $P_H$ is a prime ideal of $R \ast H$ containing $P \cap R \ast H$. Furthermore, $P_H \cap R = Q$ and hence $P_H \cap R \nsubseteq N$. Now let $L$ be any prime ideal of $R \ast H$ with $L \supset P \cap R \ast H$ and $L \cap R \nsubseteq N$. Then, by definition,

$$L \supset P \cap R \ast H \supset NP_H = (N \ast H)P_H$$

and hence, since $L \nsubseteq N \ast H$, we have $L \supset P_H$. This shows first, by taking $P_H \supset L \supset P \cap R \ast H$, that $P_H$ is a minimal covering prime of $P \cap R \ast H$. Then we see that $P_H$ is the unique such minimal covering prime with the additional property that $P_H \cap R \nsubseteq N$. This completes the proof.

**Acknowledgment**

The first author's research was supported by Deutsche Forschungsgemeinschaft Grant No. Lo. 261/1. The second author's research was partially supported by NSF Grant No. MCS 77-01775 A01.

**References**


University of Essen
4300 Essen 1, FRG

University of Wisconsin — Madison
Madison, Wisconsin 53706 USA