1. (Parts of the following have been noted in class, but not proved in detail.) Let $K$ be a field.
   (i) Prove that the intersection of all of the subfields of $K$ is itself a subfield of $K$, called the prime subfield of $K$.
   (ii) Let $F$ denote the prime subfield of $K$. Prove that $F$ is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}_p$, for some prime number $p$. We say in the first case that $K$ has characteristic zero and in the second case that $K$ has characteristic $p$.
   (iii) Let $V$ be a finite-dimensional vector space over $\mathbb{Z}_p$. Prove there exists a prime number $p$, and a positive integer $\ell$, such that $|V| = p^\ell$.
   (iv) Suppose $K$ is finite. Prove that $|K|$ is a power of a prime.

2. Let $R$ be an integral domain containing a field $K$ as a unital subring.
   (a) Prove that $R$ is a $K$-vector space (using addition and multiplication in $R$).
   (b) Let $a$ be a nonzero element of $R$. Show that the map
       \[ R \xrightarrow{r \mapsto ar} R \]
       is an injective $K$-linear transformation and is an isomorphism if and only if $r$ is invertible as an element of $R$.
   (c) Suppose that $R$ is finite dimensional as a $K$-vector space. Prove that $R$ is a field.

3. Let $L$ be a finite field extension of a field $K$, and let $R$ be a unital subring of $L$ that contains $K$ as a unital subring. Prove that $R$ is a field.

4. Let $L$ be a finite field extension of a field $K$. Prove that $L$ is algebraic over $K$.
   (We discussed this in class but without a formal proof.) Hint: First show that $K(\gamma)$ is a finite field extension of $K$ for all $\gamma \in L$.

5. Let $F \subseteq K \subseteq L$ be a tower of field extensions; that is, $K$ is a (not necessarily finite) field extension of $F$, and $L$ is a (not necessarily finite) field extension of $K$. Suppose that $K$ is algebraic over $F$ and that $L$ is algebraic over $K$. Prove that $L$ is algebraic over $F$. Hint: Consider (4) and its solution.