Inviscid limit for SQG in bounded domains

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Abstract. We prove that the limit of any weakly convergent sequence of Leray-Hopf solutions of dissipative SQG equations is a weak solution of the inviscid SQG equation in bounded domains.

1. Introduction

The behavior of high Reynolds number fluids is a broad, important and mostly open problem of nonlinear physics and of PDE. Here we consider a model problem, the surface quasi-geostrophic equation, and the limit of its viscous regularizations of certain types. We prove that the inviscid limit is rigid, and no anomalies arise in the limit.

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Denote

\[
\Lambda = \sqrt{-\Delta}
\]

where \(-\Delta\) is the Laplacian operator with Dirichlet boundary conditions. The dissipative surface quasi-geostrophic (SQG) equation in \( \Omega \) is the equation

\[
\partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu + \nu \Lambda^s \theta^\nu = 0, \quad \nu > 0, \ s \in (0, 2],
\]

where \( \theta^\nu = \theta^\nu(x, t) \) with \((x, t) \in \Omega \times [0, \infty)\) and with the velocity \( u^\nu \) given by

\[
u = R\frac{\partial}{\partial z} \partial_\nu \theta^\nu := \nabla \Lambda^{-1} \theta^\nu, \quad \nabla = (-\partial_2, \partial_1).
\]

We refer to the parameter \( \nu \) as “viscosity”. Fractional powers of the Laplacian \(-\Delta\) are based on eigenfunction expansions. The inviscid SQG equation has zero viscosity

\[
\partial_t \theta + u \cdot \nabla \theta = 0, \quad u = R\frac{\partial}{\partial z} \theta.
\]

The dissipative SQG (1.1) has global weak solutions for any \( L^2 \) initial data:

**Theorem 1.1.** For any initial data \( \theta_0 \in L^2(\Omega) \) there exists a global weak solution \( \theta \)

\[
\theta \in C_w(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; D(\Lambda^{\frac{1}{2}}))
\]

to the dissipative SQG equation (1.1). More precisely, \( \theta \) satisfies the weak formulation

\[
\int_0^\infty \int_\Omega \theta \phi(x) dx d\theta(t) dt + \int_0^\infty \int_\Omega u \cdot \nabla \phi(x) dx d\theta(t) dt - \nu \int_0^\infty \int_\Omega \Lambda^{\frac{1}{2}} \partial_\nu \phi(x) dx d\theta(t) dt = 0
\]

for any \( \phi \in C_c^\infty((0, \infty)) \) and \( \phi \in D(\Lambda^{\frac{1}{2}}) \). Moreover, \( \theta \) obeys the energy inequality

\[
\frac{1}{2} \|\theta(\cdot, t)\|^2_{L^2(\Omega)} + \nu \int_0^t \int_\Omega \Lambda^{\frac{1}{2}} \|\partial_\nu \phi\|^2 dx dr \leq \frac{1}{2} \|\theta_0\|^2_{L^2(\Omega)}
\]

and the balance

\[
\frac{1}{2} \|\theta(\cdot, t)\|^2_{D(\Lambda^{-\frac{1}{2}})} + \nu \int_0^t \int_\Omega \Lambda^{\frac{1}{2}} \|\theta\|^2 dx dr = \frac{1}{2} \|\theta_0\|^2_{D(\Lambda^{-\frac{1}{2}})}
\]

for a.e. \( t > 0 \). In addition, \( \theta \in C([0, \infty); D(\Lambda^{-\frac{1}{2}})) \) for any \( \epsilon > 0 \) and the initial data \( \theta_0 \) is attained in \( D(\Lambda^{-\epsilon}) \).

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We refer to any weak solutions of (1.1) satisfying the properties (1.4), (1.5), (1.6) as a “Leray-Hopf weak solution”.

**Remark 1.2.** Theorem 1.1 for critical dissipative SQG $s = 1$ was obtained in [6].

**Remark 1.3.** Note that $C_c^\infty(\Omega)$ is not dense in $D(\Lambda^2)$ since the $D(\Lambda^2)$ norm is equivalent to the $H^2(\Omega)$ norm and $C_c^\infty(\Omega)$ is dense in $H^2_0(\Omega)$ which is strictly contained in $D(\Lambda^2)$.

The existence of $L^2$ global weak solutions for inviscid SQG (1.3) was proved in [8]. More precisely, (see Theorem 1.1, [8]) for any initial data $\theta_0 \in L^2(\Omega)$ there exists a global weak solution $\theta \in C_w(0, \infty; L^2(\Omega))$ satisfying
\[
\int_0^\infty \int_\Omega \theta \partial_t \varphi dx dt + \int_0^\infty \int_\Omega u \varphi \cdot \nabla \varphi dx dt = 0 \quad \forall \varphi \in C_c^\infty(\Omega \times (0, \infty)),
\]
and such that the Hamiltonian
\[
H(t) := \|\theta(t)\|_{D(\Lambda^{-\frac{1}{2}})}^2
\]
is constant in time. Moreover, the initial data is attained in $D(\Lambda^{-\varepsilon})$ for any $\varepsilon > 0$.

Our main result in this note establishes the convergence of weak solutions of the dissipative SQG to weak solutions of the inviscid SQG in the inviscid limit $\nu \to 0$.

**Theorem 1.4.** Let $\{\nu_n\}$ be a sequence of viscosities converging to 0 and let $\{\theta_0^{\nu_n}\}$ be a bounded sequence in $L^2(\Omega)$. Any weak limit $\theta$ in $L^2(0, T; L^2(\Omega))$, $T > 0$, of any subsequence of $\{\theta_0^{\nu_n}\}$ of Leray-Hopf weak solutions of the dissipative SQG equation (1.1) with viscosity $\nu_n$ and initial data $\theta_0^{\nu_n}$ is a weak solution of the inviscid SQG equation (1.3) on $[0, T]$. Moreover, $\theta \in C(0, T; D(\Lambda^{-\varepsilon}))$ for any $\varepsilon > 0$, and the Hamiltonian of $\theta$ is constant on $[0, T]$.

**Remark 1.5.** The same result holds true on the torus $\mathbb{T}^2$. The case of the whole space $\mathbb{R}^2$ was treated in [11].

**Remark 1.6.** With more singular constitutive laws $u = \nabla^\perp \Lambda^{-\alpha} \theta$, $\alpha \in [0, 1)$, $L^2$ global weak solutions of the inviscid equations were obtained in [3, 15]. Theorem 1.4 could be extended to this case. It is also possible to consider $L^p$ initial data in light of the work [12].

It is worth noting that in order for a given weak solution $\theta$ of the inviscid SQG to conserve the Hamiltonian, the Onsager-type critical condition requires $\theta \in L^3_t L^\infty_x$ (see [14] for $\Omega = \mathbb{T}^2$). On the other hand, the vanishing viscosity solutions obtained in Theorem 1.4 conserve the Hamiltonian even though they are only in $L^\infty_t L^2_x$. In [4], a result in the same spirit has been obtained regarding the energy conservation of weak solutions of the Euler equation on the torus $\mathbb{T}^2$.

As a corollary of the proof of Theorem 1.4 we have the following weak rigidity of inviscid SQG in bounded domains:

**Corollary 1.7.** Any weak limit in $L^2(0, T; L^2(\Omega))$, $T > 0$, of any sequence of weak solutions of the inviscid SQG equation (1.3) is a weak solution of (1.3). Here, weak solutions of (1.3) are interpreted in the sense of (1.7).

**Remark 1.8.** On tori, this result was proved in [14]. If the weak limit occurs in $L^\infty(0, T; L^2(\Omega))$ and the sequence of weak solutions conserves the Hamiltonian then so is the limiting weak solution.

The paper is organized as follows. Section 2 is devoted to basic facts about the spectral fractional Laplacian and results on commutator estimate. The proofs of Theorems 1.1 and 1.4 are given respectively in sections 3 and 4. Finally, an auxiliary lemma is given in Appendix A.
2. Fractional Laplacian and commutators

Let \( \Omega \subset \mathbb{R}^d, \ d \geq 2, \) be a bounded domain with smooth boundary. The Laplacian \(-\Delta\) is defined on \(D(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega)\). Let \( \{w_j\}_{j=1}^{\infty} \) be an orthonormal basis of \(L^2(\Omega)\) comprised of \(L^2\)–normalized eigenfunctions \(w_j\) of \(-\Delta\), i.e.

\[
-\Delta w_j = \lambda_j w_j, \quad \int_{\Omega} w_j^2 dx = 1,
\]

with \(0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_j \to \infty\).

The fractional Laplacian is defined using eigenfunction expansions,

\[
\Lambda^s f \equiv (-\Delta)^{\frac{s}{2}} f := \sum_{j=1}^{\infty} \frac{\lambda_j^{\frac{s}{2}} f_j w_j}{\|f_j\|_{L^2}}
\]

with \(f = \sum_{j=1}^{\infty} f_j w_j, \quad f_j = \int_{\Omega} f w_j dx\)

for \(s \geq 0\) and \(f \in D(\Lambda^s)\) where

\[
D(\Lambda^s) := \{f \in L^2(\Omega) : (\lambda_j^{\frac{s}{2}} f_j) \in \ell^2(\mathbb{N})\}.
\]

The norm of \(f\) in \(D(\Lambda^s)\) is defined by

\[
\|f\|_{D(\Lambda^s)} := \|(\lambda_j^{\frac{s}{2}} f_j)\|_{\ell^2(\mathbb{N})}.
\]

It is also well-known that \(D(\Lambda)\) and \(H^1_0(\Omega)\) are isometric. In the language of interpolation theory,

\[
D(\Lambda^\alpha) = [L^2(\Omega), D(-\Delta)]_{\frac{\alpha}{2}} \quad \forall \alpha \in [0, 2].
\]

As mentioned above,

\[
H^1_0(\Omega) = D(\Lambda) = [L^2(\Omega), D(-\Delta)]_{\frac{1}{2}},
\]

hence

\[
D(\Lambda^\alpha) = [L^2(\Omega), H^1_0(\Omega)]_{\alpha} \quad \forall \alpha \in [0, 1].
\]

Consequently, we can identify \(D(\Lambda^\alpha)\) with usual Sobolev spaces (see Chapter 1, [17]):

\[
D(\Lambda^\alpha) = \begin{cases} 
H^1_0(\Omega) \cap H^\alpha(\Omega) & \text{if } \alpha \in (1, 2], \\
H^\alpha_0(\Omega) & \text{if } \alpha \in (\frac{1}{2}, 1], \\
H^1_0(\Omega) \cap \{u \in H^\alpha_0(\Omega) : u/\sqrt{d(x)} \in L^2(\Omega)\} & \text{if } \alpha = \frac{1}{2}, \\
H^\alpha(\Omega) & \text{if } \alpha \in [0, \frac{1}{2}).
\end{cases}
\]

Here and below \(d(x)\) denote the distance from \(x\) to the boundary \(\partial \Omega\).

Next, for \(s > 0\) we define

\[
\Lambda^{-s} f = \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j
\]

if \(f = \sum_{j=1}^{\infty} f_j w_j \in D(\Lambda^{-s})\) where

\[
D(\Lambda^{-s}) := \left\{\sum_{j=1}^{\infty} f_j w_j \in \mathcal{D}'(\Omega) : f_j \in \mathbb{R}, \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j \in L^2(\Omega)\right\}.
\]

The norm of \(f\) is then defined by

\[
\|f\|_{D(\Lambda^{-s})} := \|\Lambda^{-s} f\|_{L^2(\Omega)} = \left(\sum_{j=1}^{\infty} \lambda_j^{-s} f_j^2\right)^{\frac{1}{2}}.
\]

It is easy to check that \(D(\Lambda^{-s})\) is the dual of \(D(\Lambda^s)\) with respect to the pivot space \(L^2(\Omega)\).
LEMMA 2.1 (Lemma 2.1, [15]). The embedding
\[ D(\Lambda^s) \subset H^s(\Omega) \]  
(2.2)
is continuous for all \( s \geq 0 \).

LEMMA 2.2. For \( s, r \in \mathbb{R} \) with \( s > r \), the embedding \( D(\Lambda^s) \subset D(\Lambda^r) \) is compact.

PROOF. Let \( \{u_n\} \) be a bounded sequence in \( D(\Lambda^s) \). Then \( \{\Lambda^r u_n\} \) is bounded in \( D(\Lambda^{s-r}) \). Choosing \( \delta > 0 \) smaller than \( \min(s-r, \frac{1}{2}) \), we have \( D(\Lambda^{s-r}) \subset D(\Lambda^{\delta}) = H^{\delta}(\Omega) \subset L^2(\Omega) \) where the first embedding is continuous and the second is compact. Consequently the embedding \( D(\Lambda^{s-r}) \subset L^2(\Omega) \) is compact and thus there exist a subsequence \( n_j \) and a function \( f \in L^2(\Omega) \) such that \( \Lambda^r u_{n_j} \) converges to \( f \) strongly in \( L^2(\Omega) \). Then \( u_{n_j} \) converges to \( u := \Lambda^{-r} f \) strongly in \( D(\Lambda^r) \) and the proof is complete. \( \square \)

A bound for the commutator between \( \Lambda \) and multiplication by a smooth function was proved in [6] using the method of harmonic extension:

THEOREM 2.3 (Theorem 2, [6]). Let \( \chi \in B(\Omega) \) with \( B(\Omega) = W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega) \) if \( d \geq 3 \), and \( B(\Omega) = W^{2,p}(\Omega) \) with \( p > 2 \) if \( d = 2 \). There exists a constant \( C(d, p, \Omega) \) such that
\[ \|[\Lambda, \chi]\psi\|_{D(\Lambda^{\frac{1}{2}})} \leq C(d, p, \Omega)\|\chi\|_{B(\Omega)}\|\psi\|_{D(\Lambda^{\frac{1}{2}})} \]

Pointwise estimates for the commutator between fractional Laplacian and differentiation were established in [8]:

THEOREM 2.4 (Theorem 2.2, [8]). For any \( p \in [1, \infty] \) and \( s \in (0, 2) \) there exists a positive constant \( C(d, s, p, \Omega) \) such that for all \( \psi \in C_c^\infty(\Omega) \) we have
\[ \|[\Lambda^s, \nabla]\psi(x)\| \leq C(d, s, p, \Omega) d(x)^{-s-1-\frac{d}{p}} \|\psi\|_{L^p(\Omega)} \]
holds for all \( x \in \Omega \).

This pointwise bound implies the following commutator estimate in Lebesgue spaces.

THEOREM 2.5. Let \( p, q \in [1, \infty] \), \( s \in (0, 2) \) and \( \varphi \) satisfy
\[ \varphi(\cdot)d(\cdot)^{-s-1-\frac{d}{p}} \in L^q(\Omega). \]
Then the operator \( \varphi[\Lambda^s, \nabla] \) can be uniquely extended from \( C_c^\infty(\Omega) \) to \( L^p(\Omega) \) such that there exists a positive constant \( C = C(d, s, p, \Omega) \) such that
\[ \|\varphi[\Lambda^s, \nabla]\psi\|_{L^q(\Omega)} \leq C\|\varphi(\cdot)d(\cdot)^{-s-1-\frac{d}{p}}\|_{L^q(\Omega)}\|\psi\|_{L^p(\Omega)} \]
(2.3)
holds for all \( \psi \in L^p(\Omega) \).

(2.3) is remarkable in that the commutator between an operator of order \( s \in (0, 2) \) and an operator of order 1 is an operator of order 0.

3. Proof of Theorem 1.1

We use Galarkin approximations. Denote by \( \mathbb{P}_m \) the projection in \( L^2(\Omega) \) onto the linear span \( L^2_{m_0} \) of eigenfunctions \( \{w_1, \ldots, w_m\} \), i.e.
\[ \mathbb{P}_m f = \sum_{j=1}^m f_j w_j \quad \text{for } f = \sum_{j=1}^\infty f_j w_j. \]  
(3.1)
The $m$th Galerkin approximation of (1.1) is the following ODE system in the finite dimensional space $L^2_m$:

$$
\begin{align*}
\dot{\theta}_m + P_m(u_m \cdot \nabla \theta_m) + \nu \Lambda^s \theta_m &= 0, & t > 0, \\
\theta_m &= \tilde{P}_m \theta_0, & t = 0,
\end{align*}
$$

with $\theta_m(x, t) = \sum_{j=1}^m \theta_j^{(m)}(t) w_j(x)$ and $u_m = R_D \theta_0$ satisfying $\text{div} \ u_m = 0$. Note that (3.2) is equivalent to

$$
\frac{d\theta_l^{(m)}}{dt} + \sum_{j,k=1}^m \gamma_{jkl}^{(m)} \theta_j^{(m)} \theta_k^{(m)} + \nu \Lambda^s \theta_l^{(m)} = 0, \quad l = 1, 2, \ldots, m,
$$

with

$$
\gamma_{jkl}^{(m)} = \lambda_j^{\frac{1}{2}} \int_\Omega \left( \nabla \cdot w_j \cdot \nabla w_k \right) w_l dx.
$$

The local existence of $\theta_m$ on some time interval $[0, T_m]$ follows from the Cauchy-Lipschitz theorem. On the other hand, the antisymmetry property $\gamma_{jkl}^{(m)} = -\gamma_{lkj}^{(m)}$ yields

$$
\frac{1}{2} \| \theta_m(\cdot, t) \|^2_{L^2(\Omega)} + \nu \int_0^t \int_\Omega |\Lambda^{\frac{1}{2}} \theta_m|^2 dx dr = \frac{1}{2} \| P_m \theta_0 \|^2_{L^2(\Omega)} \leq \frac{1}{2} \| \theta_0 \|^2_{L^2(\Omega)}
$$

for all $t \in [0, T_m]$. This implies that $\theta_m$ is global and (3.4) holds for all positive times. The sequence $\theta_m$ converges weakly-* in $L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; D(\Lambda^{\frac{1}{2}}))$. Upon extracting a subsequence, we have $\theta_m \rightarrow \theta$ weakly-* in $L^\infty(0, \infty; L^2(\Omega))$ and weakly in $L^2(0, \infty; D(\Lambda^{\frac{1}{2}}))$. In particular, $\theta$ obeys the same energy inequality as in (3.4). On the other hand, if one multiplies (3.3) by $\lambda_j^{-1/2} \theta_l^{(m)}$ and uses the fact that $\gamma_{jkl}^{(m)} \lambda_l^{-1/2} = -\gamma_{lkj}^{(m)} \lambda_j^{-1/2}$, one obtains

$$
\frac{1}{2} \| \theta_m(\cdot, t) \|^2_{D(\Lambda^{-\frac{1}{2}})} + \nu \int_0^t \int_\Omega |\Lambda^{\frac{1}{2}} \theta_m|^2 dx dr = \frac{1}{2} \| P_m \theta_0 \|^2_{D(\Lambda^{-\frac{1}{2}})}.
$$

We derive next a uniform bound for $\partial_t \theta_m$. Let $N > 0$ be an integer to be determined. For any $\varphi \in D(\Lambda^{2N})$ we integrate by parts to get

$$
\int_\Omega \partial_t \theta_m \varphi dx = -\int_\Omega P_m \text{div}(u_m \theta_m) \varphi dx - \int_\Omega \nu \Lambda^s \theta_m \varphi dx = \int_\Omega (u_m \theta_m) \cdot \nabla (P_m \varphi) dx - \int_\Omega \nu \theta_m \Lambda^s \varphi dx.
$$

The first term is controlled by

$$
\left| \int_\Omega (u_m \theta_m) \cdot \nabla (P_m \varphi) dx \right| \leq \| u_m \theta_m \|_{L^1(\Omega)} \| \nabla P_m \varphi \|_{L^\infty(\Omega)} \leq C \| P_m \varphi \|_{H^3(\Omega)}.
$$

According to Lemma A.1 for $N$ and $k$ satisfying $N > \frac{k}{2} + 1$ there exists a positive constant $C_{N,k}$ such that

$$
\| P_m \varphi \|_{H^k(\Omega)} \leq C_{N,k} \| \varphi \|_{D(\Lambda^{2N})} \quad \forall m \geq 1, \quad \forall \varphi \in D(\Lambda^{2N}).
$$

With $k = 3$ and $N = 3$ we have

$$
\left| \int_\Omega (u_m \theta_m) \cdot \nabla (P_m \varphi) dx \right| \leq C \| \varphi \|_{D(\Lambda^6)}.
$$

On the other hand,

$$
\left| \int_\Omega \nu \theta_m \Lambda^s \varphi dx \right| \leq C \| \theta_m \|_{L^2(\Omega)} \| \varphi \|_{D(\Lambda^2)}.
$$

We have proved that

$$
\left| \int_\Omega \partial_t \theta_m \varphi dx \right| \leq C \| \varphi \|_{D(\Lambda^6)} \quad \forall \varphi \in D(\Lambda^6).
$$
Because $L^2(\Omega) \times D(\Lambda^6) \ni (f,g) \mapsto \int_{\Omega} f g dx$ extends uniquely to a bilinear from on $D(\Lambda^{-6}) \times D(\Lambda^6)$, we deduce that $\partial_t \theta_m$ are uniformly bounded in $L^\infty(0,\infty; D(\Lambda^{-6}))$. Note that we have used only the uniform regularity $L^\infty(0,\infty; L^2(\Omega))$ of $\theta_m$. We have the embeddings $D(\Lambda^2) \subset D(\Lambda^{(s-1)/2}) \subset D(\Lambda^{-6})$ where the first one is compact by virtue of Lemma 2.2, and the second is continuous. Fix $T > 0$. Aubin-Lions’ lemma (see \[16\]) ensures that for some function $f$ and along some subsequence $\theta_m$ converge to $f$ weakly in $L^2(0,T; D(\Lambda^{\frac{2}{3}}))$ and strongly in $L^2(0,T; D(\Lambda^{(s-1)/2}))$. Apriori, both $f$ and the subsequence depend on both $T$. However, we already know that $\theta_m \to \theta$ weakly in $L^2(0,\infty; D(\Lambda^{\frac{2}{3}}))$. Therefore, $f = \theta$ and the convergences to $\theta$ hold for the whole sequence. Similarly, applying Aubin-Lions’ lemma with the embeddings $L^2(\Omega) \subset D(\Lambda^{-6}) \subset D(\Lambda^{-6})$ for sufficiently small $\varepsilon > 0$ we obtain that $\theta_m \to \theta$ strongly in $C([0,T]; D(\Lambda^{-\varepsilon}))$. Integrating (3.2) against an arbitrary test function of the form $\phi(t) \varphi(x)$ with $\varphi \in C_\infty((0,T))$, $\varphi \in D(\Lambda^6)$ yields

$$\int_0^T \int_\Omega \theta_m \varphi(x) dx \partial_t \phi(t) dt + \int_0^T \int_\Omega u_m \theta_m \cdot \nabla \mathbb{P}_m \varphi(x) dx \phi(t) dt - \nu \int_0^T \int_\Omega \Lambda^\frac{2}{3} \theta_m \Lambda^\frac{2}{3} \varphi(x) dx \phi(t) dt = 0.$$ 

By Lemma \[A.1\],  

$$\|(\mathbb{I} - \mathbb{P}_m) \varphi\|_{L^\infty(\Omega)} \leq C \|\|(\mathbb{I} - \mathbb{P}_m) \varphi\|_{H^3(\Omega)} \to 0 \quad \text{as} \quad m \to \infty.$$ 

The weak convergence of $\theta_m$ in $L^2(0,T; D(\Lambda^{\frac{2}{3}}))$ allows one to pass to the limit in the two linear terms. The strong convergence of $\theta_m$ in $L^2(0,T; D(\Lambda^{\frac{2}{3}}))$ together with the weak convergence of $u_m$ in the same space allows one to pass to the limit in the nonlinear term and conclude that $\theta$ satisfies the weak formulation (1.4) with $\varphi \in D(\Lambda^6)$. In fact, $\theta \in L^2(0,\infty; D(\Lambda^{\frac{2}{3}})) \subset L^2(0,\infty; L^p(\Omega))$ for some $p > 2$, hence $u \theta \in L^2(0,\infty; L^q(\Omega))$ for some $q > 1$. In addition, if $\varphi \in D(\Lambda^2)$ then $\nabla \varphi \in L^r$ for all $r < \infty$, and thus the nonlinearity $\int_\Omega u \theta \cdot \nabla \varphi dx$ makes sense. Then because $D(\Lambda^2)$ is dense in $D(\Lambda^6)$, (1.4) holds for $\varphi \in D(\Lambda^2)$.

We now pass to the limit in (3.3). The strong convergence $\theta_m \to \theta$ in $C([0,T]; D(\Lambda^{-\varepsilon}))$ gives the convergence of the first term. On the other hand, the strong convergence $\theta_m \to \theta$ in $L^2(0,T; D(\Lambda^{(s-1)/2}))$ yields the convergence of the second term. The right hand side converges to $\frac{1}{2} \|\theta_0\|^2_{D(\Lambda^{-\frac{s}{2}})}$ since $\mathbb{P}_m \theta_0$ converge to $\theta_0$ in $L^2(\Omega)$. We thus obtain (1.6).

Since $\theta_m \to \theta$ in $C([0,T]; D(\Lambda^{-\varepsilon}))$ we deduce that  

$$\theta_0 = \lim_{m \to \infty} \mathbb{P}_m \theta_0 = \lim_{m \to \infty} \theta_m |_{t=0} = \theta |_{t=0} \quad \text{in} \quad D(\Lambda^{-\varepsilon}).$$

For a.e. $t \in [0,T]$, $\theta_m(t)$ are uniformly bounded in $L^2(\Omega)$, and thus along some subsequence $m_j$, a priori depending on $t$, we have $\theta_{m_j}(t)$ converge weakly to some $f(t)$ in $L^2(\Omega)$. But we know $\theta_m(t) \to \theta(t)$ in $D(\Lambda^{-\varepsilon})$. Thus, $f(t) = \theta(t)$ and $\theta_m(t) \to \theta(t)$ in $L^2(\Omega)$ as a whole sequence for a.e. $t \in [0,T]$. Recall that $\frac{d}{dt} \theta_m$ are uniformly bounded in $L^\infty(0,T; D(\Lambda^{-6}))$. For all $\varphi \in D(\Lambda^6)$ and $t \in [0,T]$ we write  

$$\langle \theta_m(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_m(0), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} + \int_0^t \langle \frac{d}{dt} \theta_m(r), \varphi \rangle_{D(\Lambda^{-6}), D(\Lambda^6)} dr.$$ 

Because $\frac{d}{dt} \theta_m$ converge to $\frac{d}{dt} \theta$ weakly-$*$ in $L^\infty(0,T; D(\Lambda^{-6}))$, letting $m \to \infty$ yields  

$$\langle \theta(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_0, \varphi \rangle_{L^2(\Omega), L^2(\Omega)} + \int_0^t \langle \frac{d}{dt} \theta(r), \varphi \rangle_{D(\Lambda^{-6}), D(\Lambda^6)} dr$$ 

for a.e. $t \in [0,T]$. Taking the limit $t \to 0$ gives  

$$\lim_{t \to 0} \langle \theta(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_0, \varphi \rangle_{L^2(\Omega), L^2(\Omega)}$$ 

for all $\varphi \in D(\Lambda^6)$. Finally, since $D(\Lambda^6)$ is dense in $L^2(\Omega)$ and $\theta \in L^\infty(0,T; L^2(\Omega))$ we conclude that $\theta \in C_w(0,T; L^2(\Omega))$ for all $T > 0$.  

4. Proof of Theorem 1.4

First, using approximations and commutator estimates we justify the commutator structure of the SQG nonlinearity derived in [8].

**Lemma 4.1.** For all $\psi \in H^1_0(\Omega)$ and $\varphi \in C^\infty_c(\Omega)$ we have

$$
\int_\Omega \Lambda \psi \nabla \psi \cdot \nabla \varphi dx = \frac{1}{2} \int_\Omega \left[ [\Lambda, \nabla^\perp] \psi \cdot \nabla \varphi dx - \frac{1}{2} \int_\Omega \nabla^\perp \psi \cdot [\Lambda, \nabla \varphi] \psi dx. \right. \tag{4.1}
$$

Here, the commutator $[\Lambda, \nabla^\perp] \psi \cdot \nabla \varphi$ is understood in the sense of the extended operator defined in Theorem 2.5.

**Proof.** Let $\psi_n \in C^\infty_c(\Omega)$ converging to $\psi$ in $H^1_0(\Omega)$. Integrating by parts and using the fact that $\nabla^\perp \cdot \nabla \varphi = 0$ gives

$$
\int_\Omega \Lambda \psi_n \nabla^\perp \psi_n \cdot \nabla \varphi dx = - \int_\Omega \psi_n \nabla^\perp \Lambda \psi_n \cdot \nabla \varphi dx,
$$

Because $\psi_n$ is smooth and has compact support inside $\Omega$, $\nabla^\perp \psi_n \in D(\Lambda)$, and thus we can commute $\nabla^\perp$ with $\Lambda$ to obtain

$$
\int_\Omega \Lambda \psi_n \nabla^\perp \psi_n \cdot \nabla \varphi dx
\begin{aligned}
&= - \int_\Omega \psi_n [\nabla^\perp, \Lambda] \psi_n \cdot \nabla \varphi dx - \int_\Omega \psi_n \Lambda \nabla^\perp \psi_n \cdot \nabla \varphi dx \\
&= - \int_\Omega \psi_n [\nabla^\perp, \Lambda] \psi_n \cdot \nabla \varphi dx - \int_\Omega \nabla^\perp \psi_n \cdot [\Lambda, \nabla \varphi] \psi_n dx - \int_\Omega \nabla^\perp \psi_n \cdot \nabla \varphi \Lambda \psi_n dx.
\end{aligned}
$$

Noticing that the last term on the right-hand side is exactly the negative of the left-hand side, we deduce that

$$
\int_\Omega \Lambda \psi_n \nabla^\perp \psi_n \cdot \nabla \varphi dx = \frac{1}{2} \int_\Omega [\Lambda, \nabla^\perp] \psi_n \cdot \nabla \varphi \psi_n dx - \frac{1}{2} \int_\Omega \nabla^\perp \psi_n \cdot [\Lambda, \nabla \varphi] \psi_n dx.
$$

The commutator estimates in Theorems 2.3 and 2.5 then allow us to pass to the limit in the preceding representation and conclude that (4.1) holds. □

Now let $\nu_n \to 0^+$ and let $\theta_0^{\nu_n}$ be a bounded sequence in $L^2(\Omega)$. For each $n$ let $\theta_n \equiv \theta^{\nu_n}$ be a Leray-Hopf weak solution of (1.1) with viscosity $\nu_n$ and initial data $\theta_0^{\nu_n}$. In view of the energy inequality (1.5), $\theta_n$ are uniformly bounded in $L^\infty(0, \infty; L^2(\Omega))$ and satisfies

$$
\int_0^\infty \int_\Omega \psi(x)dx \partial_t \phi(t) dt + \int_0^\infty \int_\Omega u_n \theta_n \cdot \nabla \varphi(x) dx \phi(t) dt - \nu_n \int_0^\infty \int_\Omega \Lambda \xi \theta_n \Lambda \xi \varphi(x) dx \phi(t) dt = 0 \tag{4.2}
$$

for all $\phi \in C^\infty_c((0, \infty))$ and $\varphi \in D(\Lambda^2)$. Fix $T > 0$. Assume that along a subsequence, still labeled by $n$, $\theta_n$ converge to $\theta$ weakly in $L^2(0, T; L^2(\Omega))$. We prove that $\theta$ is a weak solution of the inviscid SQG equation. We first prove a uniform bound for $\partial_t \theta_n$ provided only the uniform regularity $L^\infty(0, \theta; L^2(\Omega))$ of $\theta_n$. To this end, let us define for a.e. $t \in [0, T]$ the function $f_n(t) \in H^{-3}(\Omega)$ by

$$
\langle f_n(t), \varphi \rangle_{H^{-3}(\Omega), H_0^3(\Omega)} := \int_\Omega (u_n(x, t) \theta_n(x, t) \cdot \nabla \varphi(x) - \nu_n \theta_n(x, t) \Lambda \xi \varphi(x)) dx
$$

for all $\varphi \in H_0^3(\Omega) \subset D(\Lambda^2)$, where $H_0^3(\Omega)$ is the closure of $C^{\infty}_c(\Omega)$ in $H^3(\Omega)$ for any $\mu > 0$. Indeed, we have

$$
\left| \int_\Omega (u_n(x, t) \theta_n(x, t) \cdot \nabla \varphi(x) - \nu_n \theta_n(x, t) \Lambda \xi \varphi(x)) dx \right| \leq C \left( \| \theta_n(t) \|_{L^2(\Omega)}^{1.5} + 1 \right) \| \varphi \|_{H^3(\Omega)}.
$$
This shows that \( f_n \) are uniformly bounded in \( L^\infty(0, T; H^{-3}(\Omega)) \). Then for any \( \phi \in C_c^\infty((0, T)) \), it follows from (4.2) that
\[
\int_0^T \theta_n \partial_t \phi dt = - \int_0^T f_n \phi dt
\]
in \( H^{-3}(\Omega) \). In other words, \( \partial_t \theta_n = f_n \) and the desired uniform bound for \( \partial_t \theta_n \) follows. Fix \( \varepsilon \in (0, \frac{1}{2}) \). Aubin-Lions’ lemma applied with the embeddings \( L^2(\Omega) \subset D(\Lambda^{-\varepsilon}) \subset H^{-3}(\Omega) \) then ensures that \( \theta_n \) converge to \( \theta \) strongly in \( C(0, T; D(\Lambda^{-\varepsilon})) \). Consequently \( \psi_n \) converge to \( \psi := \Lambda^{-1} \theta \) strongly in \( C(0, T; D(\Lambda^{1-\varepsilon})) \).

Now we take \( \phi \in C_c^\infty((0, \infty)) \) and \( \varphi \in C_c^\infty(\Omega) \). By virtue of Lemma 4.1 the weak formulation (1.4) gives
\[
\int_0^T \int_\Omega \theta_n \varphi(x) dx \partial_t \phi(t) dt + \frac{1}{2} \int_0^T \int_\Omega [\Lambda, \nabla^\perp] \psi_n \cdot \nabla \varphi(x) \psi_n dx \phi(t) dt
\]
\[
- \frac{1}{2} \int_0^T \int_\Omega \nabla^\perp \psi_n \cdot [\Lambda, \nabla \varphi(x)] \psi_n dx \phi(t) dt - \nu_n \int_0^T \int_\Omega \theta_n \Lambda^s \varphi(x) dx \phi(t) dt = 0,
\]
where \( \psi_n := \Lambda^{-1} \theta_n \) are uniformly bounded in \( L^\infty(0, T; H^1_0(\Omega)) \). The weak convergence \( \theta_n \to \theta \) in \( L^2(0, T; L^2(\Omega)) \) readily yields
\[
\lim_{n \to \infty} \int_0^T \int_\Omega \theta_n \varphi(x) dx \partial_t \phi(t) dt = \int_0^T \int_\Omega \theta \varphi(x) dx \partial_t \phi(t) dt
\]
and
\[
\lim_{n \to \infty} \nu_n \int_0^T \int_\Omega \theta_n \Lambda^s \varphi(x) dx \phi(t) dt = 0.
\]

Next we pass to the limit in the two nonlinear terms. Applying the commutator estimate in Theorem 2.3 we have
\[
\left| \int_0^T \int_\Omega \nabla^\perp \psi_n \cdot [\Lambda, \nabla \varphi] \psi_n dx \phi dt - \int_0^T \int_\Omega \nabla^\perp \psi \cdot [\Lambda, \nabla \varphi] \psi dx \phi dt \right|
\]
\[
\leq \int_0^T \int_\Omega \nabla^\perp (\psi_n - \psi) \cdot [\Lambda, \nabla \varphi] \psi dx \phi dt + \| \phi \nabla^\perp \psi_n \|_{L^2(0, T; L^2(\Omega))} \| [\Lambda, \nabla \varphi] (\psi_n - \psi) \|_{L^2(0, T; L^2(\Omega))}
\]
\[
\leq \int_0^T \int_\Omega \nabla^\perp (\psi_n - \psi) \cdot [\Lambda, \nabla \varphi] \psi dx \phi dt + C \| \psi_n - \psi \|_{L^2(0, T; D(\Lambda^{\frac{1}{2}}))}.
\]
The first term converges to 0 due to the weak convergence of \( \psi_n \) to \( \psi \) in \( L^2(0, T; H^1_0(\Omega)) \) and the fact that \( [\Lambda, \nabla \varphi] \psi \in D(\Lambda^{\frac{1}{2}}) \subset L^2(\Omega) \) in view of Theorem 2.3. The second term also converges to 0 due to the strong convergence of \( \psi_n \) to \( \psi \) in \( C(0, T; D(\Lambda^{1-\varepsilon})) \) with \( \varepsilon \in (0, \frac{1}{2}) \). Finally, we apply the commutator estimate in Theorem 2.3 to obtain
\[
\left| \int_0^T \int_\Omega [\Lambda, \nabla^\perp] \psi_n \cdot \nabla \varphi \psi_n dx \phi dt - \int_0^T \int_\Omega [\Lambda, \nabla^\perp] \psi \cdot \nabla \varphi \psi dx \phi dt \right|
\]
\[
\leq \| \nabla \varphi [\Lambda, \nabla^\perp] (\psi_n - \psi) \|_{L^2(0, T; L^2(\Omega))} \| \phi \psi_n \|_{L^2(0, T; L^2(\Omega))} + \| [\Lambda, \nabla^\perp] \psi \cdot \nabla \varphi \psi \|_{L^2(0, T; L^2(\Omega))} \| \phi (\psi_n - \psi) \|_{L^2(0, T; L^2(\Omega))}
\]
\[
\leq C \| \psi_n - \psi \|_{L^2(0, T; L^2(\Omega))}
\]
which converges to 0. Putting together the above considerations leads to
\[
\int_0^T \int_\Omega \theta \varphi(x) dx \partial_t \phi(t) dt + \int_0^T \int_\Omega u \theta \cdot \nabla \varphi(x) dx \phi(t) dt = 0, \quad \forall \phi \in C_c^\infty((0, T)), \ \varphi \in C_c^\infty(\Omega).
\]
Therefore, \( \theta \) is a weak solution of the inviscid SQG equation on \([0, T] \).
Finally, let us show the Hamiltonian conservation of $\theta$. We have the energy balance (1.6) for each $\theta_n$. If $s \leq 1$, then the uniform boundedness of $\theta_n$ in $L^\infty(0, T; L^2(\Omega))$ implies

$$
\lim_{n \to \infty} \nu_n \int_0^t \int_\Omega |\Lambda^{s-\frac{1}{2}} \theta_n|^2 \, dx \, dr = 0, \quad t \in [0, T].
$$

(4.3)

In addition, $\theta_n \to \theta$ strongly in $C(0, T; D(\Lambda^{-\varepsilon})) \subset C(0, T; D(\Lambda^{-\frac{1}{2}}))$. Letting $\nu = \nu_n \to 0$ in the balance (1.6) we conclude that the Hamiltonian of $\theta$ is constant on $[0, T]$. Consider next the case $s \in (1, 2]$. Then since $\Lambda^{s-\frac{1}{2}} \in (0, \frac{s}{2})$ it follows by interpolation that

$$
\|\Lambda^{s-\frac{1}{2}} \theta_n\|_{L^2(\Omega)}^2 \leq \|\theta_n\|_{L^2(\Omega)}^{2(1-\lambda)} \|\Lambda^\lambda \theta_n\|_{L^2(\Omega)}^{2\lambda} \leq C \|\Lambda^\lambda \theta_n\|_{L^2(\Omega)}^{2\lambda}
$$

for some $\lambda \in (0, 1)$ depending only on $s$. Thus, for any $\delta > 0$,

$$
\nu_n \int_0^t \|\Lambda^{s-\frac{1}{2}} \theta_n\|_{L^2(\Omega)}^2 \, dt \leq C t \nu_n \delta^{-\frac{1}{1-\lambda}} + C \delta \nu_n \int_0^T \|\Lambda^\lambda \theta_n\|_{L^2(\Omega)}^2 \, dr, \quad t \in [0, T].
$$

By virtue of (1.5), the energy dissipations $\nu_n \int_0^t \int_\Omega |\Lambda^\lambda \theta_n|^2 \, dx \, dt$, $t \in [0, T]$, are uniformly bounded. Sending $\nu_n \to 0$ and then $\delta \to 0$ yields (4.3) for this case. This completes the proof.

**Appendix A. A bound on $P_m$**

Recall the definition (3.1) of $P_m$. The following lemma is essentially taken from [8]. We include the proof for the sake of completeness.

**Lemma A.1.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary. For every $N$ and $k \in \mathbb{N}$ satisfying $N > \frac{k}{2} + \frac{d}{2}$ there exists a positive constant $C_{N, k}$ such that

$$
\|P_m \varphi\|_{H^k(\Omega)} \leq C_{N, k} \|\varphi\|_{D(\Lambda^{2N})}
$$

(A.1)

for all $m \geq 1$ and $\varphi \in D(\Lambda^{2N})$; moreover, we have

$$
\lim_{m \to \infty} \|(I - P_m) \varphi\|_{H^k(\Omega)} = 0.
$$

(A.2)

**Proof.** As $\varphi \in D(\Lambda^{2N})$, we have $\Delta^\ell \varphi \in H^\ell_0(\Omega)$ for all $\ell = 0, 1, \ldots, N - 1$. This allows repeated integration by parts with $w_j$ using the relation $-\Delta w_j = \lambda_j w_j$. Using Hölder’s inequality and the fact that $w_j$ is normalized in $L^2$, we obtain

$$
|\varphi_j| \leq \lambda_j^{-N} \|\Delta^N \varphi\|_{L^2}, \quad \varphi_j = \int_\Omega \varphi w_j \, dx.
$$

By elliptic regularity estimates and induction, we have for all $k \in \mathbb{N}$ that

$$
\|w_j\|_{H^k(\Omega)} \leq C_k \lambda_j^{\frac{k}{2}}.
$$

We know from the easy part of Weyl’s asymptotic law that $\lambda_j \geq C j^{\frac{2}{N}}$. Consequently, with $N > \frac{k}{2} + \frac{d}{2}$ we deduce that

$$
\sum_{j=1}^\infty |\varphi_j| \|w_j\|_{H^k(\Omega)} \leq C_k \|\Delta^N \varphi\|_{L^2} \sum_{j=1}^\infty \lambda_j^{-N + \frac{1}{2}}
$$

$$
\leq C_k \|\varphi\|_{D(\Lambda^{2N})} \sum_{j=1}^\infty j^{(-N + \frac{1}{2}) \frac{2}{N}}
$$

$$
= C_{N, k} \|\varphi\|_{D(\Lambda^{2N})}
$$
where $C_{N,k} < \infty$ depends only on $N$ and $k$. Because

$$(1 - P_m) \varphi = \sum_{j=m+1}^{\infty} \varphi_j w_j,$$

this proves both (A.1) and (A.2). The proof is complete. \hfill \Box

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**References**


