1. Let \( R \) be a principal ideal domain, let \( a \) be a nonzero element of \( R \), and let \( I = (a) \) be the principal ideal of \( R \) generated by \( a \). Prove that \( R/I \) has only finitely many ideals.

2. Prove that the intersection of all of the maximal ideals of \( \mathbb{Q}[x] \) must equal the zero ideal.

3. A ring isomorphism from a ring to itself is referred to as a ring automorphism. Now choose \( \alpha \) and \( \beta \) in \( \mathbb{Q} \), with \( \alpha \neq 0 \).
   
   (i) Prove that the map from \( \mathbb{Q}[x] \) to itself, sending each \( f(x) \in \mathbb{Q}[x] \) to \( f(\alpha x + \beta) \), is a ring automorphism of \( \mathbb{Q}[x] \).
   
   (ii) Prove that every ring automorphism of \( \mathbb{Q}[x] \) must be of the form described in (i), for suitable choices of \( \alpha \) and \( \beta \). (Note that you need to verify that any ring automorphism of \( \mathbb{Q}[x] \) must restrict to the identity map on \( \mathbb{Q} \).)

4. Let \( p \) be a prime, and set \( f(x) := x^p - 1 \) in \( \mathbb{Z}_p[x] \). (Henceforth, for convenience, we will simply use \( r \) itself to also denote the congruence class in \( \mathbb{Z}_p \) of an integer \( r \).) Factor \( f(x) \) into irreducible polynomials in \( \mathbb{Z}_p[x] \). (Hint: To start, divide \( f(x) \) by \( x - 1 \).)

5. Let \( K \) be a field, let \( f(x) \) be a nonscalar polynomial in \( K[x] \), and let \( \alpha \in K \) be a root of \( f(x) \). Recall that \( \alpha \) being a root of \( f(x) \) is equivalent to saying that there exists a polynomial \( g(x) \in K[x] \) such that
   \[
   f(x) = (x - \alpha)g(x).
   \]
   We say that \( \alpha \in K \) is a multiple root of \( f(x) \) provided
   \[
   f(x) = (x - \alpha)^2h(x),
   \]
   for some polynomial \( h(x) \in K[x] \). Now set \( K = \mathbb{R} \), let \( f(x) \in \mathbb{R}[x] \), and let \( \alpha \) be a real root of \( f(x) \). Let \( f'(x) \) denote the usual derivative of \( f(x) \) (viewed as a real-valued function). Prove that \( \alpha \) is a multiple root of \( f(x) \) if and only if both \( f(\alpha) \) and \( f'(\alpha) \) are equal to zero. (You may use any of the properties of derivatives of polynomials established in elementary calculus.)