1. Let $N_\alpha$, for $\alpha$ in some index set $I$, be a family of normal subgroups of a group $G$. Prove that

$$\bigcap_{\alpha \in I} N_\alpha$$

is also a normal subgroup of $G$. (You may use the following characterization of normality proved in class: A subgroup $N$ of $G$ is normal if and only if $gng^{-1} \in N$ for all $n \in N$ and $g \in G$.)

2. Let $N$ be a subgroup of a group $G$, and suppose that $N$ is generated by the set $S$. Prove: If $gsg^{-1} \in N$ for all $s \in S$ and $g \in G$, then $N$ is normal in $G$.

3. Let $G$ be a group. The *commutator subgroup* of $G$, often denoted $G'$, is the group generated by the set

$$\{aba^{-1}b^{-1} : a, b \in G\}.$$

Prove that $G$ is abelian if and only if $G'$ is equal to the trivial subgroup of $G$. Note: Elements of the form $aba^{-1}b^{-1}$, for $a, b \in G$, are referred to as *commutators*.

4. Let $G$ be a group. Prove that the commutator subgroup $G'$ is a normal subgroup of $G$.

5. Calculate the commutator subgroup of $S_3$. Note that this will be the smallest subgroup of $S_3$ containing all of the commutators. It may also be useful for you to note that a commutator in $S_3$ has to be an even permutation.

6. Let $\varphi : G \to H$ be a surjective homomorphism of groups. Prove that $H$ is abelian if and only if the commutator subgroup $G'$ is contained in the kernel of $\varphi$. 