1. Let $R$ be a principal ideal domain, let $a$ be a nonzero element of $R$, and let $I = (a)$ be the principal ideal of $R$ generated by $a$. Prove that there exist only finitely many ideals $J$ such that $I \subseteq J$. (Hint: To start, if $a$ is a unit, then $I = R$ and there is nothing to prove. So now suppose $a$ is not a unit and that $J$ is an ideal of $R$ containing $I$. Since $R$ is a PID, we know that $J = (b)$ for some nonzero $b \in R$, and we also know that $b$ divides $a$. Next, use unique factorization in $R$ to prove that there are only finitely many pairwise non-associate divisors of $a$.)

2. Prove that the intersection of all of the maximal ideals of $\mathbb{Q}[x]$ must equal the zero ideal. (You may use results established in previous homework assignments, if you quote them in your solution.)

3. Let $p$ be a prime, and let $R$ be a commutative, unital ring of characteristic $p$. Let $a, b \in R$. Show that $(a + b)^p = a^p + b^p$.

Also show that $(a - b)^p = a^p - b^p$.

Note: You may use – without proof – the binomial theorem, modulo $p$, to expand $(a + b)^p$ and $(a - b)^p$. For example, if $p = 3$, then

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3,$$

modulo 3.

4. Let $p$ be a prime, and set $f(x) := x^p - 1$ in $\mathbb{Z}_p[x]$. (Henceforth, for convenience, we will simply use $r$ itself to also denote the congruence class in $\mathbb{Z}_p$ of an integer $r$.) Factor $f(x)$ into irreducible polynomials in $\mathbb{Z}_p[x]$. (Hint: Use the preceding problem and unique factorization in $\mathbb{Z}_p[x]$.)

5. Let $f(x)$ be a nonscalar polynomial in $\mathbb{Q}[x]$, and let $\alpha$ be a root of $f(x)$ in $\mathbb{R}$. (Note that $f(x) \in \mathbb{R}[x]$.) Recall that $\alpha$ being a root of $f(x)$ in $\mathbb{R}[x]$ is equivalent to saying that there exists a polynomial $g(x) \in \mathbb{R}[x]$ such that

$$f(x) = (x - \alpha)g(x).$$

We say that $\alpha$ is a multiple root of $f(x)$ provided

$$f(x) = (x - \alpha)^2h(x),$$

for some polynomial $h(x) \in \mathbb{R}[x]$. Now let $f'(x)$ denote the usual derivative of $f(x)$ (viewed as a real-valued function). Prove that $\alpha$ is a multiple root of $f(x)$ if and only if both $f(\alpha)$ and $f'(\alpha)$ are equal to zero. You may use any of the differentiation rules for real-valued polynomials established in elementary calculus. (Hint: First consider what happens when the product rule is applied to $(x - \alpha)^2h(x)$.)