1. Let $K$ be a field, and let $\sigma$ be an automorphism of $K$. Prove that $\sigma$ pointwise fixes (i.e., restricts to the identity on) the prime subfield of $K$.

2. Consider the subfield, 
   
   $$K = \mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \},$$

   of $\mathbb{R}$. (i) Show that the assignment 
   
   $$a + b\sqrt{2} \mapsto a - b\sqrt{2},$$

   for $a, b \in \mathbb{Q}$, produces an automorphism $\sigma$ of $K$.

   (ii) Prove that $\sigma$ and the identity map are the only automorphisms of $K$.

3. Let $L$ be a finite field extension of $K$.
   (i) Prove that there exists a finite set $\{ f_1(x), \ldots, f_m(x) \}$ of polynomials in $K[x]$ such that $L = K(\alpha_1, \ldots, \alpha_n)$, where $S = \{ \alpha_1, \ldots, \alpha_n \}$ is the set of all roots in $L$ of the $f_1(x), \ldots, f_m(x)$.

   (ii) Retaining the notation from (i), prove that every automorphism of $L$ fixing $K$ restricts to a permutation of the set $S$; let $\text{Sym}(S)$ denote the group of permutations of $S$.

   (iii) Prove that the map $\Phi : \text{Aut}(L/K) \to \text{Sym}(S)$, sending each automorphism $\sigma \in \text{Aut}(L/K)$ to the restriction map $\sigma|_S$, is a group homomorphism.

   (iii) Prove that $\Phi$ is injective. (That is, prove that two automorphisms of $L$ over $K$ whose restrictions to permutations of $S$ coincide must be equal to each other.) Conclude that $\text{Aut}(L/K)$ is isomorphic to a subgroup of $\text{Sym}(S)$.

4. Let $L$ be a field extension of a field $K$. If $\alpha \in L$ is algebraic over $K$, then we refer to the degree of the minimal polynomial for $\alpha$ over $K$ as the degree of $\alpha$ over $K$. Now suppose that $L = K(\alpha_1, \ldots, \alpha_n)$, for elements $\alpha_1, \ldots, \alpha_n \in L$ algebraic over $K$, and further suppose that $\ell_1, \ldots, \ell_n$ are the degrees, respectively, over $K$, for $\alpha_1, \ldots, \alpha_n$. Prove that 

   $$[L : K] \leq \ell_1 \cdot \ell_2 \cdots \ell_n.$$

5. Let $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt[5]{2}$ respectively denote the positive real 2nd, 3rd, and 5th roots of 2. Set $K = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[5]{2})$. Prove that $[K : \mathbb{Q}] = 30$. 