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Primitive Ideals of Group Algebras of Supersoluble Groups

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Introduction

A group G (not necessarily finite) is *polycyclic* if G has a subnormal series $\langle 1\rangle = G_0 \subset G_1 \subset \ldots \subset G_{n-1} \subset G_n = G$ such that $G_{i-1} \lhd G_i$ and G_i/G_{i-1} is cyclic ($i=1,2,\ldots,n$). If in addition the groups G_i appearing in the above series are normal in the whole group G then G is called *supersoluble*. The group G is *polycyclic-by-finite* if G has a normal polycyclic subgroup of finite index. It is well-known that the group ring $R[G]$ of a polycyclic-by-finite group G over a noetherian ring R is noetherian (Hall [5], Theorem 1).

This note aims to characterize the primitive ideals (i.e. the kernels of the simple left modules) of the group algebra $\mathscr{k}[G]$ of a supersoluble group G over a perfect field $\mathscr{k}$. The methods used and the statement of the results have been very much influenced by the great success of Lie algebra theory on this subject, because in a sense, supersoluble groups can be considered as the formal group theoretic analogue of completely solvable finite-dimensional Lie algebras [i.e. Lie algebras $\mathfrak{g}$ having a series $0=\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}$ of ideals $\mathfrak{g}_i$ such that $\dim(\mathfrak{g}_i)=i$].

Section 1 deals with the group algebra $\mathscr{k}[G]$ of a general polycyclic-by-finite group G. Using methods developed by Hall in [6] we prove that the endomorphism ring of a simple $\mathscr{k}[G]$-module is finite-dimensional over the ground field $\mathscr{k}$ if the latter is perfect. We then give a short proof of the well-known fact that $\mathscr{k}[G]$ is a Jacobson ring (i.e. the Jacobson radical of every homomorphic image is nilpotent, cf. [8, 4]).

In Section 2 the notion of the semicentre for factor algebras of group algebras is defined. The corresponding notion has been a very useful tool in the study of enveloping algebras of solvable Lie algebras (cf. [2], § 6). The main result of this section is a version of Smith's Theorem A in [9] and states that: given two ideals $I \subsetneqq J$ in the group algebra of a supersoluble group G over an algebraically closed field $\mathscr{k}$, it is possible to find a homomorphism $\lambda \in \mathrm{Hom}(G, \mathscr{k}^{\cdot})$ and an element $\alpha \in J \setminus I$ such that $\lambda(G)$ is finite and $\alpha^x - \lambda(x)\alpha \in I$ for all $x \in G$.

Section 3 uses the above to prove the main result of this note (Theorem 3.3): Let G be a supersoluble group, $\mathscr{k}$ a perfect field and I a prime ideal of the group algebra $\mathscr{k}[G]$. Then the following are equivalent: (i) I is primitive. (ii) The centre

$Z(k[G]/I)$ of $k[G]/I$ is a finite algebraic field extension of k. (iii) I is maximal. (iv) I is locally closed in $\operatorname{Spec} k[G]$. – Here as usual $\operatorname{Spec} k[G]$ denotes the set of prime ideals in $k[G]$, endowed with the Jacobson topology (see [2], 1.2). I is called locally closed if $\{I\}$ is a locally closed subset of $\operatorname{Spec} k[G]$ in this topology, that is the intersection of all prime ideals strictly containing I is distinct from I. Theorem 3.3 generalizes results of Zalesskij on group algebras of finitely generated nilpotent groups ([10], Theorem 1, Theorem 3).

The final Section 4 gives a method how to construct counterexamples to Theorem 3.3 in the case of general polycyclic-by-finite groups.

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1. Endomorphism Rings of Simple Modules

Throughout this note "module" will mean "left module" and "ideal" stands for "two-sided ideal".

(1.1) A suitable adaption of the proof (due to Gabriel) given in ([3], 2.6.9) yields the following technical

Lemma. *Let k be a field and A a k-algebra such that for any extension field K of k the K-algebra $A \bigotimes_k K$ is noetherian. Furthermore let V be a completely reducible A-module of finite length. Then for any separable algebraic field extension K/k the $\left(A \bigotimes_k K\right)$-module $V \bigotimes_k K$ is completely reducible of finite length.*

(1.2) **Theorem.** *Let G be a polycyclic-by-finite group, k a field and V an irreducible $k[G]$-module. Then every element of the endomorphism ring $D := \operatorname{End}_{k[G]}(V)$ is algebraic over k. Furthermore, if k is perfect, $\dim_k(D) < \infty$.*

Proof. (1) Recall the following result of Hall ([6], Lemma 3): Let $J := k[\langle t \rangle]$ be the group algebra of the infinite cyclic group $\langle t \rangle$ over k. Then a finitely generated $J[G]$-module cannot contain a J-submodule which is isomorphic with the field of fractions $Q(J)$ of J.

(2) Let $0 \neq x \in D$. We show that x is algebraic over k. If not, the subalgebra $J := k[x, x^{-1}]$ of the division ring D generated by $1, x, x^{-1}$ can be considered as the group algebra $k[\langle x \rangle]$ of the infinite cyclic group $\langle x \rangle$ over k. The $k[G]$-module V can be viewed as a module over the group ring $J[G] = J \bigotimes_k k[G]$ via the action $(j \otimes \alpha) \cdot v = j(\alpha \cdot v) = \alpha \cdot j(v) (j \in J, \alpha \in k[G], v \in V)$. The simplicity of ${}_{k[G]}V$ implies V to be a cyclic $J[G]$-module.

The field of fractions $Q(J)$ of J is contained in D and hence acts on V. Let $0 \neq v \in V$. Then $Q(J) \cdot v$ is a J-submodule of V that is isomorphic to $Q(J)$ via the map $q \mapsto q(v)$ $(q \in Q(J))$. This contradicts the result mentioned in (1) and concludes the proof of the first part of the theorem.

(3) As to the second assertion let $\hat{k}$ be an algebraic closure of k. By (1.1) $V \bigotimes_{k} \hat{k}$ is a direct sum of finitely many irreducible $\hat{k}[G]$-modules $\hat{V}_i$ $(i=1,2,\ldots,n)$. Therefore

$$D \bigotimes_{k} \hat{k} \subset \mathrm{End}_{\hat{k}[G]}\left(V \bigotimes_{k} \hat{k}\right) = \prod_{i,j=1}^{n} \mathrm{Hom}_{\hat{k}[G]}(\hat{V}_i, \hat{V}_j).$$

Since by (2) endomorphisms of simple $\hat{k}[G]$-modules are scalar, it follows that $\dim_k(D) = \dim_{\hat{k}}\left(D \bigotimes_{k} \hat{k}\right)$ is finite. Thus the theorem is proved.

(1.3) An ideal I of the ring R is called *semiprime* if $I \neq R$ and the factor R/I has no nonzero nilpotent ideals. I is *primitive* if R/I has a faithful simple module, i.e. a simple module $M \neq 0$ such that $a \cdot M = 0$ for $a \in R/I$ implies $a = 0$. The intersection of all primitive ideals of R containing a given ideal I is just the inverse image in R of the Jacobson radical of R/I. – From (1.2) one derives the following

Corollary. *Let G be a polycyclic-by-finite group and k a field. Then any semiprime ideal of the group algebra $k[G]$ is an intersection of primitive ideals of $k[G]$.*

Proof. Let I be a semiprime ideal of $k[G]$. We have to show that the Jacobson radical J of the algebra $A := k[G]/I$ is zero. Consider the direct product $H := G \times \langle x \rangle$ of G with an infinite cyclic group $\langle x \rangle$ and let $I' := Ik[H]$ be the two-sided ideal of $k[H]$ generated by I. Then the factor $k[H]/I'$ is isomorphic to the group ring $B := A[\langle x \rangle]$. (The isomorphism is given by $\sum \alpha_i x^i + I' \mapsto \sum [\alpha_i + I] x^i, \alpha_i \in k[G]$.) Choose $a \in J$. It will suffice to show that a is nilpotent, for then J is a nil ideal of the semiprime noetherian ring A and hence zero.

Claim. $B(1-ax) = B$.

Pf. If not, $B(1-ax)$ is contained in a maximal left ideal L of B. Let V be the simple B-module $V := B/L$, $v_0 := 1 + L \in V$ and x_V the endomorphism $v \mapsto x \cdot v$ of V. Since x is central in B it follows that $x_V \in \mathrm{End}_B(V)$. Therefore, by (1.2), x_V is algebraic over k. Furthermore being induced by a group element x_V is obviously invertible in $\mathrm{End}_B(V)$. Set $y := x_V^{-1} \in \mathrm{End}_B(V)$. For some polynomial $f \in k[X]$, the polynomial algebra in one indeterminate X over k, one has $x_V = f(y)$. The equation $a \cdot x_V(v_0) = v_0$ gives $a \cdot v_0 = y(v_0)$ and therefore $(1 - af(a)) \cdot v_0 = (1 - yf(y))(v_0) = 0$, a contradiction to the fact that $1 - af(a)$ is contained in $1 + J$ and hence is invertible.

Thus we may write $1 = (a_{-m}x^{-m} + \ldots + a_{-1}x^{-1} + a_0 + a_1 x + \ldots + a_n x^n)(1 - ax)$ for suitable $a_i \in A$. Comparing degrees one obtains $a_{-m} = a_{-m+1} = \ldots = a_{-1} = 0$, $a_0 = 1$, $a_1 = a$, $a_2 = a^2, \ldots, a_n = a^n$, $a^{n+1} = 0$. This finishes the proof.

2. Semicentres

(2.1) Let G be a group, k a field and I an ideal of the group algebra $k[G]$. Then G acts on the factor $A := k[G]/I$ according to $(\alpha + I)^x = \alpha^x + I$ $(x \in G, \alpha \in k[G])$. Set $G^* := \mathrm{Hom}(G, k^{\cdot})$, and for $\lambda \in G^*$ set $A^\lambda := \{a \in A : a^g = \lambda(g) a \text{ for all } g \in G\}$. In case $A^\lambda \neq \{0\}$ λ is called an *eigenvalue* of G in A and an element $0 \neq a \in A^\lambda$ is called *semiinvariant*. Collect the eigenvalues of G in A in the subset $\mathscr{E}(A)$ of G^* and define the *semicentre* $Sc(A)$ of A to be the sum of the eigenspaces A^λ, $\lambda \in \mathscr{E}(A)$, in A. Using standard arguments one shows that this sum is direct, hence

$$Sc(A) = \sum_{\lambda \in \mathscr{E}(A)} \oplus A^{\lambda}.$$

(2.2) *Remarks.* a) $Sc(A)$ is a subalgebra of A: For $\lambda, v \in \mathscr{E}(A)$ one easily verifies the inclusion $A^{\lambda} A^{v} \subset A^{\lambda \cdot v}$. Here $\lambda \cdot v \in G^*$ is defined by $(\lambda \cdot v)(x) = \lambda(x) v(x)$ $(x \in G)$.

b) Obviously the centre $Z(A)$ of A is just the eigenspace $A^{\mathbb{1}}$ corresponding to the character of G given by $\mathbb{1}(x) = 1$ for all $x \in G$. Thus $Z(A) \subset Sc(A)$. In case $G = [G, G]$ one has the equality $Z(A) = Sc(A)$, while in general the semicentre can be strictly greater than the centre of A.

c) For any semiinvariant element $e \in A$ one has $eA = Ae$. If in addition A is prime, e is a regular element of A: $eb = 0$ for some $b \in A$ implies $0 = Aeb = eAb$ and hence $b = 0$. Analoguously $be = 0$ implies $b = 0$. Thus $\{1, e, e^2, \ldots\}$ is an Ore subset of A (see [2], 2.2). In the same way one shows that a semiinvariant element $e \in A$ is not nilpotent provided I is semiprime.

d) If $k[G]$ is noetherian and I is prime one can form the (classical) quotient ring $Q(A)$ of $A = k[G]/I$. Suppose for some $\lambda \in \mathscr{E}(A)$ elements $a, b \in A^{\lambda}$ are given, with $b \neq 0$. Then by c) b is invertible in $Q(A)$. Furthermore remarks a) and b) show that $ab^{-1} \in Z(Q(A))$ (cf. 3.2b).

(2.3) The following theorem is based on the proof of Smith's result ([9], Theorem A).

Theorem (P. F. Smith). *Let k be an algebraically closed field and G a supersoluble group. If $I \subsetneq J$ are ideals of the group algebra $k[G]$ then there exists $\lambda \in G^*$ such that $\lambda(G)$ is finite and $J/I \cap (k[G]/I)^{\lambda} \neq 0$.*

Proof. Let $\langle 1 \rangle = G_0 \subset G_1 \subset \ldots \subset G_n = G$ be a series such that the groups G_i are normal in G and the factors G_i/G_{i-1} are cyclic. The proof is by induction on the subscript i of the groups G_i in the above series. For any normal subgroup V of G such that G/V is abelian and any index $i \in \{0, 1, \ldots, n\}$ let (V, i) denote the following statement:

(V, i): If $I \subsetneq J$ are V-stable ideals of $k[G_i]$ then there exist $\gamma \in J \setminus I$ and $\lambda \in V^*$ such that $\lambda(V)$ is finite, $\lambda | [G, G] = \mathbb{1}$ and $\gamma^x - \lambda(x)\gamma \in I$ for all $x \in V$.

Thus the assertion of the theorem is just (G, n) and our i^{th} induction statement will be:

For every normal subgroup V of G such that G/V is finite abelian the assertion (V, i) is true.

The case $i = 0$ being trivial we proceed to prove the induction step. Choose $V \lhd G$ such that G/V in finite abelian and set $U := V \cap K_i$, where K_i denotes the kernel of the natural map $G \to \mathrm{Aut}(G_{i+1}/G_i)$. Then U is a normal subgroup of G and, since $\mathrm{Aut}(G_{i+1}/G_i)$ is a finite abelian group, the factor G/U is finite abelian. In order to prove $(V, i+1)$ we proceed in two steps

(1) $(U, i+1)$ is true.

(2) If $D \subset E$ are normal subgroups of G such that G/D is finite abelian and E/D is cyclic then $(D, i+1)$ implies $(E, i+1)$.

Consider a series $U = U_0 \subset U_1 \subset \ldots \subset U_s = V$ of normal subgroups of G such that U_i/U_{i-1} is cyclic. Then $(U, i+1)$ together with (2) finally yields $(V, i+1)$.

Set $R := k[G_{i+1}]$, $S := k[G_i]$.

Proof of (1). Let $\{g_l\}_{l\in M}$ be a transversal for G_i in G_{i+1}, M some index set such that $1\in M$, $g_1=1$. The elements of R are uniquely expressible in the form $\alpha=\sum_{l\in M}\alpha_l g_l$, where $\alpha_l\in S$. Set $S(\alpha):=\{g_l: l\in M, \alpha_l\neq 0\}$ and call card $S(\alpha)$ the length of α. Choose an element $\alpha\in J\setminus I$ of minimal length among the elements of $J\setminus I$. Eventually multiplying on the right by a suitable group element we can clearly assume that $1\in S(\alpha)$. Using the definition of U one easily sees that

$$C_\alpha(I):=\Big\{\gamma_1\in S:\exists\gamma=\gamma_1+\sum_{1\neq l\in M}\gamma_l g_l\in I, \gamma_l\in S, S(\gamma)\subset S(\alpha)\Big\}$$

and the analoguously defined $C_\alpha(J)$ are U-stable ideals of S such that $C_\alpha(J)\supset C_\alpha(I)$. Clearly $\alpha_1\in C_\alpha(J)$, and the minimality of α implies that $\alpha_1\notin C_\alpha(I)$. Therefore by assumption (U,i), there are $\delta_1\in C_\alpha(J)\setminus C_\alpha(I)$ and $\lambda\in U^*$ such that $\lambda(U)$ is finite, $\lambda|[G,G]=\mathbb{1}$ and $\delta_1^x-\lambda(x)\delta_1\in C_\alpha(I)$ for all $x\in U$. Choose $\delta\in J$ such that $\delta=\delta_1+\sum_{1\neq l\in M}\delta_l g_l$, $\delta_l\in S$, $S(\delta)\subset S(\alpha)$ and for each $x\in U$ choose $\gamma_x\in I$ such that $\gamma_x=\delta_1^x-\lambda(x)\delta_1+\sum_{1\neq l\in M}\gamma_{x,l}g_l$, $\gamma_{x,l}\in S$, $S(\gamma_x)\subset S(\alpha)$. Then $\delta\notin I$ since $\delta_1\notin C_\alpha(I)$. For $x\in U$ one obtains

$$\delta^x-\lambda(x)\delta=\gamma_x+\Big(-\sum_{1\neq l\in M}\gamma_{x,l}g_l+\sum_{1\neq l\in M}(\delta_l^x[x,g_l^{-1}]-\lambda(x)\delta_l)g_l\Big).$$

Since $S(\gamma_x)$, $S(\delta)\subset S(\alpha)$ and $[x,g_l^{-1}]\in G_i$, the term in the brackets has shorter length than α. The minimality of α implies $\delta^x-\lambda(x)\delta\in I$. Thus (1) is proved.

Proof of (2). Write $E=\langle D,e\rangle$, where $e^m\in D$. Consider E-stable ideals $I\subsetneqq J$ of R. In particular I and J are D-stable and hence by the assumption $(D,i+1)$ there exist $\gamma\in J\setminus I$ and $\lambda\in D^*$ such that $\lambda(D)$ is finite, $\lambda|[G,G]=\mathbb{1}$ and $\gamma^x-\lambda(x)\gamma\in I$ for all $x\in D$. Let L be the finite-dimensional k-vector space $L:=(k\gamma+k\gamma^e+k\gamma^{e^2}+\dots+k\gamma^{e^{m-1}}+I)/I$ and let $\Phi\in\operatorname{End}_k(L)$ be induced by the automorphism $r\mapsto r^e$ of R. Since k is algebraically closed, Φ has an eigenvalue $\xi\in k$. If $0\neq\sum_{i=0}^{m-1}k_i\gamma^{e^i}+I$ $(k_i\in k)$ is a corresponding eigenvector in L, then $\delta:=\sum_{i=0}^{m-1}k_i\gamma^{e^i}\in J\setminus I$ and $\delta^e-\xi\delta\in I$. For $x\in D$ one obtains $\gamma^{e^i x}=\gamma^{[e^{-i},x^{-1}]xe^i}=\gamma^{xe^i}=\lambda(x)\gamma^{e^i}\bmod I$ since $[G,G]\subset\operatorname{Ker}\lambda$. Therefore for all $x\in D$: $\delta^x-\lambda(x)\delta\in I$. Thus one obtains a homomorphism $\tilde\lambda\in E^*$ such that $\delta^x-\tilde\lambda(x)\delta\in I$ for all $x\in E$, $\tilde\lambda|D=\lambda$, $\tilde\lambda(e)=\xi$. Since $e^m\in D$ one has $\xi^m\in D$ $\xi^m\in\tilde\lambda(D)=\lambda(D)$ and hence $\xi^n=1$ for a suitable n. It follows that $\tilde\lambda(E)$ is finite. This concludes the proof of (2) and of (2.3).

3. Primitive Ideals

(3.1) **Lemma.** *Let G be a polycyclic-by-finite group, k a perfect field and I an ideal of the group algebra $k[G]$. For an extension field k' of k let $I':=I\otimes_k k'$ be the ideal of $k'[G]$ generated by I. If I is semiprime then so is I'.*

Proof. With the obvious notational changes the proof given in [3], 3.4.2 (see also [2], 3.10) carries over.

(3.2) **Proposition.** *Let G be a supersoluble group, k a perfect field and I a semiprime ideal of the group algebra $k[G]$. Set $A := k[G]/I$.*

(a) *If J is an ideal of $k[G]$ strictly containing I then $J/I \cap Z(A) \neq 0$.*

(b) *If in addition I is prime and $Q(A)$ is the (classical) quotient ring of A then $Z(Q(A)) = Q(Z(A))$.*

Proof. (a) First suppose k to be algebraically closed. Then for any ideal J of $k[G]$ strictly containing I there exists $\lambda \in G^*$ such that $\operatorname{card} \lambda(G) =: n < \infty$ and $J/I \cap A^\lambda \neq 0$ (2.3). Choose $0 \neq a \in J/I \cap A^\lambda$. Then $a^n \neq 0$ since I is semiprime (2.2c). Furthermore $\lambda^n(x) = 1$ for all $x \in G$ and hence by (2.2a) $0 \neq a^n \in A^{\mathbb{1}} = Z(A)$ and clearly $a^n \in J/I$.

In the general case let $\hat{k}$ be an algebraic closure of k and let $\hat{I} := I \bigotimes_k \hat{k}$, $\hat{J} := J \bigotimes_k \hat{k}$ be the ideals of $\hat{k}[G]$ generated by I, J. Then $\hat{I} \subsetneqq \hat{J}$, and by (3.1), $\hat{I}$ is semiprime. Therefore there exists $0 \neq \hat{a} = \hat{\alpha} + \hat{I} \in \hat{J}/\hat{I} \cap Z(\hat{k}[G]/\hat{I})$ $(\hat{\alpha} \in \hat{J})$. Using a k-basis $\{k_i\}_{i \in M}$ of $\hat{k}$ write $\hat{\alpha}$ in the form $\hat{\alpha} = \sum_{i \in M_0} \alpha_i k_i$, $M_0 \subset M$ a finite set, $\alpha_i \in J \setminus I$. Then obviously for all $i \in M_0$: $0 \neq \alpha_i + I \in J/I \cap Z(A)$.

(b) Let c be a central element of $Q(A)$. The set of all elements $a \in A$ such that $ac \in A$ forms a nonzero two-sided ideal of A. Therefore, by (a), we can find $0 \neq z \in Z(A)$ such that $zc \in A$. Obviously $zc \in Z(A)$. Now since I is prime, z is regular in A and hence invertible in $Q(A)$. Thus $Z(Q(A)) \subset Q(Z(A))$. The other inclusion is trivial.

(3.3) **Theorem.** *Let G be a supersoluble group and k a perfect field. Then for any prime ideal I of the group algebra $k[G]$ the following properties are equivalent:*

(i) *I is primitive.*

(ii) *The centre $Z(k[G]/I)$ of $k[G]/I$ is a finite algebraic field extension of k.*

(iii) *I is maximal.*

(iv) *I is locally closed in* $\operatorname{Spec} k[G]$.

Proof. (i)⇒(ii). Let V be an irreducible $k[G]$-module with kernel I. Then $Z(k[G]/I)$ is in a natural way embedded in $Z(\operatorname{End}_{k[G]}(V))$. Application of (1.2) yields the result. (ii)⇒(iii). This follows immediately from (3.2a). Finally (iii)⇒(iv) is trivial and (iv)⇒(i) is a consequence of (1.3).

4. Counterexamples

(4.1) In the present form (3.3) does not extend to group algebras of general polycyclic-by-finite groups: Let G be a polycyclic group having all nontrivial conjugacy classes of infinite order and let k be an absolute field, i.e. a field that is algebraic over a finite field. Then by a result of Roseblade ([8], Theorem A), $k[G]$ is certainly not primitive. On the other hand $k[G]$ is prime and $Z(k[G]) = k$. Thus the ideal $I = 0$ satisfies (ii) but not (i).

(4.2) Another more interesting example dealing with non absolute fields will be given below. We first state a slightly more general result suggested by the referee. Recall that if H is a group acting on a ring S, then the *crossed product* $S_\alpha[H]$ of S and H with respect to the action $\alpha: H \to \operatorname{Aut}(S)$ is a ring that is free as a right S-module with basis the elements of H. The multiplication is defined distributively extending the rule $xr \cdot ys = xyr^{\alpha(y)}s$ $(x, y \in H, r, s \in S)$.

Proposition. *Let k be an algebraically closed field and let X be an irreducible affine k-variety with coordinate ring S. Furthermore let H be a group acting faithfully on X and let $R := S_\alpha[H]$ be the crossed product of S and H with respect to the induced action $\alpha: H \to \mathrm{Aut}(S)$ of H on S. Then:*

(a) *R is prime.*

(b) *If X contains a dense H-orbit, then R is primitive.*

(c) *If R is noetherian, then the ideal $I=0$ of R is locally closed in* $\mathrm{Spec}\, R$ *if and only if the union of all H-orbits that are not dense in X is not dense.*

Proof. (1) *If J is a nonzero ideal of R, then $J \cap S \neq 0$.* Every element $\alpha \in R$ can be written uniquely in the form $\alpha = \sum_{i=1}^{n} h_i s_i$, where $h_i \in H$ are distinct and s_i are nonzero elements of S. Call the number of summands occuring in such an expression the length of α and choose $0 \neq \alpha \in J$ of minimal length n among the nonzero elements of J. After multiplying with a suitable element of H if necessary, we may assume that $h_1 = 1$. The assertion will be proved if we can show that $n=1$. Assume $n > 1$. Then $h_n \neq 1$ and there exists an element $s \in S$ such that $s^{h_n} \neq s$. (We write s^h instead of $s^{\alpha(h)}$.) The element $\beta := s\alpha - \alpha s = \sum_{i=2}^{n} h_i(s^{h_i} - s)s_i \in J$ is nonzero, because $s^{h_n} - s$, $s_n \neq 0$ and S has no zero divisors by the irreducibility of X. Since β has shorter length than α we have the desired contradiction.

(2) *Proof of (a).* If A, B are nonzero ideals of R, then by (1) $A \cap S$ and $B \cap S$ are nonzero. Therefore $(A \cap S)(B \cap S) \neq 0$ and $AB \neq 0$.

(3) *Proof of (b).* The existence of a dense H-orbit is equivalent to the existence of a maximal ideal I of S such that $\bigcap_{h \in H} I^h = 0$. Consider the left ideal RI of R. Since R is free over S, it follows that $RI \neq R$. Hence we can choose a maximal left ideal L of R containing RI. Let V be the irreducible R-module $V := R/L$. Then as S-modules $V \supset S + L/L \cong S/S \cap L = S/I$. Therefore $A \cap S \subset I$, where $A := Ann_R(V)$. Since $A \cap S$ is clearly an H-stable ideal of S, it follows that $A \cap S \subset \bigcap_{h \in H} I^h = 0$. Finally $A = 0$, by (1). Thus V is a faithful irreducible R-module and R is primitive.

(4) *If J is a semiprime ideal of R, then $J \cap S$ is a semiprime ideal of S.* Let $\mathcal{M} := \{P_1, \ldots, P_n\}$ be the set of minimal prime ideals of S containing $J_S := J \cap S$. Since J_S is H-stable, H operates on $\mathcal{M}$. It follows that the radical $\sqrt{J_S} = \bigcap_{i=1}^{n} P_i$ of the ideal J_S is H-stable. Hence $R\sqrt{J_S}$ is a two-sided ideal of R. Furthermore there exists an n such that $\sqrt{J_S}^n \subset J_S$. It follows that $(R\sqrt{J_S})^n = R\sqrt{J_S}^n \subset RJ_S \subset J$. Therefore, since J is semiprime, $R\sqrt{J_S} \subset J$ and $\sqrt{J_S} \subset J \cap S = J_S$.

(5) *Proof of (c).* First suppose that the union of all H-orbits in X that are not dense is dense, i.e. there are maximal ideals I_α, $\alpha \in A$, of S such that $D(I_\alpha) := \bigcap_{h \in H} I_\alpha^h \neq 0$ for all $\alpha \in A$ but $\bigcap_{\alpha \in A} D(I_\alpha) = 0$. Each $D(I_\alpha)$ is a semiprime H-stable ideal of S. Let I'_α be the two-sided ideal $I'_\alpha := RD(I_\alpha)$ of R and let $J_\alpha := \sqrt{I'_\alpha}$ be the radical of I'_α. Since R is noetherian, J_α is a semiprime ideal of R such that $J_\alpha^n \subset I'_\alpha$ for some n. Therefore $(J_\alpha \cap S)^n \subset I'_\alpha \cap S = D(I_\alpha)$ and hence $J_\alpha \cap S = D(I_\alpha)$. It follows that $\left(\bigcap_{\alpha \in A} J_\alpha\right) \cap S = \bigcap_{\alpha \in A} D(I_\alpha) = 0$ and, by (1), $\bigcap_{\alpha \in A} J_\alpha = 0$. Thus the ideal $I=0$ is the intersection of nonzero prime ideals and therefore is not locally closed in $\mathrm{Spec}\, R$.

Conversely, suppose $\bigcap_{\alpha\in A} J_\alpha = 0$, where $\{J_\alpha\}_{\alpha\in A}$ denotes the set of nonzero prime ideals of R. Then $0 = \bigcap_{\alpha\in A} (J_\alpha \cap S)$ and, by (1) and (4), each $J_\alpha \cap S$ is a nonzero semiprime ideal of S. The Jacobson property of S implies that $J_\alpha \cap S$ is the intersection of all maximal ideals of S containing $J_\alpha \cap S$. Collect these ideals in $\mathscr{V}$. Certainly H operates on $\mathscr{V}$ and hence $J_\alpha \cap S = \bigcap_{M\in\mathscr{V}} \bigcap_{h\in H} M^h$. Let x_M be such that $\{x_M\}$ is the set of zeros of M and let $\mathcal{O}_M$ be the H-orbit $\mathcal{O}_M := x_M^H$. Then $\mathcal{O}_M$ is not dense, since its annihilating ideal $\mathscr{I}(\mathcal{O}_M) = \bigcap_{h\in H} M^h$ is nonzero. But the union $\bigcup_{\alpha\in A} \bigcup_{M\in\mathscr{V}} \mathcal{O}_M$ is dense, because $\mathscr{I}\left(\bigcup_{\alpha\in A} \bigcup_{M\in\mathscr{V}} \mathcal{O}_M\right) = \bigcap_{\alpha\in A} \bigcap_{M\in\mathscr{V}} \mathscr{I}(\mathcal{O}_M) = \bigcap_{\alpha\in A} (J_\alpha \cap S) = 0$.

This finishes the proof.

(4.3) We close with the promised example: Let $A = \langle x\rangle \times \langle y\rangle$ be a free abelian group of rank 2 and let $z \in \mathrm{Aut}(A)$ be defined by $x^z = x^2 y$, $y^z = xy$. Consider the group algebra $k[G]$ of the semidirect product $G := A \otimes_\sigma \langle z\rangle$ over the algebraically closed field k. Then $k[A]$ can be considered as the coordinate ring S of the variety $X := k^{\cdot} \times k^{\cdot}$. If we let $\langle z\rangle$ act on X according to $(c, d)^z := (cd^{-1}, c^{-1}d^2)$ $(c, d \in k^{\cdot})$, then $k[G]$ is isomorphic to the crossed product of S and $\langle z\rangle$. The orbits of the action of $\langle z\rangle$ on X are easily described (We omit the verifications):

(1) All infinite $\langle z\rangle$-orbits are dense in X.

(2) If $E \subset k^{\cdot}$ denotes the set of roots of unity in k then $E \times E$ is the union of all finite $\langle z\rangle$-orbits in X.

Now suppose k to be non absolute. Then $E \neq k^{\cdot}$ and hence there are infinite $\langle z\rangle$-orbits in X. By (4.2b) together with (1), we conclude that $k[G]$ is primitive. Finally, since $E \times E$ is dense in X, (4.2c) and (2) show that the ideal $I = 0$ is not locally closed in Spec $k[G]$. – We remark that the primitivity of $k[G]$ also follows from a result of Passman ([7], Corollary 7.9) that is based on Bergman's work in [1].

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