1. (Parts of the following have been noted in class, but not proved in detail.) Let \( K \) be a field.
   (i) Prove that the intersection of all of the subfields of \( K \) is itself a subfield of \( K \), called the prime subfield of \( K \).
   (ii) Let \( F \) denote the prime subfield of \( K \). Prove that \( F \) is isomorphic to either \( \mathbb{Q} \) or \( \mathbb{Z}_p \), for some prime number \( p \). We say in the first case that \( K \) has characteristic zero and in the second case that \( K \) has characteristic \( p \).
   (iii) Let \( V \) be a finite-dimensional vector space over \( \mathbb{Z}_p \). Prove there exists a prime number \( p \), and a positive integer \( \ell \), such that \( |V| = p^\ell \).
   (iv) Suppose \( K \) is finite. Prove that \( |K| \) is a power of a prime.

2. Let \( R \) be an integral domain containing a field \( K \) as a unital subring. (a) Prove that \( R \) is a \( K \)-vector space (using addition and multiplication in \( R \)). (b) Suppose that \( R \) is finite dimensional as a \( K \)-vector space. Prove that \( R \) is a field.

3. Let \( L \) be a finite field extension of a field \( K \), and let \( R \) be a unital subring of \( L \) that contains \( K \) as a unital subring. Prove that \( R \) is a field.

4. Set \( L = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \), a subfield of \( \mathbb{R} \).
   (i) Prove that \( \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \) and \( \mathbb{Q}(\sqrt{6}) \) are distinct subfields of \( L \).
   (ii) Show that \( \mathbb{Q}(\sqrt{2}) \not\subseteq L, \mathbb{Q}(\sqrt{3}) \not\subseteq L, \) and \( \mathbb{Q}(\sqrt{6}) \not\subseteq L \).
   (iii) Show that \( \mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\sqrt{6}) \cap \mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\sqrt{6}) \cap \mathbb{Q}(\sqrt{2}) = \mathbb{Q} \).
   (v) Determine \([L : \mathbb{Q}]\) and justify your result.

5. Let \( F \subseteq K \subseteq L \) be a tower of field extensions; that is, \( K \) is a (not necessarily finite) field extension of \( F \), and \( L \) is a (not necessarily finite) field extension of \( K \). Suppose that \( K \) is algebraic over \( F \) and that \( L \) is algebraic over \( K \). Prove that \( L \) is algebraic over \( F \).