1. Let $K$ be a field. (a) Let $f(x)$ and $g(x)$ be nonscalar polynomials of $K[x]$ for which $f(x) | g(x)$. Show that $f(x)$ is a proper factor of $g(x)$ if and only if $\deg f(x) < \deg g(x)$. Further show that every polynomial of degree 1 is irreducible in $K[x]$.

(b) Let $f(x) \in K[x]$, and recall that $\alpha \in K$ is a root of $f(x)$ provided $f(\alpha) = 0$ in $K$. Use the Division Algorithm to prove that $\alpha$ is a root of $f(x)$ if and only if $(x - \alpha) | f(x)$ in $k[x]$.

(c) Show that two polynomials of degree 1 in $K[x]$ are associate if and only if they have the same root.

(d) Prove that a polynomial $f(x)$ in $K[x]$ of degree two or three is irreducible if and only if $f(x)$ has no roots in $K$.

(e) Let $f(x)$ be a nonscalar polynomial in $K[x]$ of degree $n$. Prove that $f(x)$ has at most $n$ distinct roots. (Note that you have to use unique factorization in your proof. Also, don’t worry about “multiplicities” of the roots.)

2. List the irreducible polynomials of $\mathbb{Z}_2[x]$ having degree less than or equal to 4.

3. Let $R$ be a commutative ring with identity, and let $I$ be an ideal of $R$. Let $I[x]$ denote the set of polynomials in $R[x]$ having coefficients in $I$. (a) Prove that $I[x]$ is an ideal of $R[x]$. (b) Show that the quotient homomorphism $R \to R/I$ extends to a homomorphism

$$R[x] \hookrightarrow (R/I)[x]$$

with kernel $I[x]$. (c) Deduce that

$$R[x]/I[x] \cong (R/I)[x].$$

4. Let $R$ be an integral domain.

(a) Let $r$ and $s$ be nonzero elements of $R$. Show that $(r) = (s)$ if and only if $r$ and $s$ are associate.

(b) Let $I$ be an ideal of $R$. Recall that

$$I^2 = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n : a_1, b_1, a_2, b_2, \ldots, a_n, b_n \in I, \ n = 1, 2, 3, \ldots \}.$$ 

Now suppose that $R$ is a PID, and let $I$ be an ideal of $R$. Prove that $I^2 = I$ if and only if $I = (0)$ or $I = R$. (Hint: Use part (a) and unique factorization.)